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# MAX-MIN CONTROLLABILITY OF DELAY-DIFFERENTIAL GAMES IN HILBERT SPACES

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ABSTRACT. We consider a linear differential game described by the delay-differential equation in a Hilbert space $H$;

$$(*)\quad \begin{aligned} \frac{d}{dt}x(t) &= A_0x(t) + A_1x(t-h) + \int_{-h}^{0} a(s)A_2x(t+s)ds \\ &\quad + B(t)u(t) + C(t)v(t) \ a.e.\ t > 0 \\ x(0) &= g^0,\ x(s) = g^1(s)\ \in [-h, 0), \end{aligned}$$

where $g = (g^0, g^1) \in M_2 = H \times L_2([-h,0);Y)$, $u \in L_2^{loc}(R^+;U), v \in L_2^{loc}(R^+;V)$, $U$ and $V$ are Hilbert spaces, and $B(t)$ and $C(t)$ are families of bounded operators on $U$ and $V$ to $H$, respectively. $A_0$ generates an analytic semigroup $T(t) = e^{tA_0}$ in $H$.

The control variables $g, u$ and $v$ are supposed to be restricted in the norm bounded sets $\{g : ||g||_{M_2} \le \rho\}$, $\{u : ||u||_{L_2([0,t];U)} \le \delta\}$ and $\{v : ||v||_{L_2([0,t];V)} \le \gamma\}$ $(\rho, \delta, \gamma \ge 0)$. For given $x^0 \in H$ and a given time $t > 0$, we study $\epsilon$- approximate controllability to determine $x(\cdot)$ for a given $g$ and $v(\cdot)$ such that the corresponding solution $x(t)$ satisfies $||x(t) - x^0|| \le \epsilon$ ($\epsilon > 0$ :a given error).

## 0. Introduction

In the Euclidean space, various types of differential games of pursuit and evasion have been studied extensively (cf. Hájek[4]). Our main concern is to study max-min controllability problems in games theory, where we are concerned with selection of pursuer's controls from an admissible set against evader's controls. The max-min controllability has been investigated by Chan and Li[1] in the Euclidean space and in the Banach space, Park et al.[6] were studied in the case $A_0$ generates a $C_0$-semigroup and $A_I(\cdot)$ instead of $a(s)A_2$ in (*) is in $L_1([-h,0];\mathcal{L}(X))$. But we deal with the case that $A_0$ generates an analytic semigroup, $a(\cdot) \in L^2([-h,0];R)$. and $A_2 \in \mathcal{L}(Y,Y^*)$. Recently, this system has been studied by many authors[3,5].

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Here controls are assumed to belong to some norm bounded constraint sets and in such constraint sets, we want to find these controls steering a given initial state to a desired state. In this paper, we study the existence of optimal solutions which are the minimum of control times and the minimum norm of controls for the delay-differential equation (*). We derive necessary and sufficient conditions for a max-min controllability problem in game theory.

## 1. Preliminaries

We give the description of a linear delay-differential game in a Hilbert space. Let $C$ and $R$ be the sets of complex and real numbers, respectively and let $R^+$ be the set of non-negative numbers. Let $\Omega$ be bounded smooth on $R^n$ and $Y = H_0^1(\Omega), H = L^2(\Omega)$. The norms of $H, Y$ and the inner product of $H$ are denoted by $|\cdot|, ||\cdot||$ and $\langle\cdot,\cdot\rangle$ respectively. By identifying the antidual of $H$ with $H$ we may consider $Y \hookrightarrow H \equiv H^* \hookrightarrow Y^*$. The norm of the dual space $Y^*$ is denoted by $||\cdot||_*$.

We consider a linear game described by an abstract delay-differential system(s) on $H$;

$$(1.1) \qquad \frac{dx(t)}{dt} = A_0x(t) \quad +A_1x(t-h) + \textstyle\int_{-h}^0 a(s)A_2x(t+s)ds$$
$$+B(t)u(t) + C(t)v(t), a.e.\ t > 0.$$

$$(1.2) \qquad x(o) = g^0,\ x(s) = g^1(s)\ a.e.\ s \in [-h, 0).$$

where $g = (g^0, g^1) \in M_2 = H \times L_2([-h, 0]; Y), u \in L_2^{loc}(R^+; U), v \in L_2^{loc}(R^+; V)$, $\{B(t) : t \geq 0\} \subset \mathcal{L}(U, H)$ is a strongly continuous family of bounded operators from $U$ into $H$, $\{C(t); t \geq 0\} \subset \mathcal{L}(V, H)$ is also a strongly continuous family of bounded operators from $V$ into $H$, $A_0$ generates an analytic semigroup $T(t) = e^{tA_0}$ both in $H$ and $Y^*$ and that $T(t) : Y^* \to Y$ for each $t > 0$ and $\eta$ is a stieltjes measure given by

$$(1.3) \qquad \eta(s) = -\chi_{(-\infty,-h]}(s)A_1 - \int_s^0 a(\xi)d\xi A_2 : Y \to Y^*, s \in [-h, 0],$$

where $\chi_{(-\infty,-h]}(\cdot)$ denotes the characteristic function of $(-\infty, -h]$.

The delayed terms in (1.1) are written simply by $\displaystyle\int_{-h}^0 d\eta(s)x(t+s)$.

Let $a(x_1, x_2)$ be a bounded sesqulinear form defined in $Y \times Y$ satisfying Gårding's inequality

$$(1.4) \qquad Re\ a(x, x) \geq c_0||x||^2 - c_1|x|^2,$$

where $c_0$ and $c_1$ are real constants. Let$A_0$ be the operator associated with the sesquilinear form

$$(1.5)\qquad \langle x_1, A_0x_2\rangle = -a(x_2, x_1), x_1, x_2 \in Y,$$

where $\langle\cdot,\cdot\rangle_{Y,Y^*}$ denotes also the duality pairing between $Y$ and $Y^*$. The operator $A_0$ is bounded linear from $Y$ into $Y^*$. The realization of $A_0$ in $H$, which is the restriction of $A_0$ to the domain $\mathcal{D}(A_0) = \{x \in Y : A_0x \in H\}$ is also denoted by $A_0$.

