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THE CROSSED COPRODUCT HOPF ALGEBRAS

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Abstract. We find the necessary and sufficient conditions for the smash product algebra structure and the crossed coproduct coalgebra structure with the dual cocycle α to afford a Hopf algebra $B \bowtie^{\alpha} H$. If B and H are finite algebra and Hopf algebra, respectively, then the linear dual $(B \bowtie^{\alpha} H)^*$ is also a Hopf algebra. We show that the weak coaction admissible mapping system characterizes the new Hopf algebras $B \bowtie^{\alpha} H$.

Introduction

The smash product algebra and the smash coproduct coalgebra are well known in the context of Hopf algebras [1, 7, 9, 10, 11, 12] and these notions can be viewed as being motivated by the semidirect product construction in the theory of groups and in the theory of affine group schemes, respectively.

The main construction we use is one which transforms Hopf algebras in the category ${}^{H}\mathcal{M}$ of H-comodules, for any Hopf algebra H, to (usual) Hopf algebras; this is Radford's biproduct [11]. Recently, D. Fischman and S. Montgomery [6] have reviewed this structure of the Hopf algebra from the Yetter-Drinfeld category point of view. Also, when B is an algebra with weak action on the Hopf algebra H and a coalgebra in ${}^{H}\mathcal{M}$, Z. Jiao, S. Wang and W. Zhao [7] constructed the biproduct $B \bowtie_{\sigma} H$ which is a (usual) Hopf algebra. $B \bowtie_{\sigma} H$ is the crossed product $B \#_{\sigma} H$ with a cocycle σ as an algebra and the smash coproduct $B \rtimes H$ as a coalgebra.

Now, it is very natural to consider what conditions the smash product algebra structure and the crossed coproduct coalgebra structure will inherit a Hopf algebra structure. In this paper we discuss this problem

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and find the necessary and sufficient conditions for it to be true. This generalizes the cases in [11].

In Section 2 we will find a necessary and sufficient conditions for the smash product algebra structure and the crossed coproduct coalgebra structure with the dual cocycle on $B \otimes H$ to afford a bialgebra structure. (Theorem 2.7.) (we denote the resulting bialgebra by $B \bowtie^{\alpha} H$.)

In Section 3 we introduce the Hopf algebra H with respect to the dual cocycle $\alpha : B \to H \otimes H$ and so derive necessary and sufficient conditions for the new bialgebra $B \bowtie^{\alpha} H$ to be a Hopf algebra. (Proposition 3.3 and Proposition 3.4.) In particular if B and H are finite dimensional algebra and Hopf algebra, respectively, then the linear dual $(B \bowtie^{\alpha} H)^*$ is exactly the structure of [7].

In Section 4 let $B \bowtie^{\alpha} H$ be a bialgebra. We show that the weak coaction admissible mapping system $B \leftrightarrows^{p_B}_{j_B} B \bowtie^{\alpha} H \rightleftarrows^{\pi_H}_{i_H} H$ (j_B and i_H are the canonical algebra inclusions, p_B and π_H are the canonical coalgebra projections) characterizes $B \bowtie^{\alpha} H$. (Theorem 4.7 and Theorem 4.11.)

1. Preliminaries

Throughout the paper we freely use the results and notions of [1, 10, 12]. In particular, all vector spaces will be over a field k. We adopt the usual sigma notions for the comultiplications of coalgebras.

First of all recall that the notion of the smash product of k-algebra A by a Hopf algebra H over k was introduced in [2, 9, 12] as follows: A is a left H-module algebra if A is a left H-module via $\tau : h \otimes A \to A$, $h \otimes a \mapsto h \cdot a$ and the following identies hold;

$$h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b) \quad \text{and} \quad h \cdot 1_A = \varepsilon(h) 1_A$$

for all $a, b \in A$ and $h \in H$. Now let A be a left H-module algebra. One can then define the smash product $A\#H$ is defined as follow; as k-space, $A\#H$ is $A \otimes H$ with elements $a \otimes h$ written as $a\#h$. Multiplication is defined by

$$(a\#h)(b\#l) = \sum a(h_1 \cdot b)\#h_2 l.$$

This makes $A\#H$ into a k-algebra with unit element $1 = 1_A\#1_H$.

