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CHARACTERIZATIONS OF A KRULL RING $R[X]$

GYU WHAN CHANG

Dedicated to my father WHA SIK CHANG

ABSTRACT. We show that $R[X]$ is a Krull (resp. factorial) ring if and only if R is a normal Krull (resp. factorial) ring with a finite number of minimal prime ideals if and only if R is a Krull (resp. factorial) ring with a finite number of minimal prime ideals and R_M is an integral domain for every maximal ideal M of R. As a corollary, we have that if $R[X]$ is a Krull (resp. factorial) ring and if D is a Krull (resp. factorial) overring of R, then $D[X]$ is a Krull (resp. factorial) ring.

1. Introduction

Throughout this paper, R denotes a commutative ring with identity, $T(R)$ its total quotient ring. $Z(R)$ will be the set of zero divisors in R. An overring of R is a ring between R and $T(R)$. For technical reason we assume that $R \subsetneq T(R)$. An element which is not a zero divisor is said to be regular and an ideal is called a regular ideal if it contains a regular element. A ring R is called a Marot ring if each regular ideal of R is generated by its set of regular elements (cf. [4, Ch. 7]). For a fractional ideal A of R, let $A^{-1} = \{x \in T(R) | xA \subseteq R\}$ and $A_v = (A^{-1})^{-1}$. A fractional ideal A is divisorial if $A = A_v$.

It is well known that R is a Krull (resp. factorial) domain if and only if $R[X]$ is a Krull (resp. factorial) domain. Anderson et al. [1, p.113] gave a factorial ring R (and hence a Krull ring) such that $R[X]$ is not a Krull ring (and hence not a factorial ring). In [1, Theorem 5.7], they also showed that $R[X]$ is a Krull ring if and only if R is a finite direct sum of Krull domains. Anderson and Markanda [2] showed that R is a

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finite direct sum of UFDs if and only if $R[X]$ is a UFR if and only if $R[X]$ is a factorial ring.

In this paper, we will prove that $R[X]$ is a Krull (resp. factorial) ring if and only if R is a normal Krull (resp. factorial) ring with a finite number of minimal prime ideals if and only if R is a Krull (resp. factorial) ring with a finite number of minimal prime ideals and R_M is an integral domain for every maximal ideal M of R. As a corollary, we have that if $R[X]$ is a Krull (resp. factorial) ring and if D is an overring of R, then D is a Krull (resp. factorial) ring if and only if $D[X]$ is a Krull (resp. factorial) ring. Moreover, if R is a reduced Noetherian ring and if D is a Krull overring of R, then $D[X]$ is a Krull ring. We also show that if R is a regular Noetherian ring, then $R[X]$ is a Krull ring. For undefined notations and definitions, the reader can be referred to [4, 5, 6].

2. Main Results

Recall from [6, p. 116] that a ring R is said to be *normal* if R_P is an integrally closed domain for each prime ideal P of R. It is easy to show that R is a normal ring if and only if $R[X]$ is a normal ring (cf. [6, Proposition 17.B(2)]). R is said to be *reduced* if it has no nonzero nilpotent elements. It is clear that for each maximal ideal M of R, if R_M is an integral domain then R is a reduced ring. Thus a normal ring is reduced.

LEMMA 1. (cf. [5, Theorem 168]) *A ring R has a finite number of minimal prime ideals and R_M is an integral domain for every maximal ideal M of R if and only if R is a finite direct sum of integral domains.*

Proof. ($\Rightarrow$) Let $\{P_1, \dots, P_k\}$ be the set of minimal prime ideals of R. Let M be a maximal ideal of R. Since R_M is an integral domain, M contains exactly one of the P_i's. Thus if $i \neq j$, then $P_i + P_j = R$. Note that R is reduced since R_M is an integral domain for each maximal ideal M of R, that is, $P_1 \cap \cdots \cap P_k = 0$. By the Chinese Remainder Theorem, $R \cong (R/P_1) \oplus \cdots \oplus (R/P_k)$.