Throughout this paper it is assumed that each $A_i (i = 1, 2)$ is bounded and linear from $Y$ to $Y^*$ (i.e. $A_i \in \mathcal{L}(Y, Y^*)$) such that $A_i$ maps $\mathcal{D}(A_0)$ endowed with the graph norm of $A_0$ to $H$ continuously. The real valued scalar function $a(s)$ is assumed to be $L^2$-integrable on $[-h, 0]$, that is $a(\cdot) \in L^2([-h, 0]; R)$. Let $W(t)$ be the fundamental solution of (s), which is a unique solution of the equation

$$(1.6)\qquad W(t) = \begin{cases} T(t) + \int_0^t T(t-s)\int_{-h}^0 d\eta(\xi)W(\xi+s)ds, & t \geq 0, \\ 0, & t < 0 \end{cases}$$

i.e.

(1.7)

$$W(t) = \begin{cases} T(t) + \int_0^t (A_1W(s-h) + \int_{-h}^0 a(\sigma)A_2W(\sigma+s)d\sigma)ds, & t \geq 0 \\ 0\ , & t < 0. \end{cases}$$

Then $W(t) \in \mathcal{L}(H)$ for each $t \geq 0$ and $W(t)$ is strongly continuous in $R^+ = [0, \infty)$ and $AW(t)$ and $\frac{d}{dt}W(t)$ are strongly continuous except at $t = nh, n = 0, 1, 2, \ldots.$ Therefore we may assume that

$$(1.8)\qquad |W(t)| \leq M, t \geq 0, \text{ where } M \text{ is a constant.}$$

The solution of (1.1) is expressed by

(1.9)

$$x(t, g, u, v) = \begin{cases} W(t)g^0 + \int_{-h}^0 U_t(s)g^1(s)ds + \int_0^t W(t-s)B(s)u(s)ds \\ \quad + \int_0^t W(t-s)C(s)v(s)ds, t \geq 0 \\ g^1(t), a.e. t \in [-h, 0), \end{cases}$$

where

$$(1.10)\qquad U_t(s) = W(t-s-h)A_1 + \int_{-h}^s W(t-s+\sigma)a(\sigma)d\sigma$$

is well defined and is an element of $C(R^+; H)$.

The function $x(t) = x(t, g, u, v)$ is a unique solution of the integrated form of (1.1), (1.2) by $T(t)$. In this sense $x(t)$ is called the mild solution of the system(s). In the system (s), $u(t), v(t)$ and $(g^0, g^1(s))$ are called a pursuer's control, an evader's control on forcing term and an evader's control on initial data, respectively.

The state space $M_2 = H \times L_2([-h,0];Y)$ of the system (s) is the product reflexive space with the norm

$$(1.11) \qquad ||g||_{M_2} = (|g^0|^2 + \int_{-h}^{0} ||g^1||^2 ds)^{\frac{1}{2}}, g = (g^0, g^1) \in M_2.$$

The dual space $M_2^*$ of $M_2$ is identified with the product space $H \times L_2([-h,0];Y^*) \equiv H \times L_2([-h,0];Y)^*$ via the duality pairing

$$(1.12) \qquad \langle g, f \rangle_{M_2} = \langle g^0, f^0 \rangle_H + \int_{-h}^{0} \langle g^1(s), f^1(s) \rangle_{Y,Y^*} ds,$$

where $g = (g^0, g^1) \in M_2, f = (f^0, f^1) \in M_2^*$ and $\langle \cdot, \cdot \rangle_{Y,Y^*}$ denotes the duality pairing between $Y$ and $Y^*$. Here we note that the pairing $\langle \cdot, \cdot \rangle_{Y,Y^*}$ is assumed to satisfy $\langle g^0, \alpha f^0 \rangle = \langle \bar{\alpha} g_0, f^0 \rangle$ for $\alpha \in \mathbf{C}, (g^0, f^0) \in H \times H, \bar{\alpha}$ being the conjugate of $\alpha$. We denote the norm in $Y^*$ by $|| \cdot ||_*$. For more detailed structural properties of the equations (1.1), (1.2) on the space $M_2$, we refer to [3].

## 2. Max-Min controllability

In this section, we study a max-min controllability problem which is noncooperative in the sense that against one evader's controls, the other pursuer can select an appropriate control. For each $t > 0, \rho \geq 0, \gamma \geq 0$, we define constraint sets

$$(2.1) \qquad U_\delta^t = \{u \in L_2([0,t];U) : ||u||_{2,[0,t]} = (\int_0^t |u(s)|_U^2 ds)^{\frac{1}{2}} \leq \delta\},$$

$$(2.2) \qquad V_\gamma^t = \{v \in L_2([0,t];V) : ||v||_{2,[0,t]} = (\int_0^t |v(s)|_V^2 ds)^{\frac{1}{2}} \leq \gamma\}$$

and

$$(2.3) \qquad G_\rho = \{g \in M_2 : ||g||_{M_2} = (|g^0|^2 + \int_{-h}^{0} ||g^1(s)||^2 ds)^{\frac{1}{2}} \leq \rho\}.$$

The set $U_\delta^t, V_\gamma^t$ and $G_\rho$ are convex and closed in $L_2([0,t];U), L_2([0,t];V)$ and $M_2$, respectively. We put $Y_{\gamma,\rho}^t = G_\rho \times V_r^t$ for evader's constraint sets and define the reachable set $\mathcal{R}_t(Y_{\gamma,\rho}^t)$ with respect to (i.e. w.r.t.) evader's controls by

$$(2.4) \qquad \mathcal{R}_t(Y_{\gamma,\rho}^t) = \{x \in H : x = x(t; g, 0, v) \text{where} (g, v) \in Y_{\gamma,\rho}^t\}.$$

