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4-DIMENSIONAL CRITICAL WEYL STRUCTURES

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ABSTRACT. We view Weyl structures as generalizations of Riemannian metrics and study the critical points of geometric functionals which involve scalar curvature, defined on the space of Weyl structures on a closed 4-manifold. The main goal here is to provide a framework to analyze critical Weyl structures by defining functionals, discussing function spaces and writing down basic formulas for the equations of critical points.

1. Introduction

A *Weyl structure* on a smooth manifold M consists of a conformal class $[g]$ of Riemannian metrics and a torsion-free connection D preserving $[g]$, i.e. for any metric g in $[g]$, $Dg = \omega \otimes g$ for a 1-form ω. This structure may be viewed as a generalization of a Riemannian metric because given a Riemannian metric one can associate its conformal class and Levi-Civita connection. More precisely, the space of Riemannian metrics with unit volume, when M is compact, is canonically embedded in the space of Weyl structures (see section 2).

The study of Weyl structures stemmed from E. Cartan's work [4] on 3-dimensional *Einstein-Weyl* structures which are generalizations of Einstein metrics in Riemannian or Lorentzian geometry. A Weyl structure is called *Einstein-Weyl* if the symmetric part of the Ricci curvature is proportional to a metric in $[g]$. Weyl geometry arises naturally in the study of almost hermitian geometry and contact geometry. And it has been recently explored much to understand Einstein-Weyl structures [5,

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6, 7, 10, 11, 12, 13]. An interesting part of Einstein-Weyl geometry is that it reveals much analogy to Einstein geometry.

Motivated by these works, we want to study *critical* Weyl structures, partly to understand Einstein-Weyl structures better and partly to search for other special Weyl structures.

In Riemannian geometry the total scalar curvature functional and the squared L^2 norm functional of scalar curvature have played important roles in the study of scalar curvature and Riemannian functionals [3, Chap. 4]. We discuss on generalizing them and define corresponding functionals on the space of Weyl structures.

To analyze critical structures of Weyl functionals in this paper and in a sequel paper [8] we needed to describe the space of Weyl structures as a Banach manifold, which is done in section 4 of this paper. This follows a standard type of argument in nonlinear Banach analysis. We do this partly for the sake of completeness and also because we could not find the exact literature clearly stating it. One may also read [3] and references therein.

Then we set up the formulas for the equations of critical structures. From analyzing them, we find that in most cases critical structures are locally conformally a metric and that under mild conditions they are Einstein metrics or metrics with zero scalar curvature (see section 3 and 5 for details).

The paper is organized as follows: In section 2 we describe definitions and basic facts in Riemannian and Weyl geometry. In section 3 we study critical points of the total conformal scalar curvature functional. In section 4 we explain the Banach manifold structure of the space of Weyl structures. In section 5, we discuss on generalizations of the squared L^2 norm functional of scalar curvature and study its critical points.

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2. Preliminaries

In this section we shall review some basics on Weyl geometry [7, 11] and recall definitions and properties concerning two functionals.

On a manifold M with a Weyl structure $([g], D)$, a choice of a metric g in $[g]$ induces a 1-form ω from the equation $Dg = \omega \otimes g$. Under a conformal change $g \mapsto f^2 g$, we have $\omega \mapsto \omega + 2d\ln(f)$. So $([g], D)$ may well be called *closed* if $d\omega = 0$ and *exact* if ω is exact. If $([g], D)$ is exact, then D is the Levi-Civita connection of some metric in $[g]$ and if closed, then D is locally a Levi-Civita connection of a metric. We will simply say that a structure $([g], D)$ is a *metric* or *locally conformally a metric* when it is exact or closed, respectively.

Any one-form ω together with the Levi-Civita connection ∇^g of a metric g determines a torsion-free connection D by $D_X Y = \nabla^g{}_X Y - \frac{1}{2}(\omega(X)Y + \omega(Y)X - g(X,Y)\omega^\#)$, which preserves $[g]$, where $\omega^\#$ is the dual vector field to ω with respect to g.

An important fact crucial to this paper is that a Weyl structure on a compact manifold has a unique, up to homothety, metric (to be called the *Gauduchon* metric) in the conformal class such that the associated 1-form ω is co-closed [5], i.e. $\delta\omega = 0$.

From above discussion we may identify a Weyl structure $([g], D)$ with a pair (g, ω) of a metric g of unit volume and its co-closed 1-form ω.