($\Leftarrow$) By a localization technique, we have that R_M is an integral domain for every maximal ideal M of R. It is clear that R has a finite number of minimal prime ideals. □

Let P be a prime ideal of R contained in a maximal ideal M of R. Since $R_P \cong (R_M)_{PR_M}$, if R_M is an integral domain, so is R_P. Thus R_M

is an integral domain for every maximal ideal M of R if and only if R_P is an integral domain for every prime ideal P of R.

LEMMA 2. *Let R be a ring such that R_P is an integral domain for each maximal ideal P of R and let D be an overring of R. If M is a maximal ideal of D, then D_M is an integral domain.*

Proof. Let $P = M \cap R$. Note that R_P is an integral domain and $R_P \hookrightarrow D_{R-P} \hookrightarrow T(R_P)$. Thus D_{R-P} and hence $D_M \cong (D_{R-P})_{MD_{R-P}}$ is an integral domain. □

A ring R is called a *Krull ring* if there is a family $\{V_\alpha\}$ of rank one discrete valuation rings (DVR) such that $R = \cap_\alpha V_\alpha$ and the intersection has finite character, or equivalently R is completely integrally closed and the ascending chain condition on regular divisorial ideals holds.

THEOREM 3. (cf. [1, Theorem 5.7]) *The following conditions are equivalent.*

(1) *R is a finite direct sum of Krull domains.*
(2) *$R[X]$ is a Krull ring.*
(3) *$R[X]$ is a normal Krull ring.*
(4) *R is a normal Krull ring with a finite number of minimal prime ideals.*
(5) *R is a Krull ring with a finite number of minimal prime ideals and R_M is an integral domain for every maximal ideal M of R.*

Proof. (1) $\Leftrightarrow$ (2): This is [1, Theorem 5.7].

(3) $\Rightarrow$ (2) and (4) $\Rightarrow$ (5): These are clear.

(1) $\Rightarrow$ (4): Let $R = D_1 \oplus \cdots \oplus D_n$, where D_i's are Krull domains. Since each prime ideal Q of R is of the form $D_1 \oplus \cdots \oplus P \oplus \cdots \oplus D_n$ and $R_Q = (D_i)_P$, where P is a prime ideal of D_i, R is a normal Krull ring with a finite number of minimal prime ideals $\{0 \oplus D_2 \cdots \oplus D_n, D_1 \oplus 0 \oplus D_2 \oplus \cdots \oplus D_n, \dots, D_1 \oplus \cdots \oplus D_{n-1} \oplus 0\}$.

(5) $\Rightarrow$ (1): Let $\{P_1, \dots, P_n\}$ be the set of minimal prime ideals of R. By Lemma 1, $R = (R/P_1) \oplus \cdots \oplus (R/P_n)$. Since R is reduced, $Z(R) = \cup_{i=1}^n P_i$ [4, Lemma 4.8] and R is a Marot ring [4, Theorem 7.2]. Thus the set $X^1(R/P_i)$ of height-one prime ideals of R/P_i is $\{P/P_i |$ reg-ht$P=$ ht$P = 1$ and $P_i \subsetneq P\}$. We claim that each R/P_i is a Krull domain.

<u>Claim 1.</u> For each $P/P_i \in X^1(R/P_i)$, $(R/P_i)_{P/P_i}$ is a rank one DVR.

Proof. Since R is normal, $P_iR_P = 0$. Hence $R_P = R_P/P_iR_P \cong (R/P_i)_{P/P_i}$. Since R is a Marot Krull ring and reg-ht$P = 1$, $P \subsetneq PP^{-1}$ [4, Theorems 8.4 and 8.6]. Since $R_P = (PP^{-1})R_P = (PR_P)(P^{-1}R_P) \subseteq (PR_P)(PR_P)^{-1} \subseteq R_P$, PR_P is invertible. Thus $R_P \cong (R/P_i)_{P/P_i}$ is a rank one DVR.

Claim 2. $R/P_i = \cap\{(R/P_i)_{P/P_i} | P/P_i \in X^1(R/P_i)\}$.