Lemma 2.1. *The set $\mathcal{R}_t(Y_{\gamma,\rho}^t)$ is closed and convex for any $t > 0, \gamma \geq 0, \rho \geq 0$.*

*Proof.* It is clear that $\mathcal{R}(Y^t_{\gamma,\rho})$ is convex. We shall prove $\mathcal{R}(Y^t_{\gamma,\rho})$ is closed. Let $x(t;g_n,0,v_n)$ strongly converge to some $x_0 \in H$ as $n \to \infty$ for $(g_n,v_n) \in Y^t_{\gamma,\rho}$. Then we have to prove that $x_0 = x(t;g,0,v)$ for some $(g,v) \in Y^t_{\gamma,\rho}$. Since $Y^t_{\gamma,\rho}$ is bounded in the reflexive product space $M_2 \times L_2([0,t];Y)$, there exists a subsequence (which we denote again by $\{(g_n,v_n)\}$ ) of $\{(g_n,v_n)\}$ weakly convergent to $(g,v)$(e.g. K.Yosida [7, p141]). Furthermore, by $||g||_{M_2} \leq \liminf_{n\to\infty} ||g_n||_{M_2}$ and $||v||_{2,[0,t]} \leq \liminf_{n\to\infty} ||v_n||_{2,[0,t]}$ (e.g. [7, p120]). We see $(g,v) \in Y^t_{\gamma,\rho}$.

Let $x^* \in H$. Then by (1.8),

$$\begin{aligned}\langle x(t;g_n,0,v_n),x^*\rangle &= \langle W(t)g_n^0 + \int_{-h}^0 U_t(s)g_n^1(s)ds \\ &\qquad + \int_0^t W(t-s)C(s)v_n(s)ds, x^*\rangle \\ &= \langle (g_n^0,g_n^1),(W(t)x^*,U_t^*(\cdot)x^*)\rangle_{M_2} \\ &\qquad + \int_0^t \langle v_n(s), C^*(s)W^*(t-s)x^*\rangle_V ds.\end{aligned}$$

Here it can be verified by the strong continuity of $W(t)$ and $C(t)$ and the equation (1.7) that $(W^*(t)x^*,U_t^*(\cdot)x^*) \in M_2^*$ and $C^*(t)W^*(t-\cdot)x^* \in (L_2([0,t];V))^* = L_2([0,t];V^*)$. Since $\{(g_n,v_n)\}$ is weakly convergent to $(g,v)$ and $x(t;g_n,0,v_n)$ is strongly convergent to $x_0$, the above equality implies, by letting $n \to \infty$, that

$$\begin{aligned}\langle x_0,x^*\rangle &= \langle (g^0,g^1),(W^*(t)x^*,U_t^*(\cdot)x^*)\rangle_{M_2} \\ &\qquad + \int_0^t \langle v(s), C^*(s)W^*(t-s)x^*\rangle_V ds \\ &= \langle W(t)g^0 + \int_{-h}^0 U_t(s)g^1(s)ds + \int_0^t W(t-s)C(s)v(s)ds, x^*\rangle\end{aligned}$$

since $x^* \in H$ is arbitrarily chosen $x_0 = W(t)g^0 + \int_{-h}^0 U_t(s)g^1(s)ds + \int_0^t W(t-s)C(s)v(s)ds$, and hence $x_0 \in \mathcal{R}_t(Y^t_{\gamma,\rho})$. This completes the proof of Lemma 2.1. □

LEMMA 2.2. ([6]) *Let $E$ and $F$ be closed convex sets in $X$. Then $E \subset F$ if and only if*

$$\sup_{x\in E}\langle x,x^*\rangle \leq \sup_{x\in F}\langle x,x^*\rangle \text{ for all } x^* \in H.$$

DEFINITION 2.1. The system(s) is said to be max-min $(\delta,\gamma,\rho)$-controllable on $[0,t]$ with respect to $B(x^0;\epsilon)$ if each evader's controls $(g,v) \in Y^t_{\gamma,\rho}$, there exists a pursuer's control $u \in U_\delta^t$ such that $x(t; g, u, v) \in B(x^0; \epsilon)$, where $B(x^0; \epsilon) = \{x \in X; |x - x^0| \leq \epsilon\}(\epsilon \geq 0)$.

Here $x^0$ is assumed to be a described state (a target point), and $B(x^0; \epsilon)$ is a target set with error $\epsilon$. Henceforth $\langle \cdot, \cdot \rangle_U$ and $\langle \cdot, \cdot \rangle_V$ denote the duality pairings between $U$ and $U^*$ and, $V$ and $V^*$, and $|\cdot|_{U^*}$ and $|\cdot|_{V^*}$ denote the norms in $U^*$ and $V^*$, respectively.

Using Lemma 2.1 and Lemma 2.2, we obtain the following result.