One can define various curvature tensors of a Weyl structure $([g], D)$ similarly to a Riemannian metric. The curvature R^D can be defined : $R^D_{X,Y}Z = D_{[X,Y]}Z - [D_X, D_Y]Z$, for $X, Y, Z \in TM$. And the *Ricci curvature* r^D is defined as $r^D(X,Y) = g(R^D_{e_i,X}Y, e_i)$ for a metric $g \in [g]$ and g-orthonormal frame e_i, $i = 1, 2, ..., n$. This definition is well defined independent of the choice of a metric in $[g]$ but r^D is not necessarily symmetric. The *conformal scalar curvature* s^D is defined as the trace of r^D with respect to $[g]$. So s^D is so-called *of conformal weight* -2, which means that if one denotes the trace of r^D with respect to g by s^D_g, then $s^D_{f^2 g} = f^{-2} s^D_g$ holds. These curvature notations follow the convention of current Weyl geometry such as in [7, 11], but it should be clear that r^D and s^D depend also on $[g]$, not only on D.

From now on we will consider only 4-dimensional closed smooth manifolds. We denote the symmetric part of a 2-tensor ϕ by $S(\phi)$ and the trace-free part of a symmetric 2-tensor ψ by $S_0(\psi)$. Let r, s, $z := r - \frac{1}{4}sg$, W and dv denote respectively the Ricci, scalar, trace-free Ricci, Weyl curvature tensor and volume form of a metric g.

Recall that there is a decomposition of Riemannian curvature tensor [3];

$$R = \frac{s}{24} g \odot g + \frac{1}{2} z \odot g + W, \tag{2.1}$$

where for 2-tensors α and g, the Kulkarni-Nomizu product $\alpha \odot g$ is the

4-tensor

$$\begin{aligned}\alpha \odot g(x,y,z,t) = \alpha(x,z)g(y,t) + \alpha(y,t)g(x,z) \\ - \alpha(x,t)g(y,z) - \alpha(y,z)g(x,t).\end{aligned}$$

For a given metric g in $[g]$, its associated 1-form ω and Levi-Civita connection ∇, the curvature tensor R^D of $([g], D)$ can be similarly decomposed as follows [11];

$$R^D = W + \frac{1}{2}S_0(r^D) \odot g + \frac{1}{24}s_g^D g \odot g + (\frac{1}{4}d\omega \odot g + \frac{1}{2}d\omega \otimes g). \tag{2.2}$$

If g is a Gauduchon metric, the following holds;

$$\begin{aligned} S(r^D) &= r_g + \frac{1}{2}(\omega \otimes \omega - |\omega|^2 g + 2\nabla\omega - d\omega), \\ s_g^D &= s_g - \frac{3}{2}|\omega|^2. \end{aligned} \tag{2.3}$$

For the rest of this section we shall explain definitions and some properties of functionals and critical points.

We shall call a functional $\mathcal{F}$ *Riemannian* [or *Weyl*] if it is defined on a (sub)space of Riemannian metrics [or Weyl structures] and invariant with respect to the action of diffeomorphism group of M. Here a diffeomorphism acts on a Weyl structure by pull-back. For such a functional $\mathcal{F}$ we say in this paper that a smooth metric [or a smooth Weyl structure] (g, ω) is a critical point of $\mathcal{F}$ functional if it satisfies $(\mathcal{F}(g_t, \omega_t))'(0) = 0$ for any C^∞ curve of metrics [or Weyl structures] (g_t, ω_t) such that $(g_0, \omega_0) = (g, \omega)$.

Recall the following two functionals defined on the space $\mathcal{M}_1$ of all smooth Riemannian metrics with unit volume on M. The first one is called the *total scalar curvature functional* and the second the *squared L^2 norm functional of scalar curvature.*

$$\mathcal{S}(g) = \int_M s_g dv, \tag{2.4}$$

$$\mathcal{S}s(g) = \int_M (s_g)^2 dv. \tag{2.5}$$

The critical points of these two functionals are well known [3];

PROPOSITION 2.1. *For a metric g of unit volume on a 4-dimensional closed manifold,*

(1) *g is $\mathcal{S}$-critical if and only if it is an Einstein metric.*
(2) *g is $\mathcal{S}s$-critical if and only if it is either an Einstein metric or has zero scalar curvature.*