For example, let $H = kG$, the group algebra. It is well-known that A is a left kG-module algebra if and only if G acts on A, that is, there exists a group homomorphsim $G \to \mathrm{Aut}_k A$. In this case, $A\#kG = A*G$, the skew group algebra; here multiplication is just $(ah)(bl) = a(h \cdot b)hl$ for all $a, b \in A$ and $h, l \in G$. So this is the generalization for Hopf algebras of the skew group algebra $A * G$, which has proved very useful in studying group actions.

Now the notion of crossed coproduct of k-coalgebra C by a Hopf algebra H was introduced in [4, 3, 5] as follows: Let H coact weakly on C to the left, that is, there is a k-linear map $\rho : C \to H \otimes C$, $c \mapsto \sum c_{(-1)} \otimes c_{(0)}$, such that the following conditions hold: for all $c \in C$,

(1) $\sum c_{(-1)} \otimes c_{(0)1} \otimes c_{(0)2} = \sum c_{1(-1)}c_{2(-1)} \otimes c_{1(0)} \otimes c_{2(0)}$,
(2) $\sum \varepsilon(c_{(0)})c_{(-1)} = \varepsilon(c)1_H$,
(3) $\sum \varepsilon(c_{(-1)})c_{(0)} = c$.

Let $\alpha : C \to H \otimes H$ be a linear map, $\alpha(c) = \sum \alpha_1(c) \otimes \alpha_2(c)$. Let $C \rtimes_\alpha H$ be the (in general noncoassociative) coalgebra (in general without a counit) whose underlying vector space is $C \otimes H$ and whose comultiplication is given by

$$\Delta_\alpha(c \rtimes h) = \sum \big(c_1 \rtimes c_{2(-1)}\alpha_1(c_3)h_1\big) \otimes \big(c_{2(0)} \rtimes \alpha_2(c_3)h_2\big)$$

and $\varepsilon_\alpha : C \rtimes_\alpha H \to k$ given by $\varepsilon_\alpha(c \rtimes h) = \varepsilon_C(c)\varepsilon_H(h)$. If $C \rtimes_\alpha H$ is a coassociative with $\varepsilon_\alpha = \varepsilon_C \otimes \varepsilon_H$ as counit element under the structures Δ_α and ε_α as above, we say $C \rtimes_\alpha H$ a crossed coproduct and σ the dual cocycle.

LEMMA 1.1. *$C \rtimes_\alpha H$ is a crossed coproduct if and only if the following three conditions hold:*

(*normal cocycle condition*) :

$$\sum \varepsilon\big(\alpha_1(c)\big)\alpha_2(c) = \sum \varepsilon\big(\alpha_2(c)\big)\alpha_1(c) = \varepsilon(c)1_H.$$

(*cocycle condition*) :

$$\begin{aligned}&\sum c_{1(-1)}\alpha_1(c_2) \otimes \alpha_1(c_{1(0)})\alpha_2(c_2)_1 \otimes \alpha_2(c_{1(0)})\alpha_2(c_2)_2\\ =&\sum \alpha_1(c_1)\alpha_1(c_2)_1 \otimes \alpha_2(c_1)\alpha_1(c_2)_2 \otimes \alpha_2(c_2).\end{aligned}$$

(*twisted comodule condition*) :

$$\begin{aligned}&\sum c_{1(-1)}\alpha_1(c_2) \otimes c_{1(0)(-1)}\alpha_2(c_2) \otimes c_{1(0)(0)}\\ =&\sum \alpha_1(c_1)c_{2(-1)1} \otimes \alpha_2(c_1)c_{2(-1)2} \otimes c_{2(0)}\end{aligned}$$

for all $h \in H$, $c \in C$.

Proof. See [5]. □

In this paper, suppose that a Hopf algebra H acts on an algebra B with the k-bilinear map $H\otimes B \to B$, $h\otimes b \mapsto h\cdot b$ and H coacts weakly on a coalgebra B with the k-linear map $\rho : B \to H \otimes B$, $b \mapsto \sum b_{(-1)} \otimes b_{(0)}$ for all $b \in B$, $h \in H$. Assume that $\alpha : B \to H \otimes H$ a k-linear map defined by $\alpha(b) = \sum \alpha_1(b) \otimes \alpha_2(b)$ which is convolution invertible with inverse denoted by $\alpha^{-1}(b) = \sum \alpha_1^{-1}(b) \otimes \alpha_2^{-1}(b)$ for all $b \in B$.