THEOREM 2.1. *The system(s) is max-min $(\delta, \gamma, \rho)$-controllable on $[0, t]$ with respect to $B(x^0; \epsilon)$ if and only if*

$$\begin{aligned} &|\langle x^0, x^* \rangle| - \epsilon|x^*| \\ (2.5) \qquad \leq\ &\delta ||B^*(\cdot)W^*(t - \cdot)x^*||_{2,[0,t]} - \gamma||C^*(\cdot)W^*(t - \cdot)x^*||_{2,[0,t]} \\ &-\rho||(W^*(t)x^*, U_t^*(\cdot)x^*)||_{M_2^*} \text{ for each } x^* \in H, \end{aligned}$$

*where*

$$(2.6) \quad ||B^*(\cdot)W^*(t - \cdot)x^*||_{2,[0,t]} = \left(\int_0^t |B^*(s)W^*(t - s)x^*|_{U^*}^2 ds\right)^{\frac{1}{2}},$$

$$(2.7) \quad ||C^*(\cdot)W^*(t - \cdot)x^*||_{2,[0,t]} = \left(\int_0^t |C^*(s)W^*(t - s)x^*|_{V^*}^2 ds\right)^{\frac{1}{2}},$$

$$(2.8) \quad ||(W^*(t)x^*, U_t^*(\cdot)x^*)||_{M_2^*} = \left(|W^*(t)x^*|^2 + \int_{-h}^0 ||U_t^*(s)x^*||_*^2 ds\right)^{\frac{1}{2}}$$

*and*
(2.9)
$$U_t^*(s) = A_1^* W^*(t - s - h) + \int_{-h}^s a(\theta)W^*(t - s - \theta)d\theta \text{ a.e. } s \in [-h, 0).$$

*By (2.5) it is evident that the max-min $(\delta, \gamma, \rho)$-controllability of the system(s) on $[0, t]$ w.r.t. $B(x^0; \epsilon)$ implies the max-min $(\delta', \gamma', \rho')$-controllability of(s) on $[0, t]$ w.r.t. $B(x^0; \epsilon')$ for any $\delta' \geq \delta, \rho' \leq \rho, \gamma' \leq \gamma$ and $\epsilon' \geq \epsilon$.*

*Proof.* For each $t > 0$, we define two operators

$\mathcal{B}_t : L_2([0, t]; U) \to H$ and $\mathcal{Z}_t : M_2 \times L_2([0, t]; V) \to H$ by

$$(2.10) \qquad \mathcal{B}_t u = \int_0^t W(t - s)B(s)u(s)ds$$

and

$$(2.11) \qquad \mathcal{Z}_t(g, v) = W(t)g^0 + \int_{-h}^0 U_t(s)g^1(s)ds + \int_{-h}^0 W(t - s)C(s)v(s)ds,$$

respectively.

It is verified as in the proof of Lemma 2.1 that $\mathcal{B}_t(U_\delta^t)$ and $\mathcal{Z}_t(Y_{\gamma,\rho}^t)$ are closed convex in $H$. By definition, the system(s) is max-min $(\delta,\gamma,\rho)$-controllable on $[0,t]$ with respect to $B(x^0;\epsilon)$ iff

$$\mathcal{Z}_t(Y_{\gamma,\rho}^t) \subset -\mathcal{B}_t(U_\delta^t) + B(x^0;\epsilon). \tag{2.12}$$

By Lemma 2.1, the set $\mathcal{Z}_t(Y_{\gamma,\rho}^t)$ is closed in $H$ and the set $-\mathcal{B}_t(U_\delta^t) + B(x^0;\epsilon) = \mathcal{B}_t(U_\delta^t)+B(x^0;\epsilon)$ is also closed.(In fact, it is obvious that $\mathcal{B}_t(U_\delta^t)+B(x^0;\epsilon)$ is convex. Since both $\mathcal{B}_t(U_\delta^t)$ and $B(x^0;\epsilon)$ are weakly closed and bounded, these sets are weakly compact (cf, [2, p.425]). Then the sum $\mathcal{B}_t(U_\delta^t) + B(x^0;\epsilon)$ is weakly compact, and hence weakly closed. Therefore by the well known theorem (cf, [2, p.442]), $\mathcal{B}_t(U_\delta^t) + B(x^0;\epsilon)$ is closed). Since both $\mathcal{Z}_t(Y_{\gamma,\rho}^t)$ and $-\mathcal{B}_t(U_\delta^t) + B(x^0;\epsilon)$ are convex, then we can apply Lemma 2.2, to obtain that (2.12) is equivalent to

$$\begin{aligned} &\sup\{\langle \mathcal{Z}_t(g,v), x^*\rangle; (g,v) \in Y_{\gamma,\rho}^t\} \\ &\le \sup\{\langle \mathcal{B}_t u + y, x^*\rangle; u \in U_\delta^t, y \in B(x^0;\epsilon)\} \text{ for each } x^* \in H. \end{aligned} \tag{2.13}$$

By (2.11), we have

$$\begin{aligned} &\sup\{\langle \mathcal{Z}_t(g,v), x^*\rangle; (g,v) \in Y_{\gamma,\rho}^t\} \\ = \ &\sup\{\langle W(t)g^0 + \int_{-h}^0 U_t(s)g^1(s)ds, x^*\rangle; (g^0,g^1) \in G_\rho\} \\ &+\sup\{\langle \int_0^t W(t-s)C(s)v(s)ds, x^*\rangle; v \in V_\gamma^t\} \\ = \ &\rho\sup\{\langle g^0, W^*(t)x^*\rangle + \int_{-h}^0 \langle g^1(s), U_t^*(s)x^*\rangle ds; ||(g^0,g^1)||_{M_2} \le 1\} \\ &+\gamma\sup\{\int_0^t \langle v(s), C^*(s)W^*(t-s)x^*\rangle_V ds; ||v||_{2,[0,t]} \le 1\} \\ = \ &\rho||(W^*(t)x^*, U_t^*(\cdot)x^*)||_{M_2^*} + \gamma||C^*(\cdot)W^*(t-\cdot)x^*||_{2,[0,t]}. \end{aligned} \tag{2.14}$$

On the other hand, by (2.10) the right side of (2.13) is calculated as follows;

$$\begin{aligned} &\sup\{\langle \int_0^t W(t-s)B(s)u(s)ds, x^*\rangle; u \in U_\delta^t\} \\ &+\langle x^0, x^*\rangle + \sup\{\langle y, x^*\rangle; |y| \le \epsilon\} \\ &= \sup\{\int_0^t \langle u(s), B^*(s)W^*(t-s)x^*\rangle_U ds; ||u||_{2,[0,t]} \le \delta\} \\ &+\langle x_0, x^*\rangle + \epsilon\sup\{\langle z, x^*\rangle; |z| \le 1\} \\ &= \delta||B^*(\cdot)W^*(t-\cdot)x^*||_{2,[0,t]} + \langle x_0, x^*\rangle + \epsilon|x^*|. \end{aligned} \tag{2.15}$$

Replacing $x^*$ by $-x^*$ in (2.15), we obtain condition (2.5). This completes the proof. □

Next we consider the continuity of max-min controllability with respect to positive times $t$, non-negative parameters $\delta, \gamma, \rho, \epsilon$ and vectors $x^0$ in $H$.