3. Total conformal scalar curvature functional

We are going to define a generalization of the total scalar curvature functional (2.4) on the space of all Weyl structures on a smooth closed 4-manifold M. The problem is that $\int_M s^D dv$ appears to be not well defined because under the conformal change of metric, $g \mapsto f^2 g$, s^D and the volume form dv changes to $f^{-2}s^D$ and $f^4 dv$ respectively. But as explained in section 2 we may identify the space of Weyl structures $([g], D)$, which we denote by $\mathcal{W}_o$, with $\mathcal{W} := \{(g,\omega)|\delta\omega = 0\}$ which is a subset of $\mathcal{M}_1 \times \Omega^1$, where Ω^1 is the space of all smooth one-forms on M.

Then we can define $\int_M s^D dv$ on $\mathcal{W}$, which we shall call the *total conformal scalar curvature functional* and denote by $\mathcal{S}^w$. Note that the total scalar curvature functional played an important role in the study of Riemannian functionals [3].

To understand the critical points of Weyl functionals we should consider a one-parameter family of Weyl structures (g_t, ω_t). Let $\frac{d}{dt} g_t|_0 = g'(0) = h$ and $\frac{d}{dt}\omega_t|_0 = \omega'(0) = \eta$. As $\delta_t \omega_t = 0$, we have a constraint equation for h and η. We denote by $\mathrm{tr}h$ the trace of h with respect to g.

LEMMA 3.1. *The infinitesimal deformation (h,η) at a Weyl structure (g,ω) tangent to a one-parameter family of Weyl structures satisfies the following;*

$$\delta_g\{\eta + h(\omega) + \frac{1}{2}(\mathrm{tr}h)\omega\} = 0. \tag{3.1}$$

Proof. For the computation we may take a local orientation and the corresponding Hodge star operator. The equation $\delta_t\omega_t = -*_t d *_t \omega_t = 0$ is equivalent to $d *_t \omega_t = 0$, where $*_t$ is the Hodge star operator of g_t. Taking derivative with respect to t gives $d(*_t)'_0\omega_0 + d*_0\eta = 0$. To compute $(*_t)'$, let ϕ_1, ϕ_2 each be any one-form. From $\phi_1 \wedge *_t\phi_2 = (\phi_1,\phi_2)_t dv_t$

where $(\cdot,\cdot)_t$ is the pointwise inner product with respect to g_t, we get

$$\begin{aligned}\phi_1 \wedge (*_t)'_0\phi_2 &= (\phi_1,\phi_2)'_t|_0 dv_0 + (\phi_1,\phi_2)_0\frac{\mathrm{tr}h}{2}dv_0\\ &= \{h(\phi_1,\phi_2) + (\phi_1,\phi_2)_0\frac{\mathrm{tr}h}{2}\}dv_0\\ &= (\phi_1, h(\phi_2) + \frac{\mathrm{tr}h}{2}\phi_2)_0 dv_0\\ &= \phi_1 \wedge *_0\{h(\phi_2) + \frac{\mathrm{tr}h}{2}\phi_2\}.\end{aligned}$$

So, $(*_t)'_0\omega = *_0\{h(\omega) + \frac{(\mathrm{tr}h)}{2}\omega\}$. We get $d *_0 \{h(\omega) + \frac{(\mathrm{tr}h)}{2}\omega + \eta\} = 0$, which implies (3.1). This completes the proof. □

Now we compute the *derivative* of $\mathcal{S}^w$ functional as a function defined on a Banach manifold, which will be explained below in section 4. We denote by $\langle\cdot,\cdot\rangle$ the L^2 inner product.

LEMMA 3.2. *$\mathcal{S}^w$ is differentiable and the derivative is as follows;*

$$(\mathcal{S}^w)'_{(g,\omega)}(h,\eta) = \langle\frac{1}{2}s^D g - r, h\rangle - \frac{3}{2}\langle\omega, h(\omega)\rangle - 3\langle\omega,\eta\rangle,$$

where (h,η) satisfies (3.1).

