Bull. Korean Math. Soc. **38** (2001), No. 3, pp. 565–573

NEW MAXIMUM THEOREMS WITH STRICT QUASI-CONCAVITY

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ABSTRACT. In this paper, we first prove the strict quasi-concavity of maximizing function, and next prove a new maximum theorem using Fan's generalization of the classical KKM theorem. Also an existence theorem of social equilibrium can be proved when an additional assumption on the constraint correspondence is assumed. Finally, we give illustrative two examples of constrained optimization problems.

1. Introduction

In 1959, Berge [2] first proved the maximum theorem which gives conditions under which a "maximizing correspondence" will be closed, and the original form is essentially as follows :

Let E and Y be topological spaces and let $u : E \times Y \to \mathbb{R}$ be a continuous real-valued function; let $F : E \to 2^Y$ be a continuous and compact valued correspondence; and, for each $x \in E$, let $M(x) := \{y \in F(x) | u(x,y) \geq u(x,z)$ for all $z \in F(x)\}$. Then the correspondence M is upper semicontinuous and non-empty compact valued.

Since then, this theorem, called Berge's maximum theorem, has become one of the most useful and powerful theorems in economics, optimization theory, and game theory. And there have been many generalizations and applications of Berge's theorem, e.g., Leininger [9], Park [11], Tian-Zhou [12] and Walker [13]. In their generalizations, continuity assumptions on u and F have been relaxed; but the properties of continuity assumptions of u and F are still needed in the different forms, e.g., graph-continuity in [9] and transfer-continuity in [12].

Received March 22, 2001.
2000 Mathematics Subject Classification: Primary 52A41, 91A40, 91B50.
Key words and phrases: maximum theorem, strictly quasi-concave, convex, social equilibrium.

On the other hand, there also have been several maximum theorems which can be comparable to Berge's theorem in different settings, e.g., [8]. And those theorems can be useful for nonlinear settings in several economic models.

The purpose of this paper is two-fold. First, we shall prove the strict quasi-concavity of maximizing functions, and next prove a new maximum theorem using Fan's generalization of the classical KKM theorem. Second, an existence theorem of social equilibrium can be proved when an additional assumption on the constraint correspondence is assumed. Finally, we give two illustrative examples of constrained optimization problems.

2. Preliminaries

Let A be a subset of a topological space X. We shall denote by 2^A the family of all subsets of A. If A is a subset of a vector space, we shall denote by $co\,A$ the convex hull of A. Let X, Y be topological vector spaces and $T : X \to 2^Y$ be a correspondence (or a multimap). Recall that a correspondence T is said to be *convex* [10] if $\lambda T(x_1) + (1 - \lambda)T(x_2) \subseteq T(\lambda x_1 + (1 - \lambda)x_2)$ for each $x_1, x_2 \in X$, and every $\lambda \in [0, 1]$. A correspondence $T : X \to 2^Y$ is said to be (1) *upper semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$ and (2) *lower semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x in X such that $T(y) \cap V \neq \emptyset$ for each $y \in U$, and (3) *continuous* if T is both upper semicontinuous and lower semicontinuous.

Next we recall the continuity definitions of the real-valued function. Let X be a non-empty subset of a topological space E and $f : X \to \mathbb{R}$. We say that f is *upper semicontinuous* if for each $t \in \mathbb{R}$, $\{x \in X \mid f(x) \geq t\}$ is closed in X, and f is *lower semicontinuous* if $-f$ is upper semicontinuous. And we say that f is *continuous* if f is both upper semicontinuous and lower semicontinuous. For the other standard notations and terminologies, we shall refer to [1, 3, 5, 10].