THEOREM 2.2. *Assume that the system (s) is max-min $(\delta_n, \gamma_n, \rho_n)$-controllable on $[0, t_n]$ w.r.t. $B(x_n^0; \epsilon)$ for each $n \geq 1$. If*

$$t_n \to t > 0,\ \delta_n \to \delta,\ \gamma_n \to \gamma,\ \rho \to \rho,\ \epsilon_n \to \epsilon \text{ in } R^+ \tag{2.16}$$

$$x_n^0 \to x^0 \text{ weakly in } H \text{ as } n \to \infty, \tag{2.17}$$

*then the system (s) is max-min $(\delta, \gamma, \rho)$-controllable on $[0, t]$ w.r.t. $B(x^0; \epsilon)$.*

*Note that we require a weak convergence $x_n^0 \to x^0$ not a strong one.*

*Proof.* Since the system(s) is max-min $(\delta_n, \gamma_n, \rho_n)$-controllable on $[0, t_n]$ w.r.t. $B(x_n^0; \epsilon_n)$, then by Theorem 2.1,

$$\begin{aligned} &|\langle x_n^0, x^* \rangle| - \epsilon_n |x^*| \\ &\leq \delta_n ||B^*(\cdot)W^*(t_n - \cdot)x^*||_{2,[0,t_n]} - \gamma_n ||C^*(\cdot)W^*(t_n - \cdot)x^*||_{2,[0,t_n]} \\ &-\rho_n ||(W^*(t_n)x^*, U_{t_n}^*(\cdot))||_{M_2^*} \end{aligned} \tag{2.18}$$

for each $x^* \in H$. Clearly by (2.17), we have

$$|\langle x_n^0, x^* \rangle| \to |\langle x^0, x^* \rangle| \text{ as } n \to \infty. \tag{2.19}$$

Let us set

$$F_1(t) = ||B^*(\cdot)W^*(t - \cdot)x^*||_{2,[0,t]}, \tag{2.20}$$

$$F_2(t) = ||C^*(\cdot)W^*(t - \cdot)x^*||_{2,[0,t]}, \tag{2.21}$$

$$F_3(t) = ||(W^*(t - \cdot)x^*, U_t^*(\cdot)x^*)||_{M_2^*}. \tag{2.22}$$

Let $T = \sup_{n \geq 1} t_n$ and $I = [0, T]$. Since $W(t) = 0$ if $t < 0$, $F_1(t_n)$ can be written as $||B^*(\cdot)W^*(t_n - \cdot)x^*||_{2,I}$. By reflexiveness of $H$, $W^*(t)$ is also strongly continuous on $R^+$ (cf, [3]), so that by (2.16)

$$\lim_{n \to \infty} W^*(t_n - s)x^* = \lim_{n \to \infty} W^*(t - s)x^* \text{ for all } s \in I \tag{2.23}$$

provided that $t - s \neq 0$. Since

$$\begin{aligned} &|F_1(t_n) - F_1(t)| \\ &\leq (\int_I |B^*(s)(W^*(t_n - s)x^* - W^*(t - s)x^*)|_{U^*}^2)^{\frac{1}{2}} \\ &\leq (\sup_{s \in I} ||B^*(s)||)(\int_I |W^*(t_n - s)x^* - W^*(t - s)x^*|_{U^*}^2 ds)^{\frac{1}{2}} \end{aligned} \tag{2.24}$$

and $\sup_{s\in I}||B^*(s)|| = \sup_{s\in I}||B(s)||$ is bounded by the strong continuity of $B(\cdot)$ and the uniform boundedness principle, thus by applying the Lebesgue dominated convergence theorem, we have

$$F_1(t_n) \to F_1(t) \text{ as } n \to \infty. \tag{2.25}$$

By similar calculations, we can verify

$$F_2(t_n) \to F_2(t) \text{ as } n \to \infty. \tag{2.26}$$

Lastly we shall show

$$F_3(t_n) \to F_3(t) \text{ as } n \to \infty. \tag{2.27}$$

By the strong continuity of $W^*(t)$, we have

$$|W^*(t_n)x^*|^2 \to |W^*(t)x^*|^2 \text{ as } n \to \infty. \tag{2.28}$$

From (2.9) it follows that

$$\begin{aligned} &U^*_{t_n}(s)x^* - U^*_t(s)x^* \\ &= A^*_1W^*(t_n - s - h)x^* + \int_{-h}^{s} A^*_2 a(\xi)W^*(t_n - s + \xi)x^* d\xi \\ &\quad -A^*_1W^*(t - s - h)x^* + \int_{-h}^{s} A^*_2 a(\xi)W^*(t - s + \xi)x^* d\xi \\ &\quad \text{a.e. } s \in [-h, 0]. \end{aligned} \tag{2.29}$$