Proof. We use (2.3) and a formula from [1, p. 120] to compute

$$\begin{aligned}(s^D)'(h,\eta) &= s'_g(h) - \frac{3}{2}g(h,\omega\otimes\omega) - 3g(\omega,\eta)\\ &= \Delta(\mathrm{tr}(h)) + \delta(\delta h) - g(h,r) - \frac{3}{2}g(h,\omega\otimes\omega) - 3g(\omega,\eta).\end{aligned}$$

Then we get

$$\begin{aligned}(\mathcal{S}^w)'_{(g,\omega)}(h,\eta) &= \int_M (s^D)'dv_g + \int_M s^D\frac{\mathrm{tr}h}{2}dv_g\\ &= -<r,h> - \frac{3}{2} <h,\omega\otimes\omega> - 3<\omega,\eta>\\ &\quad + \frac{1}{2}<s^D g, h>\\ &= <\frac{1}{2}s^D g - r, h> - \frac{3}{2}<\omega, h(\omega)> - 3<\omega,\eta>.\end{aligned}$$

This completes the proof. □

Now we prove

PROPOSITION 3.3. *A Weyl structure (g, ω) on a 4-dimensional closed smooth manifold M is critical with respect to the total conformal scalar curvature functional if and only if it is an Einstein metric.*

REMARK 3.1. In other words, on $\mathcal{W}$ there exists no critical Weyl structures away from the subspace of Riemannian metrics, and Einstein metrics are actually critical with respect to $\mathcal{S}^w$, not just with respect to (2.4).

Proof of Proposition 3.3. As we defined the functional on $\mathcal{W}$, any metric g of a Weyl structure (g, ω) is Gauduchon. So one can argue directly from (2.3) to see that a $\mathcal{S}^w$-critical structure is a metric. But this would not work for most other functionals. So to provide a prototype of argument to be used later, we will prove using Lemma 3.2.

Suppose that a structure (g, ω) is critical with respect to the total conformal scalar curvature functional. We consider the following one parameter family of Weyl structures $(g_t, \omega_t) = (g, \omega + t\omega)$. Then we get from above lemma $-3 < \omega, \omega >= 0$. So, $\omega = 0$. Now this structure $(g, 0)$, which is a Riemannian metric, is a critical point of the (restricted) total scalar curvature functional (2.4). So it must be an Einstein metric from Proposition 2.1 (1). Conversely, any Einstein metric $(g, 0)$ of unit volume can be checked to be $\mathcal{S}^w$-critical by similar argument as in [3, Theorem 4.21]. This finishes the proof. $\square$

REMARK 3.2. In this paper we mainly discuss on a 4-manifold. However, the same argument should be useful in other dimensions, too.

4. The space of Weyl structures as a Banach manifold

As illustrated in the proof of Proposition 3.3, finding an actual deformation (g_t, ω_t) to a given infinitesimal deformation (h, η) is necessary in characterizing critical structures. We can find a deformation to the desired direction in a few practical cases, but here we prefer to explain concretely the Banach manifold structure of $\mathcal{W}$ for future study of Weyl functionals. Then we shall see that for any vector $(h, \eta) \in T_{(g,\omega)}\mathcal{W}$, there exist a smooth curve tangent to it. One can find in [3, Chap. 4] how Banach space – or other infinite dimensional function spaces – theory plays an essential role in the study of Riemannian functionals.

Denote by $\mathcal{W}^{k,p}(\mathcal{M}_1)$ and $\mathcal{W}^{k,p}(\Omega^1)$ the Banach space of $W^{k,p}$ Riemannian metrics of unit volume and 1-forms on M, respectively. We define below a map π from $\mathcal{W}^{k+2,p}(\mathcal{M}_1)\times\mathcal{W}^{k,p}(\Omega^1)$ to the space $\mathcal{W}^{k-1,p}(\Gamma_0)$ of $W^{k-1,p}$ functions f on M satisfying $\int_M f dv = 0$, where k is a large natural number and $p > 2$. Set $\pi(g,\omega) = \delta_g\omega$. We shall explain that $\pi^{-1}(0)$ is a smooth Banach submanifold and so that for any element (h,η) in the kernel of the differential, $\ker(d\pi)$, there exists a smooth curve in $\pi^{-1}(0)$ tangent to it. The argument follows a standard way in nonlinear geometric analysis.

First we show

LEMMA 4.1. *The differential $d\pi_{(g,\omega)}$ is surjective at any point (g,ω).*

Proof. As done in the proof of Lemma 3.1, we take a local orientation in order to use Hodge star operator. Then $d\pi_{(g,\omega)}(h,\eta) = -*'\circ d\circ *\omega - *\circ d\circ *'\omega - *\circ d\circ *\eta$. In the proof of Lemma 3.1, we computed for 1-form ω that $(*_t)'_0\omega = *_0(h(\omega) + \frac{(\mathrm{tr}h)}{2}\omega)$. Note that $(*_t)'_0\omega$ consists of linear terms in h. Similar formula holds for 4-forms so that $*'\circ d\circ *\omega$ also has only h-linear terms.