Now we recall some concept which generalize the concavity as follows : Let X be a non-empty convex subset of a vector space E and let $f : X \to \mathbb{R}$. We say that f is *strictly quasi-concave on X* [5] if for each $x_1, x_2 \in X$ with $x_1 \neq x_2$, and every $\lambda \in (0, 1)$, we have

$$f(\lambda x_1 + (1 - \lambda)x_2) > \min\{f(x_1), f(x_2)\}.$$

Then it is easy to see that if f is strictly quasi-concave on X, then for each $t \in \mathbb{R}$, $\{x \in X \mid f(x) > t\}$ is convex. However, the converse does not hold in general. In fact, if $\{x \in X \mid f(x) > t\}$ is convex for each $t \in \mathbb{R}$, then we have $f(\lambda x_1 + (1-\lambda)x_2) \geq \min\{f(x_1), f(x_2)\}$ for each $x_1, x_2 \in X$ with $x_1 \neq x_2$, and every $\lambda \in (0,1)$; i.e., f is quasi-concave on X. It should be noted that if f, g are strictly quasi-concave, then $f+g$ is not strictly quasi-concave in general.

Let X be an arbitrary non-empty subset of a Hausdorff topological vector space E. A multimap $T : X \to 2^E$ is called a *KKM-map* [7] if $co\{x_1, \ldots, x_k\} \subseteq \cup_{i=1}^k T(x_i)$ for each finite subset $\{x_1, \ldots, x_k\} \subset X$. Note that if T is a KKM-map, then $x \in T(x)$ for each $x \in X$.

We shall need the following infinite dimensional version of the KKM theorem due to Fan [6] :

LEMMA 1. *Let X be a non-empty subset of a Hausdorff topological vector space and let $T : X \to 2^E$ be a closed valued KKM-map. If $T(x_0)$ is compact for at least one $x_0 \in X$, then $\cap_{x \in X} T(x) \neq \emptyset$.*

Economic analysis makes substantial use of theorems on the properties of maximizing functions. As an example, consider the problem of optimal choices of the vector x, as part of a plan of optimal choices of x and y in the following constrained optimization programming problem :

$$\text{Maximize} \quad f(x,y) \quad \text{subject to} \quad h(x,y) \leq 0, \quad x \geq 0, \quad y \geq 0.$$

In this paper, in order to apply the KKM theorem or fixed point theorems, we shall replace the above maximization problem with the following form :

$$\text{Maximize} \quad g(x) \quad \text{subject to} \quad x \geq 0,$$

when g is defined by

$$g(x) := \max_{y \in C(x)} f(x,y),$$

where $C(x) := \{y \in Y \mid h(x,y) \leq 0, \ y \geq 0\}$.

Maximization problems of the above type arise in analysis of intertemporal allocation, where x is the vector of present values of the variables,

y is the vector of future values of the variables, and $h(x,y)$ is a constraint relating y to x. Other analysis in which this type of formulation is used are models of the firm and various models of the consumer, where an information structure of competition technology is introduced.

Solving the above constrained optimization problem, we shall need the following :

LEMMA 2. *Let X, Y be non-empty subsets of Hausdorff topological vector spaces and let $f : X \times Y \to \mathbb{R}$ be an upper semicontinuous and strictly quasi-concave mapping on $X \times Y$. If $C : X \to 2^Y$ is an upper semicontinuous convex correspondence such that each $C(x)$ is non-empty compact, then the mapping $g(x) := \max_{y \in C(x)} f(x,y)$ is also upper semicontinuous and strictly quasi-concave on X.*

Proof. The upper semicontinuity follows directly from Berge's theorem [2, p. 122]. We shall prove the strict quasi-concavity of g. For any distinct $x_1,\ x_2 \in X$, we let

$$f(x_1,y_1) := \max_{y \in C(x_1)} f(x_1,y), \quad \text{where } y_1 \in C(x_1),$$
$$f(x_2,y_2) := \max_{y \in C(x_2)} f(x_2,y), \quad \text{where } y_2 \in C(x_2).$$