We fix $s$ such that the equality (2.29) holds. Then by (2.16) we have

$$A^*_1(W^*(t_n - s - h) - W^*(t - s - h)) \to 0 \text{ in } H, \tag{2.30}$$

provided that $t - s - h \neq 0$ ; and that

$$(W^*(t_n - s + \xi) - W^*(t - s + \xi))x^* \to \text{ in } H \tag{2.31}$$

provided that $t - s + \xi \neq 0$ for each $\xi \in [-h, s]$. By (2.31), $a(\cdot) \in L^2([-h, 0]; R)$ and the Lebesgue dominated convergence theorem, we see that

$$\begin{aligned} &|\int_{-h}^{s} A^*_2 a(\xi)(W^*(t_n - s + \xi) - W^*(t - s + \xi))x^* d\xi| \\ &\leq ||A_2||\Big(\int_{-h}^{0} |a(\xi)|^2 d\xi\Big)^{\frac{1}{2}}\Big(\int_{-h}^{s} |(W^*(t_n - s + \xi) - W^*(t - s + \xi))x^*|^2 d\xi\Big)^{\frac{1}{2}} \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{2.32}$$

This implies, by (2.30) and (2.32), that for a.e. $s \in [-h, 0]$

$$U^*_{t_n}x^* \to U^*_t(s)x^* \text{ in } H \text{ as } n \to \infty. \tag{2.33}$$

Hence from (2.33), we have

$$
\begin{aligned}
(2.34)\qquad & |(\int_{-h}^{0} |U_{t_n}^*(s)x^*|^2 ds)^{\frac{1}{2}} - (\int_{-h}^{0} |U_t^*(s)x^*|^2 ds)^{\frac{1}{2}}| \\
\leq\ & (\int_{-h}^{0} |(U_{t_n}^*(s)x^* - U_t^*(s))x^*|^2 ds)^{\frac{1}{2}} \to 0 \text{ as } n \to \infty,
\end{aligned}
$$

by applying the Lebesgue dominated convergence theorem again. Therefore we show (2.27). □

Now letting $n \to \infty$ in (2.18) we reach the desired inequality (2.5) by (2.25)-(2.27). This proves, in view of Theorem 2.1, that the system(s) is max-min $(\delta, \gamma, \rho)$-controllable on $[0, t]$ with respect to $B(x^0; \epsilon)$.

## 3. Optimal value problems

In this section we study the existence of optimal solutions. Here, being optimal means the minimality of time interval $[0, t]$ over which we can control the system (Definition in section 2), the one of bound $\delta$ of norms of pursurer's controls, the one of error $\epsilon$ for the target point $x_0$ or the maximality of bounds $\gamma, \rho$ of norms of evader's controls. Throughout this section, it is assumed that the system (s) is max-min $(\delta, \gamma, \rho)$-controllable on $[0, t]$ with respect to $B(x^0; \epsilon)$ for some $\delta, \rho, \gamma, t, x^0$ and $\epsilon$.

First we show the following theorem standing the existence of the minimal time interval $[0, t_f]$ on which max-min controllability is preserved.

THEOREM 3.1. *Let*

$$
\begin{aligned}
(3.1)\qquad \pi_T \ = \ & \{t' \in R^+ - \{0\}; \textit{the system (s) is max-min} \\
& (\delta, \gamma, \rho) - \textit{ controllable on } [0, t'] \textit{ w.r.t. } B(x^0; \epsilon)\}.
\end{aligned}
$$

*Then* $\inf \pi_T = 0$ *or there exists a minimal time* $t_f > 0$ *such that*

$$
(3.2)\qquad t_f = \min \pi_T.
$$

*In particular, if* $\inf \pi_T > 0$, *then the system (s) remains max-min* $(\delta, \gamma, \rho)$-*controllable on* $[0, t_f]$ *w.r.t.* $B(x^0; \epsilon)$, *where* $t_f$ *is given by (3.2).*

*Proof.* Obviously, $t_f = \inf \pi_T$ exists. If $t_f > 0$, let $\{t_n\} \subset \pi_T$ be a sequence such that

$$
(3.3)\qquad \lim_{n\to\infty} t_n = \inf \pi_T = t_f > 0.
$$

Then by (3.3) we can apply Theorem 2.2 to obtain the conclusion $t_f \in \pi_T$.

Now we introduce the following subsets of $H$ in order to characterize the optimal values for various optimal value problems given below;

$$H_B = \{x^* \in H; ||B^*(\cdot)W^*(t-\cdot)x^*||_{2,[0,t]} = 1\} \tag{3.4}$$

$$H_C = \{x^* \in H; ||C^*(\cdot)W^*(t-\cdot)x^*||_{2,[0,t]} = 1\} \tag{3.5}$$

$$H_G = \{x^* \in H; ||(W^*(t)x^*, U_t^*(\cdot)x^*)||_{M_2^*} = 1\}. \tag{3.6}$$

□

THEOREM 3.2. *Let*

$$\begin{aligned}(3.7)\quad \pi_D \;=\; & \{\delta' \in R^+; \textit{the system (s) is max-min} \\ & (\delta', \gamma, \rho) - \textit{ controllable on } [0, t'] \text{ w.r.t. } B(x^0; \epsilon)\}.\end{aligned}$$

*Then there exists a minimal bound $\delta_f$ such that*

$$\delta_f = \min \pi_D. \tag{3.8}$$

*In particular, the system (s) remains max-min $(\delta_f, \gamma, \rho)$-controllable on $[0, t]$ w.r.t. $B(x^0; \epsilon)$. Further if $H_B \neq \phi$, then $\delta_f$ is given by*

$$\delta_f = \max\{0, \hat{\delta}\} \tag{3.9}$$

*where*

$$\begin{aligned}(3.10)\;\hat{\delta} \;=\; & \sup\{|\langle x^0, x^*\rangle| + \gamma||C^*(\cdot)W^*(t-\cdot)x^*||_{2,[0,t]} \\ & +\rho||(W^*(t)x^*, U_t^*(\cdot)x^*)||_{M_2^*} - \epsilon|x^*|; x^* \in H_B\};\end{aligned}$$