Then, $d\pi_{(g,\omega)}(h,\eta) = h$-linear terms $+\ \delta_g\eta$. This implies that $d\pi_{(g,\omega)}$ is surjective at any (g,ω). This finishes the proof. □

Now $d\pi$ is not a Fredholm operator but still we can find easily a family of right inverses. Using elliptic regularity, define for each $(g,\omega)\in\mathcal{W}^{k+2,p}(\mathcal{M}_1)\times\mathcal{W}^{k,p}(\Omega^1)$, a map

$$\mathcal{P}^{(g,\omega)} : \mathcal{W}^{k-1,p}(\Gamma_0) \longrightarrow \mathcal{W}^{k+2,p}(\mathcal{M}_1)\times\mathcal{W}^{k,p}(\Omega^1)$$

by $\mathcal{P}^{(g,\omega)}(f) = (g,\omega + dGf)$, where G is Green's function (operator) of g. The map $\mathcal{P}^{(g,\omega)}$ is smooth and its differential at 0 is simply $d\mathcal{P}^{(g,\omega)}{}_{(0)}(k) = (0, dGk)$ for any function $k\in T_0\mathcal{W}^{k-1,p}(\Gamma_0)$. It holds that

$$\pi\circ\mathcal{P}^{(g,\omega)}(f) = \pi(g,\omega+dGf) = \delta_g(\omega+dGf) = f.$$

So the differential satisfy $d\pi_{(g,\omega)}\circ d\mathcal{P}^{(g,\omega)}{}_{(0)} = id$ on the tangent space $T_0\mathcal{W}^{k-1,p}(\Gamma_0)$, i.e. $d\mathcal{P}^{(g,\omega)}{}_{(0)}$ is a right inverse of $d\pi_{(g,\omega)}$.

The rest of Banach space argument is standard; with this simple right inverse at each $(g,\omega)\in\pi^{-1}(0)$ one can conclude from the infinite dimensional implicit function theorem, see for instance [1], that $\pi^{-1}(0)$

is an infinite dimensional Banach submanifold or, more directly, one can check similarly as in [9, Theorem 3.3.4] that for each (h, η) in the kernel of $d\pi_{(g,\omega)}$, there exist a suitably smooth (at least C^2) curve tangent to (h, η). We summarize;

PROPOSITION 4.2. *$\mathcal{W}$ has a smooth ($W^{k,p}$ for any large k) Banach manifold structure and for any element (h, η) in each tangent space $T_{(g,\omega)}\mathcal{W}$, there exists a smooth curve tangent to it.*

Note that (g, ω) is critical with respect to a functional $\mathcal{F}$ if and only if for any C^∞ curve (g_t, ω_t) in $\mathcal{W}$, $\{\mathcal{F}(g_t, \omega_t)\}'(0) = 0$. But note that this definition is not affected even if we replace C^∞ by C^2. Therefore, we do not really need to get a C^∞ curve (g_t, ω_t). For us, some large integer k would be enough to use the Sobolev embedding $W^{k,p} \subset C^2$.

REMARK 4.1. For the purpose of obtaining C^∞ curves, one could have described the space of Weyl structures as a smooth tame Frechet space or ILH space – inverse limit of Hilbert space. The choice of our right inverses as above would work it through.

5. Generalization of the squared L^2 norm functional of scalar curvature

In this section we study on some generalization of the squared L^2 norm functional of scalar curvature. One candidate arises immediately; $\int_M (s^D)^2 dv$. This functional is well defined on the original space of Weyl structures $\mathcal{W}_0$ and we do not need to use the identification of $\mathcal{W}$ and $\mathcal{W}_0$ in discussing its critical points. For computational convenience, we still prefer the $\mathcal{W}$ space description and keep using (g, ω) rather than $([g], D)$.

But we shall discuss not only $\int_M (s^D)^2 dv$ but also the following functional,

$$\mathcal{S}s^w(g, \omega) := \int_M (s^D)^2 + 18|d\omega|^2 \ dv,$$

where $|d\omega|^2 = \sum_{i<j}(d\omega_{ij})^2$ for g-orthonormal frame $e_1, ..., e_4$.