Proof. It is enough to show that each prime t-ideal of R/P_i is of height-one [3, Ex. 22, p. 52]. Let Q/P_i be a prime ideal of R/P_i such that $\text{ht}(Q/P_i) \geq 2$. Then Q is a regular prime ideal of R and ht$Q$= reg-ht$Q \geq 2$ (note that R is reduced). Since R is a Krull ring, there are some elements $a, b \in Q$ with a regular such that $(a, b)_v = R$.

Let $\overline{a} = a + P_i$, $\overline{b} = b + P_i \in R/P_i$ and let $\frac{\overline{y}}{\overline{x}} \in T(R/P_i)$ such that $\frac{\overline{y}}{\overline{x}}(\overline{a}, \overline{b}) \subseteq R/P_i$ where $x, y \in R$ and $\overline{x} = x + P_i$, $\overline{y} = y + P_i$. Since $\overline{x} \neq 0$, $x \notin P_i$. Hence $(P_i, x) \nsubseteq Z(R)$. Thus (P_i, x) is a regular ideal of R. Take a regular element $x' \in (P_i, x)$. Then $x' = xr + p$ for some $r \in R$ and $p \in P_i$. So $x' + P_i = rx + P_i$, which implies that $\overline{(\frac{yr}{x'})} = \frac{\overline{yr}}{\overline{xr}} = \frac{\overline{y}}{\overline{x}}$. Since $\overline{(\frac{yr}{x'})}(\overline{a}, \overline{b}) \subseteq R/P_i$, $\frac{yr}{x'}a \in R$ and $\frac{yr}{x'}b \in R$. So $\frac{yr}{x'} \in (a, b)^{-1} = R$. Thus $(\overline{a}, \overline{b})^{-1} = R/P_i$ and $(Q/P_i)_t = R/P_i$, which shows that each prime t-ideal of R/P_i is of height-one.

Claim 3. The intersection $R/P_i = \cap\{(R/P_i)_{P/P_i} | P/P_i \in X^1(R/P_i)\}$ has finite character.

Proof. Let $\overline{a} = a + P_i$ be a nonzero element of R/P_i. Then (P_i, a) is a regular ideal of R. Hence there exist a finite number of regular height-one prime ideals of R containing (P_i, a) (note that R is a Krull ring). Thus the number of height-one prime ideals of R/P_i containing $\overline{a}$ is finite.

By Claims 1, 2, and 3, R/P_i is a Krull domain.

(4) $\Rightarrow$ (3): If R is a normal Krull ring with a finite number of minimal prime ideals, then R is a finite direct sum of Krull domains by ((5) $\Rightarrow$ (1)). Thus $R[X]$ is a Krull ring. Since R is normal, $R[X]$ is a normal Krull ring. □

COROLLARY 4. *Let R be a Noetherian ring. Then R is a normal ring if and only if $R[X]$ is a Krull ring.*

Proof. Note that an integrally closed Noetherian ring is a Krull ring [4, Theorem 10.1], and that the number of minimal prime ideals of a

Noetherian ring is finite (cf. [5, Theorem 88]). Thus the result follows from Theorem 3. □

It is clear that if D is an overring of R, then R and D have the same number of minimal prime ideals (if R is an integral domain then the minimal prime ideal of R is (0)). Thus the following result is an immediate consequence of Lemma 2 and Theorem 3.

COROLLARY 5. *Let $R[X]$ be a Krull ring and D be an overring of R. Then D is a Krull ring if and only if $D[X]$ is a Krull ring.*

A ring R satisfies *Property A* if each finitely generated ideal $I \subseteq Z(R)$ has nonzero annihilator. [5, Theorem 82] shows that a Noetherian ring satisfies *Property A*. It follows from [4, Corollary 2.6] that each overring of a Noetherian ring satisfies *Property A*.