*and if $H_B = \phi$, then $\delta_f = 0$.*

*Proof.* By Theorem 2.2, we can readily see the existence of $\min \pi_D$. Next we have to prove (3.9). To this end, setting $\delta'_f = \max\{0, \hat{\delta}\}$, we have only to prove $\delta'_f = \delta_f$. First we consider the case $H_B \neq \phi$. Then by the definition (3.10) of $\hat{\delta}, \delta'_f$ is finite. We shall show (3.9). Since $\delta_f \in \pi_D$, the following inequality holds for each $x^* \in H$ :

$$\begin{aligned}(3.11)\quad & |\langle x^0, x^*\rangle| - \epsilon|x^*| \\ & \leq \delta_f||B^*(\cdot)W^*(t-\cdot)x^*||_{2,[0,t]} - \gamma||C^*(\cdot)W^*(t-\cdot)x^*||_{2.[0,t]} \\ & -\rho||(W^*(t)x^*, U_t^*(\cdot)x^*)||_{M_2^*}.\end{aligned}$$

Taking the supremum of (3.11) on the set $H_B$, we have $\hat{\delta} \leq \delta_f$ by definition of $\hat{\delta}$. Hence $\delta'_f \leq \delta_f$. We will divide the proof into the two cases $\delta)0$ and $\hat{\delta} \leq 0$.

First we assume $\hat{\delta} > 0$. In order to show the equality $\delta'_f = \delta_f$, assume contrary that $\delta'_f \langle \delta_f$, that is, $\hat{\delta} \leq \delta_f$. Since $\delta_f = \min \pi_D$, we have $\hat{\delta} \notin \pi_D$ and hence by Theorem 2.1, there exists a nonzero vector $x_S^* \in H$ such that

$$(3.12) \qquad \begin{aligned} &|\langle x^0, x_S^*\rangle| - \epsilon |x_S^*| \\ &> \hat{\delta}||B^*(\cdot)W^*(t-\cdot)x_S^*||_{2,[0,t]} - \gamma||C^*(\cdot)W^*(t-\cdot)x_S^*||_{2,[0,t]} \\ &\quad -\rho||(W^*(t)x_S^*, U_t^*(\cdot)x_S^*)||_{M_2^*}. \end{aligned}$$

This implies

$$(3.13) \qquad \begin{aligned} &|\langle x^0, x_S^*\rangle| - \epsilon |x_S^*| \\ &+\gamma||C^*(\cdot)W^*(t-\cdot)x_S^*||_{2,[0,t]} + \rho||(W^*(t)x_S^*, U_t^*(\cdot)x_S^*)||_{M_2^*} \\ &> \hat{\delta}||B^*(\cdot)W^*(t-\cdot)x_S^*)||_{2,[0,t]}. \end{aligned}$$

On the other hand, by substituting $x^* = x_S^*$ in (3.11) we have

$$(3.14) \qquad \begin{aligned} &|\langle x^0, x_S^*\rangle| - \epsilon |x_S^*| \\ &+\gamma||C^*(\cdot)W^*(t-\cdot)x_S^*||_{2,[0,t]} + \rho||(W^*(t)x_S^*, U_t^*(\cdot)x_S^*)||_{M_2^*} \\ &\leq \delta_f||B^*(\cdot)W^*(t-\cdot)x_S^*||_{2,[0,t]}. \end{aligned}$$

By (3.13) and (3.14), it follows that $\delta_f||B^*(\cdot)W^*(t-\cdot)x_S^*||_{2,[0,t]} > 0$. Since $\delta_f > \hat{\delta} > 0$, we have

$$(3.15) \qquad ||B^*(\cdot)W^*(t-\cdot)x_S^*||_{2,[0,t]} > 0.$$

Let

$$(3.16) \qquad y_S^* = x_S^*/||B^*(\cdot)W^*(t-\cdot)x_S^*||_{2,[0,t]},$$

then we see easily that $y_S^* \in H_B$ and

$$(3.17) \qquad \begin{aligned} &|\langle x^0, y_S^*\rangle - \epsilon |y_S^*| \\ &+\gamma||C^*(\cdot)W^*(t-\cdot)y_S^*||_{2,[0,t]} + \rho||(W^*(t)y_S^*, U_t^*(\cdot)y_S^*||_{M_2^*} > \hat{\delta}. \end{aligned}$$

The inequality (3.17) contradicts the definition (3.10) of $\hat{\delta}$. Thus, in the case of $\hat{\delta} > 0$, we see $\delta'_f = \delta_f$.

Second we assume $\hat{\delta} \leq 0$. Then we can show for each $x^* \in H$

$$(3.18) \qquad \begin{aligned} |\langle x^0, x^*\rangle| - \epsilon|x^*| \leq \; &-\gamma||C^*(\cdot)W^*(t-\cdot)x^*||_{2,[0,t]} \\ &-\rho||(W^*(t)x^*, U_t^*(\cdot)x^*)||_{M_2^*}. \end{aligned}$$

When $x^*$ satisfies $||B^*(\cdot)W^*(t-\cdot)x^*||_{2,[0,t]} \neq 0$, we set

$$y^* = x^*/||B^*(\cdot)W^*(t-\cdot)x^*||_{2,[0,t]}.$$

Then $y^* \in H_B$, so that by (3.10) we have

$$\text{(3.19)} \qquad \hat{\delta} \geq |\langle x^0, y^* \rangle| + \gamma \|C^*(\cdot)W^*(t-\cdot)y^*\|_{2,[0,t]} + \rho \|(W^*(t)y^*, U_t^*(\cdot)y^*)\|_{M_2^*} - \epsilon |y^*|,$$

and hence, by $\hat{\delta} \leq 0$, we get

$$\text{(3.20)} \qquad |\langle x^0, x^* \rangle| - \epsilon |x^*| \leq -\gamma \|C^*(\cdot)W^*(t-\cdot)x^*\|_{2,[0,t]} - \rho \|(W^*(t)x^*, U_t^*(\cdot)x^*)\|_{M_2^*}.$$

Therefore (3.18) holds true in this case. Lastly, when $x^*$ satisfies

$$\|B^*(\cdot)W^*(t-\cdot)x^*\|_{2,[0,t]} = 0,$$

we substitute this condition into (2.5) to obtain (3.18). This means that the system (s) is $(0, \gamma, \rho)$-controllable on $[0, t]$ w.r.t. $B(x^0; \epsilon)$. Thus $0 \in \pi_D$, and this proves $\delta_f = 0$ since $\delta_f = \min \pi_D$. Thus, also in the case of $\hat{\delta} \leq 0$, we see $\delta'_f = \delta_f (= 0)$.