The reason why we consider this as a generalization of (2.5) is as follows; consider the full Weyl curvature functional $\int_M |R^D|^2_\otimes dv$, where R^D is considered as a $(3,1)$-tensor and $|\cdot|^2_\otimes$ is the norm induced by g on

(3,1)-tensors. This is well defined on $\mathcal{W}_0$. If we express this functional in terms of other quadratic terms, it becomes from (2.2)

$$\int_M |R^D|^2_{\otimes} dv = \frac{1}{6}\int_M (s^D)^2 + 18|d\omega|^2 \, dv + \int 2|S_0(r^D)|^2 + \int_M 4|W|^2 dv. \tag{5.1}$$

Comparing this with the corresponding expression in Riemannian curvature integrals

$$\int_M |R|^2 dv = \frac{1}{6}\int_M s^2 dv + \int 2|z|^2 + \int_M 4|W|^2 dv,$$

we may regard the first term on the right hand side of (5.1) as a generalization of $\int_M s^2 dv$.

As functionals defined on a Banach manifold, $\mathcal{S}s^w$ and other quadratic integrals may have their derivatives, which we compute now. Note that these derivatives should coincide with directional derivatives. First we do for $\tilde{\mathcal{S}s}^w := \int_M (s^D)^2 dv$.

LEMMA 5.1. *The functional $\tilde{\mathcal{S}s}^w$ is differentiable and the derivative at (g,ω) is as follows:*

$$\begin{aligned}(\tilde{\mathcal{S}s}^w)'_{(g,\omega)}(h,\eta) &= \{\tilde{\mathcal{S}s}^w(g_t,\omega_t)\}'(0)\\ &= \langle -6s^D\omega, \eta\rangle + \langle 2\nabla ds^D + 2(\Delta_g s^D)g \\ &\quad + \frac{1}{2}(s^D)^2 g - 2s^D r_g - 3s^D \omega\otimes\omega, h\rangle,\end{aligned}$$

where (h,η) satisfies (3.1).

Proof. One may use the proof of Lemma 3.2 to compute

$$(\tilde{\mathcal{S}}^w)'_{g,\omega}(h,\eta) = \frac{1}{2}\int (s^D)^2 \mathrm{tr}(h) dv + 2\int s^D (s^D)'(h,\eta) dv. \qquad \square$$

Next, we also compute the derivative of the *additional* curvature; $\mathcal{A}^w(g,\omega) := \int_M |d\omega|^2 dv$.

LEMMA 5.2. *The following holds:*

$$(\mathcal{A}^w)'_{(g,\omega)}(h,\eta) = \langle \frac{1}{2}|d\omega|^2 g - d\omega \circ d\omega, h\rangle + \langle 2d^* d\omega, \eta\rangle$$

where (h,η) *satisfies* (3.1) *and for* g*-orthonormal frame* $e_i, i = 1, ..., 4$, $(d\omega \circ d\omega)_{ij} = (d\omega)_{ik}(d\omega)_{kj}$.

Proof. We compute from $|d\omega|^2 = \sum_{i<k,j<l} g^{ij} g^{kl} (d\omega)_{ik}(d\omega)_{jl}$ for some coordinates to get

$$(|d\omega|^2)'_{(g,\omega)}(h,\eta) = 2(d\omega, d\eta)_g - (d\omega \circ d\omega, h)_g.$$

Therefore,

$$\begin{aligned}(\mathcal{A}^w)'_{(g,\omega)}(h,\eta) &= \frac{1}{2}\int |d\omega|^2 \mathrm{tr}(h) dv \\ &\quad + \int \{2(d\omega, d\eta)_g - (d\omega \circ d\omega, h)_g\} dv \\ &= \langle \frac{1}{2}|d\omega|^2 g, h\rangle - \langle d\omega \circ d\omega, h\rangle + \langle 2d^* d\omega, \eta\rangle. \qquad \square\end{aligned}$$

So, combining Lemmas 5.1 and 5.2, we have the formula for the derivative of $\mathcal{S}s^w$.