2. The bialgebra structure on $B \bowtie^{\alpha} H$

In this section we derive necessary and sufficient conditions for $B\otimes H$ to be a bialgebra with the algebra structure of $B\#H$ and the coalgebra structure of $B \rtimes_{\alpha} H$. If $\left(B \otimes H, m_{B\#H}, \mu_{B\#H}, \Delta_{B\rtimes_{\alpha}H}, \varepsilon_{B\rtimes_{\alpha}H}\right)$ is bialgebra, we denote this bialgebra by $B \bowtie^{\alpha} H$. Here we have written $b \bowtie^{\alpha} h$ (or more informally by $b \bowtie h$) for the tensor $b \otimes h$ for all $b \in B$, $h \in H$.

Putting $\varepsilon_{B\bowtie^{\alpha}H} = \varepsilon_{B\rtimes_{\alpha}H} = \epsilon$ and $\Delta_{B\bowtie^{\alpha}H} = \Delta_{B\rtimes_{\alpha}H} = \mathbf{\Delta}$, then we will first determine ϵ and $\mathbf{\Delta}$ are algebra maps.

LEMMA 2.1. *ϵ is an algebra map if and only if ε_B is an algebra map and the equation $\varepsilon_B(h\cdot b) = \varepsilon_H(h)\varepsilon_B(b)$ holds for all $b \in B$, $h \in H$.*

Proof. Let $a, b \in B$ and $h, l \in H$. We compute

$$\begin{aligned}\epsilon\big((a\bowtie h)(b\bowtie l)\big) &= \sum \varepsilon_B(a(h_1\cdot b))\varepsilon_H(h_3)\varepsilon_H(l_2)\\ &= \sum \varepsilon_B(a)\varepsilon_B(h_1\cdot b)\varepsilon_H(h_2)\varepsilon_H(l)\\ &= \sum \varepsilon_B(a)\varepsilon_H(h)\varepsilon_B(b)\varepsilon_H(l)\\ &= \epsilon(a\bowtie h)\epsilon(b\bowtie l)\end{aligned}$$

the first equality using that ε_H is an algebra map, the second equality using that ε_B is an algebra map and the third equality using that $\varepsilon_B\sigma(h,l) = \varepsilon_{H\otimes H}(h,l)$ and the fourth equality using that $\varepsilon_B(h\cdot b) = \varepsilon_H(h)\varepsilon_B(b)$. □

Now, we check that $\mathbf{\Delta}$ is an algebra homomorphism.

LEMMA 2.2. $\mathbf{\Delta}(1_B \bowtie 1_H) = (1_B \bowtie 1_H) \otimes (1_B \bowtie 1_H)$ *if and only if* $\rho(1_B) = 1_H \otimes 1_B$, $\Delta_B(1_B) = 1_B \otimes 1_B$ *and* $\alpha(1_B) = 1_H \otimes 1_H$.

Proof. It is easy check from similar calculations to those of [11, eq. 2.3]. This completes the proof of the lemma. □

Now, we want find the conditions such that $\mathbf{\Delta}$ is multiplicative on $B \bowtie^{\alpha} H$.

First, we compute

$$
\begin{aligned}
&\mathbf{\Delta}\big((a \bowtie h)(b \bowtie l)\big)\\
=&\mathbf{\Delta}\big(\sum a(h_1 \cdot b) \bowtie h_2 l\big)\\
=&\sum \big(a(h_1 \cdot b)\big)_1 \bowtie \big(a(h_1 \cdot b)\big)_{2(-1)} \alpha_1\Big(\big(a(h_1 \cdot b)\big)_3\Big) h_2 l_1\\
&\quad \otimes \big(a(h_1 \cdot b)\big)_{2(0)} \bowtie \alpha_2\Big(\big(a(h_1 \cdot b)\big)_3\Big) h_3 l_2.
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbf{\Delta}(a \bowtie h)\mathbf{\Delta}(b \bowtie l)\\
=&\sum \big(a_1 \bowtie a_{2(-1)} \alpha_1(a_3) h_1\big)\big(b_1 \bowtie b_{2(-1)} \alpha_1(b_3) l_1\big)\\
&\quad \otimes \big(a_{2(0)} \bowtie \alpha_2(a_3) h_2\big)\big(b_{2(0)} \bowtie \alpha_2(b_3) l_2\big)\\
=&\sum a_1\Big(\big(a_{2(-1)1} \alpha_1(a_3)_1 h_1\big) \cdot b_1\Big)\\
&\quad \bowtie \big(a_{2(-1)2} \alpha_1(a_3)_2 h_2\big)\big(b_{2(-1)} \alpha_1(b_3) l_1\big)\\
&\quad \otimes a_{2(0)}\Big(\big(\alpha_2(a_3)_1 h_3\big) \cdot b_{2(0)}\Big) \bowtie \big(\alpha_2(a_3)_2 h_4\big)\big(\alpha_2(b_3) l_2\big).
\end{aligned}
$$