Next we consider the case $H_B = \phi$. Then

$$\|B^*(\cdot)W^*(t-\cdot)x^*\|_{2,[0,t]} = 0 \text{ for each } x^* \in H.$$

Therefore we have by (2.5)

$$\text{(3.21)} \qquad |\langle x^0, x^* \rangle| - \epsilon |x^*| \leq \gamma \|C^*(\cdot)W^*(t-\cdot)x^*\|_{2,[0,t]} - \rho \|(W^*(t)x^*, U_t^*(\cdot)x^*)\|_{M_2^*}$$

for each $x^* \in H$. This shows $\delta_f = 0$. Hence, the proof is completed. □

By analogous argument, we can verify the following theorem on the existence of minimal target error $\epsilon$.

THEOREM 3.3. *Let*

$$\text{(3.22)} \qquad \pi_E = \{\epsilon' \in R^+; \text{the system(s) is max-min } (\delta, \gamma, \rho) - \text{controllabe on } [0, t] \text{ w.r.t. } B(x^0; \epsilon')\}.$$

*Then there exists a minimal value $\epsilon_f$ such that*

$$\text{(3.23)} \qquad \epsilon_f = \min \pi_E.$$

*In particular, the system(s) remains max-min $(\delta, \gamma, \rho)$-controllable on $[0, t]$ w.r.t. $B(x^0; \epsilon_f)$. Further $\epsilon_f$ is given by*

$$\text{(3.24)} \qquad \epsilon_f = \max\{0, \hat{\epsilon}\}$$

*where*

$$\begin{aligned}\hat{\epsilon} = \sup\{&|\langle x^0, x^*\rangle| + \gamma\|C^*(\cdot)W^*(t-\cdot)x^*\|_{2,[0,t]} \\ &-\rho\|(W^*(t)x^*, U_t^*(\cdot)x^*\|_{M_2^*} \\ &-\delta\|B^*(\cdot)W^*(t-\cdot)x^*\|_{2.[0,t]}; |x^*| = 1\}.\end{aligned} \tag{3.25}$$

The next two theorems are related to the existence of maximal norm-bounds for the evader's controls.

THEOREM 3.4. *Let*

$$\begin{aligned}\pi_C = \{\gamma' \in R^+; &\text{ the system(s) is max-min} \\ &(\delta, \gamma', \rho) - \text{controllable on } [0,t] \text{ with respect to } B(x^0;\epsilon)\}.\end{aligned} \tag{3.26}$$

*If $H_C \neq \phi$ and $\pi_C$ is bounded, then there exists a maximal value $\gamma_f$ such that*

$$\gamma_f = \max \pi_C. \tag{3.27}$$

*In particular, the system(s) remains max-min $(\delta, \gamma_f, \rho)$-controllable on $[0,t]$ with respect to $B(x^0;\epsilon)$. Further in this case the maximal value $\gamma_f$ is given by*

$$\gamma_f = \max\{0, \hat{\gamma}\}, \tag{3.28}$$

*where*

$$\begin{aligned}\hat{\gamma} \;=\; \inf\{&\delta\|B^*(\dot{)}W^*(t-\cdot)x^*\|_{2,[0,t]} + \epsilon|x^*| \\ &-|\langle x^0, x^*\rangle| - \rho\|(W^*(t)x^*, U_t^*(\cdot)x^*)\|_{M_2^*}; x^* \in H_C\}.\end{aligned} \tag{3.29}$$

*If $H_C = \phi$ or $\pi_C$ is unbounded, then $\pi_C = R^+$.*

THEOREM 3.5. *Let*

$$\begin{aligned}\pi_R \;=\; &\{\rho' \in R^+; \text{ the system (s) is max-min} \\ &(\delta, \gamma, \rho') - \text{controllable on } [0,t] \text{ w.r.t. } B(x^0;\epsilon)\}.\end{aligned} \tag{3.30}$$

*If $H_G \neq \phi$ and $\pi_R$ is bounded, then there exists a maximal value $\rho_f$ such that*

$$\rho_f = \max \pi_R. \tag{3.31}$$

*In particular, the system (s) remains max-min $(\delta, \gamma, \rho_f)$-controllable on $[0,t]$ w.r.t $B(x^0;\epsilon)$. Further in this case the maximal value $\rho_f$ is given by*

$$\rho_f = \max\{0, \hat{\rho}\}, \tag{3.32}$$

*where*

$$(3.33) \quad \hat{\rho} = \inf\{\delta\|B^*(\cdot)W^*(t-\cdot)x^*\|_{2,[0,t]} + |x^*| \\ -\gamma\|C^*(\cdot)W^*(t-\cdot)x^*\|_{2,[0,t]} - |\langle x^0, x^*\rangle|; x^* \in H_G\}.$$

*If* $H_G = \phi$ *or* $\pi_R$ *is unbounded, then* $\pi_R = R^+$.

*Proof of Theorem 3.4 and Theorem 3.5.* We can prove these theorems in a manner similar to Theorem 3.2. □

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