Then, for any $\lambda \in (0,1)$, by the convexity of C, $\lambda y_1 + (1-\lambda)y_2 \in C(\lambda x_1 + (1-\lambda)x_2)$; so we have

$$\begin{aligned}
& g(\lambda x_1 + (1-\lambda)x_2) \\
= & \max_{y \in C(\lambda x_1 + (1-\lambda)x_2)} f(\lambda x_1 + (1-\lambda)x_2, y) \\
\geq & f(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \\
= & f(\lambda(x_1,y_1) + (1-\lambda)(x_2,y_2)) \\
> & \min\{f(x_1,y_1), f(x_2,y_2)\} \quad (\text{since } f(x,y) \text{ is strictly quasi-concave}) \\
= & \min\{\max_{y \in C(x_1)} f(x_1,y),\ \max_{y \in C(x_2)} f(x_2,y)\} \\
= & \min\{g(x_1),\ g(x_2)\},
\end{aligned}$$

which implies that g is strictly quasi-concave on X. □

3. New maximum theorems with strict quasi-concavity

Using the maximum function $g(x) := \max_{y\in C(x)} f(x,y)$, we now introduce a multimap $\phi : X \to 2^X$ defined by

$$\phi(x) := \{y \in X \mid g(x) \leq g(y)\} \quad \text{for each} \quad x \in X.$$

By assuming the compactness condition on ϕ, we shall prove the following new maximum theorem :

THEOREM 1. *Let X be a non-empty convex subset of a Hausdorff topological vector space, Y be a non-empty subset of a Hausdorff topological vector space and let $f : X \times Y \to \mathbb{R}$ be an upper semicontinuous and strictly quasi-concave mapping on $X \times Y$. If $C : X \to 2^Y$ is an upper semicontinuous convex correspondence such that each $C(x)$ is non-empty compact and $\phi(x_0)$ is compact for some $x_0 \in X$, then there exists a point $\hat{x} \in X$ such that*

$$\max_{y\in C(x)} f(x,y) \leq \max_{y\in C(\hat{x})} f(\hat{x},y) \quad \text{for all} \quad x \in X.$$

Proof. First note that $x \in \phi(x) = \{y \in X \mid g(x) \leq g(y)\}$ for each $x \in X$. Since g is upper semicontinuous on X, each $\phi(x)$ is a non-empty closed subset of X. Furthermore, ϕ is a KKM-map. In fact, for each finite subset $\{x_1, \ldots, x_k\} \subset X$, we shall show that $co\{x_1, \ldots, x_k\} \subseteq \cup_{i=1}^k \phi(x_i)$. Suppose the contrary. Then there exists a point $y_0 = \sum_{i=1}^k \lambda_i x_i \notin \cup_{i=1}^k \phi(x_i)$, where $\lambda_i \in [0,1], i = 1, \ldots, k$, and $\sum_{i=1}^n \lambda_i = 1$. Then $g(x_i) > g(y_0)$ for all $i = 1, \ldots, k$. By Lemma 2, g is strictly quasi-concave on X; hence the set $\{x \in X \mid g(x) > g(y_0)\}$ is convex. Therefore, we have

$$g(y_0) = g(\sum_{i=1}^k \lambda_i x_i) > g(y_0),$$

which is a contradiction. Hence ϕ is a KKM-map. Since $\phi(x_0)$ is compact for some $x_0 \in X$, by Lemma 1, there exists a point $\hat{x} \in X$ such that $\hat{x} \in \cap_{x\in X} \phi(x)$. This implies the conclusion

$$g(x) = \max_{y\in C(x)} f(x,y) \leq \max_{y\in C(\hat{x})} f(\hat{x},y) = g(\hat{x}) \quad \text{for all} \quad x \in X. \qquad \square$$

When X is compact, the compactness assumption on $\phi(x_0)$ is automatically satisfied since g is upper semicontinuous. In this case, if we assume the additional assumption on the constraint correspondence C in Theorem 1, we can prove a new existence theorem of social equilibrium which was introduced by Debreu [4] :

THEOREM 2. *Let X be a non-empty compact convex subset of a locally convex Hausdorff topological vector space and let $f : X \times X \to \mathbb{R}$ be an upper semicontinuous and strictly quasi-concave mapping on $X \times X$. Let $C : X \to 2^X$ be an upper semicontinuous convex correspondence such that each $C(x)$ is non-empty compact. Furthermore, assume that for each $x \in X$, $g(x) \leq g(y)$ for all $y \in C(x)$.*

Then there exists a point $\hat{x} \in X$ such that

$$\hat{x} \in C(\hat{x}) \quad \text{and} \quad \max_{y \in C(x)} f(x,y) \leq \max_{y \in C(\hat{x})} f(\hat{x},y) \quad \text{for all} \quad x \in X.$$