So we have the deduced equation as following:

$$
\begin{aligned}
&\sum \big(a(h_1 \cdot b)\big)_1 \bowtie \big(a(h_1 \cdot b)\big)_{2(-1)} \alpha_1\Big(\big(a(h_1 \cdot b)\big)_3\Big) h_2 l_1\\
&\quad \otimes \big(a(h_1 \cdot b)\big)_{2(0)} \bowtie \alpha_2\Big(\big(a(h_1 \cdot b)\big)_3\Big) h_3 l_2.\\
=&\sum a_1\Big(\big(a_{2(-1)1} \alpha_1(a_3)_1 h_1\big) \cdot b_1\Big)\\
&\quad \bowtie \big(a_{2(-1)2} \alpha_1(a_3)_2 h_2\big)\big(b_{2(-1)} \alpha_1(b_3) l_1\big)\\
&\quad \otimes a_{2(0)}\Big(\big(\alpha_2(a_3)_1 h_3\big) \cdot b_{2(0)}\Big) \bowtie \big(\alpha_2(a_3)_2 h_4\big)\big(\alpha_2(b_3) l_2\big)
\end{aligned}
\tag{1}
$$

for all $a, b \in B$ and $h, l \in H$.

To holds the above equation (1), it is enough to show [7, 11] that for all $a, b \in B$ and $h, l \in H$,

1. $\mathbf{\Delta}\big((a \bowtie 1)(b \bowtie 1)\big) = \mathbf{\Delta}(a \bowtie 1)\mathbf{\Delta}(b \bowtie 1)$,
2. $\mathbf{\Delta}\big((a \bowtie 1)(1 \bowtie l)\big) = \mathbf{\Delta}(a \bowtie 1)\mathbf{\Delta}(1 \bowtie l)$,
3. $\mathbf{\Delta}\big((1 \bowtie h)(b \bowtie 1)\big) = \mathbf{\Delta}(1 \bowtie h)\mathbf{\Delta}(b \bowtie 1)$,
4. $\mathbf{\Delta}\big((1 \bowtie h)(1 \bowtie l)\big) = \mathbf{\Delta}(1 \bowtie h)\mathbf{\Delta}(1 \bowtie l)$.

Since the case (2) is automatically holds, we will consider for the othercases. At first, we will find the sufficient conditions for the othercases.

LEMMA 2.3. *Suppose that the following conditions hold:*

(1) $\rho(ab) = \rho(a)\rho(b)$,

(2) $\sum(h \cdot a)_{(-1)} \otimes (h \cdot a)_{(0)} = \sum h_1 a_{(-1)} \otimes h_2 \cdot a_{(0)}$,

(3) $\sum a_{(-1)} \otimes a_{(0)(-1)} \otimes a_{(0)(0)} = \sum a_{(-1)1} \otimes a_{(-1)2} \otimes a_{(0)}$,

(4) $\sum(ab)_1 \otimes (ab)_2\alpha_1\big((ab)_3\big) \otimes \alpha_2\big((ab)_3\big)$
$= \sum a_1\big(a_{2(-1)}\alpha_1(a_3)_1 \cdot b_1\big) \otimes a_{2(0)}\big(\alpha_1(a_3)_2 \cdot b_2\big)\alpha_1(b_3) \otimes \alpha_2(a_3)\alpha_2(b_3)$.

Then the equation $\mathbf{\Delta}\big((a \bowtie 1)(b \bowtie 1)\big) = \mathbf{\Delta}(a \bowtie 1)\mathbf{\Delta}(b \bowtie 1)$ *holds for all* $a, b \in B$.