COROLLARY 6. *Let R be a reduced Noetherian ring and let R' be the integral closure of R. Then $R'[X]$ is a Krull ring. Moreover, if D is a Krull overring of R, then $D[X]$ is a Krull ring.*

Proof. Since R is a reduced Noetherian ring, R' is a reduced integrally closed ring satisfying *Property A*. By [4, Theorem 13.11], $R'[X]$ is integrally closed. Hence $R'[X]$ is the integral closure of $R[X]$. Since R is a Noetherian ring, so is $R[X]$. Thus $R'[X]$ is a Krull ring [4, Theorem 10.1]. Moreover, if D is a Krull overring of R, then D is an overring of R'. Thus by Corollary 5, $D[X]$ is a Krull ring. □

Let R be a local Noetherian ring with maximal ideal M. Note that M/M^2 is an R-module annihilated by M and hence a vector space over the field R/M. Recall from [5, p. 116] that R is a *regular local ring* if $\dim_{R/M}(M/M^2) = \dim R$, where $\dim_{R/M}(M/M^2)$ is the dimension of a vector space M/M^2 over R/M and $\dim R$ is the Krull dimension of R. A Noetherian ring R is called a *regular ring* if R_P is a regular local ring for every prime ideal P of R [6, p. 140].

Since a regular local ring is a UFD [6, Theorem 48], a regular domain is an integrally closed Noetherian domain and hence a Krull domain [5, Theorems 103 and 104].

COROLLARY 7. *If R is a regular ring, then R is a finite direct sum of Krull domains. Thus $R[X]$ is a Krull ring.*

Proof. Since R is a regular ring, R is a finite direct sum of integral domains [5, Theorem 168]. By a localization technique, we can easily show that each direct summand of R is a regular domain and hence a Krull domain. □

A ring R is a *unique factorization ring* (UFR) if every principal ideal of R is a product of principal prime ideals and a ring R is called a *factorial ring* if each regular element of R is a product of (regular) prime elements. Hence a UFR is a factorial ring and a factorial ring is a Krull ring.

THEOREM 8. (cf. [1, Corollary 5.8]) *The following conditions are equivalent.*

(1) *R is a finite direct sum of UFDs.*
(2) *$R[X]$ is a UFR.*
(3) *$R[X]$ is a factorial ring.*
(4) *$R[X]$ is a normal UFR.*
(5) *$R[X]$ is a normal factorial ring.*
(6) *R is a normal factorial ring with a finite number of minimal prime ideals.*
(7) *R is a factorial ring with a finite number of minimal prime ideals and R_M is an integral domain for every maximal ideal M of R.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) and (4) $\Leftrightarrow$ (5): These are in [1, Corollary 5.8].

(4) $\Rightarrow$ (2) and (6) $\Rightarrow$ (7): These are clear.

(3) $\Rightarrow$ (5): Since a factorial ring is a Krull ring, $R[X]$ is normal by Theorem 3.

(3) $\Rightarrow$ (6): By Theorem 3, R is a normal Krull ring with a finite number of minimal prime ideals. Since $R[X]$ is a factorial ring, so is R. Thus R is a normal factorial ring with a finite number of minimal prime ideals.

(7) $\Rightarrow$ (1): By Theorem 3, R is a finite direct sum of Krull domains. Let $R = D_1 \oplus \cdots \oplus D_n$. We show that each D_i is a UFD. Let P be a prime ideal of D_i and let a be a nonzero element of P. Then $(1, \ldots, a, \ldots, 1)$ is a regular element of R. Since R is a factorial ring, $(1, \ldots, a, \ldots, 1)$ is a finite product of prime elements of R. Hence a is a finite product of prime elements of D_i. Thus P contains a prime element of D_i and hence D_i is a UFD by [5, Theorem 5]. □

The following corollary follows directly from Lemma 2 and Theorem 8.

COROLLARY 9. *Let $R[X]$ be a factorial ring and D be an overring of R. Then D is a factorial ring if and only if $D[X]$ is a factorial ring.*

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DEPARTMENT OF MATHEMATICS, KANGWON NATIONAL UNIVERSITY, CHUNCHON 200-701, KOREA

E-mail: whan@postech.ac.kr