LEMMA 5.3. *The following holds:*

$$\begin{aligned}(\mathcal{S}s^w)'_{(g,\omega)}(h,\eta) = \langle -6s^D\omega + 36d^* d\omega, \eta\rangle \ &+ \langle 2\nabla ds^D + 2(\Delta_g s^D)g \\ &+ \frac{1}{2}(s^D)^2 g - 2s^D r_g - 3s^D \omega \otimes \omega, h\rangle \\ &+ \langle 9|d\omega|^2 g - 18 d\omega \circ d\omega, h\rangle\end{aligned}$$

where (h,η) *satisfies* (3.1).

From this lemma we get;

THEOREM 5.4. *An* $\mathcal{S}s^w$*-critical Weyl structure* (g,ω) *on a closed smooth 4-manifold* M *is closed, i.e.* ω *is a harmonic 1-form.*

Suppose that, moreover, s^D *is either non-positive or non-negative, but nonzero somewhere. Then,* (g,ω) *is either an Einstein metric or a metric with zero scalar curvature.*

Proof. Suppose (g,ω) is critical with respect to the $\mathcal{S}s^w$ functional. We consider any one-parameter deformation (g_t,ω_t) of (g,ω) such that $(g_0',\omega_0') = (h,\eta) = (0,\omega)$. From above lemma we get $\int_M 36|d\omega|^2 - 6s^D|\omega|^2 dv = 0$.

Next, consider any one-parameter deformation such that $(h,\eta) = (g,0)$. Again from above lemma,

$$\int_M 6\Delta s^D + 2(s^D)^2 - 2s^D s - 3s^D|\omega|^2 + 72|d\omega|^2 dv = 0.$$

From (2.3), above reduces to $\int_M -6s^D|\omega|^2 + 72|d\omega|^2 dv = 0$. So we get $d\omega = 0$, which proves the first half of theorem, and $\int_M s^D|\omega|^2 dv = 0$. If s^D is either non-positive or non-negative, but nonzero somewhere, then from $\int_M s^D|\omega|^2 dv = 0$ and the unique continuation theorem for harmonic forms, see Remark 5.1 below, we get $\omega = 0$. By proposition 2.1 (2), (g,ω) is either an Einstein metric or a metric with zero scalar curvature. This finishes the proof of theorem. □

REMARK 5.1. In the proof of above theorem, we need the following *weak* unique continuation Theorem of Aronszajn [2]; Let L be a second order elliptic operator with C^l coefficients, for large l. Suppose $Lu = 0$ on a domain D and $u = 0$ on a nonempty open subset of D. Then $u = 0$ on D.

From Theorem 5.4, the study of $\mathcal{S}s^w$-critical Weyl structures has to do with the first cohomology of the manifold. In particular, when there is no harmonic 1-form on a manifold, we have

COROLLARY 5.5. *If the first Betti number of M vanishes, then every $\mathcal{S}s^w$-critical Weyl structure is either an Einstein metric or a metric with zero scalar curvature.*

Now we treat $\tilde{\mathcal{S}s}^w$-critical structures. The proof is similar to that of theorem 5.4.

PROPOSITION 5.6. *An $\tilde{\mathcal{S}s}^w$-critical Weyl structure (g,ω) on a closed smooth 4-manifold M satisfies that $\int_M s^D|\omega|^2 dv = 0$.*

Suppose that, moreover, s^D is either non-positive or non-negative, but nonzero somewhere. Then, (g, ω) is either an Einstein metric or a metric with zero scalar curvature.

REMARK 5.2. Theorem 5.4 and Proposition 5.6 is an extension of Proposition 2.1 (2) to Weyl geometry. It follows from Theorem 5.4 that there can be only one (family of) $\mathcal{S}s^w$-critical Einstein-Weyl structures which are not (global) metrics [7]. See Example 5.1 below.

EXAMPLE 5.1. On $S^1 \times S^3$, let g_t be the product of the metric $t^2 d\theta^2$on $S^1 = \{\exp(i\theta) | \theta \in [0, 2\pi)\}$ and the canonical metric g_{can} on S^3 with sectional curvature one. Let ω_t be the 1-form $2td\theta$, which is harmonic with respect to g_t. For each t, (g_t, ω_t) is an Einstein-Weyl structure with $s^{D_t} = 0$. Any closed non-exact 4-dimensional Einstein-Weyl structure is locally equivalent to this structure.

REMARK 5.3. From Lemma 5.2, a Weyl structures (g, ω) is critical with respect to the $\mathcal{A}^w$ functional if and only if it is closed.

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