Proof. We compute for all $a, b \in B$,

$$\begin{aligned}
&\mathbf{\Delta}\big((a \bowtie 1)(b \bowtie 1)\big)\\
=&\sum (ab)_1 \bowtie (ab)_{2(-1)}\alpha_1\big((ab)_3\big)\\
&\quad \otimes (ab)_{2(0)} \bowtie \alpha_2\big((ab)_3\big)\\
=&\sum a_1\big(a_{2(-1)}\alpha_1(a_3)_1 \cdot b_1\big) \bowtie a_{2(0)(-1)}\alpha_1(a_3)_2 b_{2(-1)}\alpha_1(b_3)\\
&\quad \otimes a_{2(0)(0)}\big(\alpha_2(a_3)_1 \cdot b_{2(0)}\big) \bowtie \alpha_2(a_3)_2\alpha_2(b_3)\\
=&\sum a_1\big(a_{2(-1)1}\alpha_1(a_3)_1 \cdot b_1\big) \bowtie a_{2(-1)2}\alpha_1(a_3)_2 b_{2(-1)}\alpha_1(b_3)\\
&\quad \otimes a_{2(0)}\big(\alpha_2(a_3)_1 \cdot b_{2(0)}\big) \bowtie \alpha_2(a_3)_2\alpha_2(b_3)\\
=&\mathbf{\Delta}(a \bowtie 1)\mathbf{\Delta}(b \bowtie 1)
\end{aligned}$$

the second equality using that the conditions (1), (2), and (4), the third equality using that the condition (3). This completes the proof of the lemma. □

LEMMA 2.4. *Suppose that the following conditions hold:*

(1) $\alpha(1) = 1 \otimes 1$,

(2) $\sum(h \cdot a)_1 \otimes (h \cdot a)_2 = \sum(h_1 \cdot a_1) \otimes (h_2 \cdot a_2)$,

(3) $\sum(h_1 \cdot a_1)_{(-1)}\alpha_1(h_2 \cdot a_2)h_3 \otimes (h_1 \cdot a_1)_{(0)} \otimes \alpha_2(h_2 \cdot a_2)$
$= \sum h_1 a_{1(-1)}\alpha_1(a_2) \otimes h_2 \cdot a_{1(0)} \otimes \alpha_2(a_2)$.

Then the equation $\mathbf{\Delta}\big((1 \bowtie h)(b \bowtie 1)\big) = \mathbf{\Delta}(1 \bowtie h)\mathbf{\Delta}(b \bowtie 1)$ *holds for all* $a \in B$, $h \in H$.

Proof. We compute for all $b \in B$, $h \in H$,

$$\begin{aligned}
&\mathbf{\Delta}\big((1 \bowtie h)(b \bowtie 1)\big) \\
=&\sum (h_1 \cdot b)_1 \bowtie (h_1 \cdot b)_{2(-1)}\alpha_1\big((h_1 \cdot b)_3\big)h_2 \\
&\quad \otimes (h_1 \cdot b)_{2(0)} \bowtie \alpha_2\big((h_1 \cdot b)_3\big)h_3 \\
=&\sum (h_1 \cdot b_1) \bowtie (h_2 \cdot b_2)_{(-1)}\alpha_1(h_3 \cdot b_3)h_4 \\
&\quad \otimes (h_2 \cdot b_2)_{(0)} \bowtie \alpha_2(h_3 \cdot b_3)h_5 \\
=&\sum (h_1 \cdot b_1) \bowtie h_2 b_{2(-1)}\alpha_1(b_3) \otimes h_3 \cdot b_{2(0)} \bowtie \alpha_2(b_3)h_4 \\
=&\mathbf{\Delta}(1 \bowtie h)\mathbf{\Delta}(b \bowtie 1)
\end{aligned}$$

the second equality using that the condition (2), the third equality using that the condition (1) and (3). This completes the proof of the lemma. □

LEMMA 2.5. *Suppose that the identity $\alpha_(1_B) = 1_H \otimes 1_H$holds. Then the equation $\mathbf{\Delta}\big((1 \bowtie h)(1 \bowtie l)\big) = \mathbf{\Delta}(1 \bowtie h)\mathbf{\Delta}(1 \bowtie l)$ holds for all $h, l \in H$.*