Proof. Since X is compact and g is upper semicontinuous on X, each $\phi(x)$ is non-empty compact in X. Hence, as we have seen in the proof of Theorem 1, let $K := \cap_{x \in X} \phi(x)$ be a non-empty compact subset of X. Now we shall show that K is convex. Let x_1, x_2 be distinct points in K and $\lambda \in [0,1]$ be arbitrary. Then for every $x \in X$, we have

$$\max_{y \in C(x_1)} f(x_1,y) \geq \max_{y \in C(x)} f(x,y) \quad \text{and} \quad \max_{y \in C(x_2)} f(x_2,y) \geq \max_{y \in C(x)} f(x,y);$$

hence $\max_{y \in C(x_1)} f(x_1,y) = \max_{y \in C(x_2)} f(x_2,y)$. Let $x_0 := \lambda x_1 + (1-\lambda)x_2$. Then we shall show that $\max_{y \in C(x_0)} f(x_0,y) \geq \max_{y \in C(x)} f(x,y)$ for every $x \in X$. Suppose the contrary. Then there exists a point $\bar{x} \in X$ such that $t := \max_{y \in C(x_0)} f(x_0,y) < \max_{y \in C(\bar{x})} f(\bar{x},y)$. Note that

$$t < \max_{y \in C(\bar{x})} f(\bar{x},y) \leq \max_{y \in C(x_1)} f(x_1,y) = \max_{y \in C(x_2)} f(x_2,y).$$

Since g is strictly quasi-concave, the set $\mathcal{A} := \{x \in X \mid g(x) > t\}$ is convex and $x_1, x_2 \in \mathcal{A}$, so that $x_0 = \lambda x_1 + (1-\lambda)x_2 \in \mathcal{A}$. Hence $\max_{y \in C(x_0)} f(x_0,y) > t$, which is a contradiction. Therefore K is convex. In order to obtain the conclusion, it remains to show that there exists a point $\hat{x} \in K$ such that $\hat{x} \in C(\hat{x})$. We next claim that $C(K) \subseteq K$. Suppose the contrary. Then there exists a point $y_0 \in K$ such that $C(y_0) \nsubseteq K$, i.e., there exists a point $z_0 \in C(y_0)$ such that $z_0 \notin \phi(x_1)$ for some $x_1 \in X$. Therefore,

$$\max_{y \in C(z_0)} f(z_0,y) < \max_{y \in C(x_1)} f(x_1,y) \leq \max_{y \in C(y_0)} f(y_0,y).$$

However, by the assumption, $g(y_0) \leq g(z_0)$, and so we have a contradiction. Therefore, C maps the non-empty compact convex subset K of a locally convex Hausdorff topological vector space into itself. By the Fan-Glicksberg fixed point theorem [6], there exists a point $\hat{x} \in K$ such that $\hat{x} \in C(\hat{x})$. This completes the proof. □

REMARKS. (1) Theorem 2 is different from Berge's maximum theorem in the following aspects :

(i) f need not be continuous on $X \times Y$, but f need to be strictly quasi-concave ;

(ii) C need not be continuous on X, but C is assumed to be convex and satisfy the additional assumption.

(2) As we mentioned, in maximization problems of economic analysis of intertemporal allocation included models of the firm and various models of the consumer, when f is strictly quasi-concave and C is assumed to be convex, then the maximizing function g is suitable to Theorem 2; but the previous maximum theorems in [4, 9, 11, 12, 13] can not be applied.

Finally, we shall give two illustrative examples of Theorems 1 and 2.

EXAMPLE 1. Consider the following constrained optimization problem :

Maximize $f(x,y) = xy$ subject to $x+4y \leq 1$, and $x \in (0,1]$, $y \in [0,1]$.