Proof. We compute for all $h, l \in H$,

$$\begin{aligned}
\mathbf{\Delta}\big((1 \bowtie h)(1 \bowtie l)\big) &= \sum 1 \bowtie h_1 l_1 \otimes 1 \bowtie h_2 l_2 \\
&= \mathbf{\Delta}(1 \bowtie h)\mathbf{\Delta}(1 \bowtie l)
\end{aligned}$$

the equality using that the above condition. This completes the proof of the lemma. □

Now, we have directly needed following lemmas for the above cases to be the necessary conditions;

LEMMA 2.6. *Let $B\#H$ be a smash product and $B \rtimes_\alpha H$ a crossed coproduct. If the identity $\varepsilon_B(h \cdot a) = \varepsilon_H(h)\varepsilon_B(a)$ holds for all $a \in B$, $h \in H$. Then the following are equivalent: for all $a \in B$ and $h \in H$,*

(1) (i) $\sum (h \cdot a)_1 \otimes (h \cdot a)_2 = \sum (h_1 \cdot a_1) \otimes (h_2 \cdot a_2)$,
(ii) $\sum (h_1 \cdot a_1)_{(-1)}\alpha_1(h_2 \cdot a_2)h_3 \otimes (h_1 \cdot a_1)_{(0)} \otimes \alpha_2(h_2 \cdot a_2)$
$= \sum h_1 a_{1(-1)}\alpha_1(a_2) \otimes h_2 \cdot a_{1(0)} \otimes \alpha_2(a_2)$.

(2) $\sum (h_1 \cdot b)_1 \bowtie (h_1 \cdot b)_{2(-1)}\alpha_1\big((h_1 \cdot b)_3\big)h_2$
$\otimes (h_1 \cdot b)_{2(0)} \bowtie \alpha_2\big((h_1 \cdot b)_3\big)h_3$
$= \sum (h_1 \cdot b_1) \bowtie h_2 b_{2(-1)}\alpha_1(b_3) \otimes h_3 \cdot b_{2(0)} \bowtie \alpha_2(b_3)h_4$.

Proof. We note that the condition (2) is equivalent to the case 3. □

Consequently, we now have necessary and sufficient conditions for $B \otimes H$ to be a bialgebra with the structures indicated above as the following statements:

THEOREM 2.7. *Let H be a bialgebra over a field k. Suppose that B is an algebra with action via $\tau : H \otimes B \to B$, $a \otimes h \mapsto a \cdot h$ and a coalgebra with weak coaction via $\rho : B \to H \otimes B$, $a \mapsto \sum a_{(-1)} \otimes a_{(0)}$ for all $a \in B$, $h \in H$. Assume that $B \# H$ is a smash product and $B \rtimes_\alpha H$ a crossed coproduct with the dual cocycle α. Then the following conditions are equivalent:*

(1) $B \bowtie^\alpha H = \big(B \otimes H, m_{B\#H}, \mu_{B\#H}, \Delta_{B \rtimes_\alpha H}, \varepsilon_{B \rtimes_\alpha H}\big)$ *is a bialgebra.*
(2) *The following identities hold; for all $a, b \in B$ and $h, l \in H$,*
  (B1) ε_B *and* ρ *are algebra maps,*
  (B2) $\varepsilon_B(h \cdot a) = \varepsilon_H(h)\varepsilon_B(a)$,
  (B3) $\sum (h \cdot a)_1 \otimes (h \cdot a)_2 = \sum (h_1 \cdot a_1) \otimes (h_2 \cdot a_2)$,
  (B4) $\sum a_{(-1)} \otimes a_{(0)(-1)} \otimes a_{(0)(0)} = \sum a_{(-1)1} \otimes a_{(-1)2} \otimes a_{(0)}$,
  (B5) $\alpha(1) = 1 \otimes 1$,
  (B6) $\Delta_B(1_B) = 1_B \otimes 1_B$,
  (B7) $$\sum (ab)_1 \otimes (ab)_2 \alpha_1\big((ab)_3\big) \otimes \alpha_2\big((ab)_3\big) = \sum a_1\big(a_{2(-1)}\alpha_1(a_3)_1 \cdot b_1\big) \otimes a_{2(0)}\big(\alpha_1(a_3)_2 \cdot b_2\big)\alpha_1(b_3) \otimes \alpha_2(a_3)\alpha_2(b_3),$$
  (B8) $$\sum (h_1 \cdot a_1)_{(-1)}\alpha_1(h_2 \cdot a_2)h_3 \otimes (h_1 \cdot a_1)_{(0)} \otimes \alpha_2(h_2 \cdot a_2) = \sum h_1 a_{1(-1)}\alpha_1(a_2) \otimes h_2 \cdot a_{1(0)} \otimes \alpha_2(a_2).$$
(3) *B is an algebra in ${}^H\mathcal{M}$ and a coalgebra in ${}_H\mathcal{M}$, ε_B is an algebra map, and the conditions* (B5) – (B8) *of* (2) *hold.*