Then $X := (0,1]$ is non-compact convex and $Y := [0,1]$ is compact convex, and we have the corresponding maximizing function

$$g(x) := \max_{y \in C(x)} f(x,y),$$

where $C(x) := \{y \in Y \mid x + 4y \leq 1\} = \{y \in Y \mid y \leq \frac{1}{4}(1-x)\}$ for each $x \in X$ for this optimization problem. Then we can easily check that $C : X \to 2^Y$ is upper semicontinuous and convex on X and each $C(x)$ is non-empty compact convex in Y. In fact, for the convexity of C, we let two distinct points $x_1, x_2 \in X$ and $\lambda \in [0,1]$ be arbitrary. Then we can see that $\lambda C(x_1) + (1-\lambda)C(x_2) \subseteq C(\lambda x_1 + (1-\lambda)x_2)$; hence C is convex. Also the following calculation shows the strict quasi-concavity of $f(x,y)$: for distinct two points $(x_1,y_1),(x_2,y_2)\in X\times Y$ and every $\lambda\in(0,1)$, we first let $\alpha=\min\{x_1y_1,x_2y_2\}$; then

$$\begin{aligned}
&f\Big(\lambda(x_1,y_1)+(1-\lambda)(x_2,y_2)\Big)\\
&=(\lambda x_1+(1-\lambda)x_2)\cdot(\lambda y_1+(1-\lambda)y_2)\\
&=\lambda^2x_1y_1+(1-\lambda)^2x_2y_2+\lambda(1-\lambda)(x_1y_2+x_2y_1)\\
&\geq\lambda^2x_1y_1+(1-\lambda)^2x_2y_2+\lambda(1-\lambda)2\sqrt{x_1y_1x_2y_2}\\
&>\alpha[\lambda^2+(1-\lambda)^2+2\lambda(1-\lambda)]=\alpha\,;
\end{aligned}$$

which implies the strict quasi-concavity of $f(x,y)$ on $X\times Y$. Finally, we have that

$$\phi\Big(\frac{1}{2}\Big)=\Big\{x\in X\mid\frac{1}{16}=\max_{y\in C(\frac{1}{2})}f\Big(\frac{1}{2},y\Big)\leq\max_{y\in C(x)}f(x,y)\Big\}=\Big\{\frac{1}{2}\Big\}$$

is non-empty compact. Therefore, all the hypotheses of Theorem 1 are satisfied so that we can obtain the optimal solution $\frac{1}{2}\in X$ such that

$$\max_{y\in C(x)}f(x,y)\leq\max_{y\in C(\frac{1}{2})}f\Big(\frac{1}{2},y\Big)=\frac{1}{16}\quad\text{for all}\quad x\in X.$$

However, C does not satisfy the additional assumption of Theorem 2, and we can confirm that $\frac{1}{2}\notin C(\frac{1}{2})$.

EXAMPLE 2. Next we consider the following constrained optimization problem :

$$\text{Maximize}\ \ f(x,y)=xy\ \ \text{subject to}\ \ x-y\leq 0;\ \ \text{and}\ \ x,\,y\in[0,1].$$

Let $X=Y:=[0,1]$ be a compact convex set, and we have the corresponding maximizing function $g(x):=\max_{y\in C(x)}f(x,y)$, where for each $x\in[0,1]$,

$$C(x):=[x,1]$$

for this optimization problem. Then we can easily check that $C:[0,1]\to 2^{[0,1]}$ is upper semicontinuous such that each $C(x)$ is non-empty compact convex in X. And we can see that C is a convex correspondence on X. In fact, for the convexity of C, we let two distinct points $x_1,x_2\in X$ and $\lambda\in[0,1]$ be arbitrary. Then the simple calculations show that

$\lambda C(x_1) + (1-\lambda)C(x_2) \subseteq C(\lambda x_1 + (1-\lambda)x_2)$; hence C is convex. And it is easy to check that for each $x \in X$, $g(x) \le g(y)$ for all $y \in C(x)$. Finally, as in Example 1, we can see the strict quasi-concavity of $f(x, y)$. Therefore, all the hypotheses of Theorem 2 are satisfied, so that we can obtain the social equilibrium solution $1 \in X$ such that

$$1 \in C(1) \text{ and } \max_{y \in C(x)} f(x,y) \le \max_{y \in C(1)} f(1,y) = 1 \quad \text{for all} \quad x \in [0,1].$$

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