Proof. (2) $\Longleftrightarrow$ (3) is clear. (1) $\Rightarrow$ (2) follows from the lemmas 2.1 – 2.5. (2) $\Rightarrow$ (1) follows from the lemma 2.6 and similar calculations to those of [11]. This completes the proof of the theorem. □

REMARK 2.8. If α is trivial, that is, $\alpha(a) = \varepsilon_B(a)1_H$ for all $a \in B$, then this theorem is that of [11].

Now, we introduce some examples of this new bialgebra $B \bowtie^\alpha H$. Let H be any finite-dimensional Hopf algebra over k with antipode $\mathcal{S}$ and let $\mathcal{S}^*$ denote the antipode of H^*. We have the identification of $H \otimes H^*$ with $(H^* \otimes H)^*$. The composition inverses of $\mathcal{S}$ and $\mathcal{S}^*$ are denoted by $\mathcal{S}^{-1}$ and $\mathcal{S}^{*-1}$, respectively.

We need comodule actions correspondence to module actions as the following [7, 8, 10]: for all $p, q, f, g \in H^*$ and $h, l \in H$,

$$\leftharpoonup: H \otimes H^* \to H, \quad h \leftharpoonup p = \sum < p, h_1 > h_2,$$
$$\psi^r : H^* \otimes H \to H^*, \quad \psi^r(p \otimes h) = \sum \mathcal{S}^{-1}h_1 \rightharpoonup p \leftharpoonup h_2,$$
$$\psi^l : H^* \otimes H \to H, \quad \psi^l(p \otimes h) = \sum p_1 \rightharpoonup h \leftharpoonup \mathcal{S}^{-1}p_2,$$

and

$$\rho : H \to H \otimes H^*, \quad < \rho(h), f \otimes l > = < f, lh >,$$
$$\rho^r : H \to H \otimes H^*, \quad < \rho^r(h), f \otimes l > = \sum < f, h_2\mathcal{S}^{-1}(h_1)l >,$$
$$\rho^l : H^* \to H \otimes H^*, \quad < \rho^l(f), h \otimes g > = \sum < h, \mathcal{S}^{*-1}(g_2) f g_1 >,$$

respectively.

EXAMPLE 2.9. We have the following results under the above notations: Assume that H is cocommutative. Then both $H^{*cop} \bowtie^{\rho^l \psi^r} H$ and $H \bowtie^{\rho^r \psi^l} H^{*cop}$ are bialgebras. In particular there is a bialgebra isomorphism

$$\left(H^{*cop} \bowtie^{\rho^l \psi^r} H\right)^* \to H \bowtie^{\rho^r \psi^l} H^{*cop}.$$

3. The Hopf algebra $B \bowtie^\alpha H$

In this section we derive the necessary conditions for the new bialgebra $B \bowtie^\alpha H$ to be a Hopf algebra. Firstly, we introduce the Hopf algebra H with respect to the dual cocycle $\alpha : B \to H \otimes H$.

DEFINITION 3.1. Let H be a bialgebra over the field k. Suppose that B is both an algebra and a coalgebra over k, $\alpha : B \to H \otimes H$ and $\mathcal{S} : H \to H$ are linear maps. Then $\mathcal{S}$ is called a *α-antipode of H* if
(1) $m_H(\mathcal{S} \otimes 1_H)m_{H\otimes H}(\alpha \otimes \Delta_H) = (\varepsilon_B \otimes \varepsilon_H)(1_B \otimes 1_H)$,
(2) $m_H(1_H \otimes \mathcal{S})m_{H\otimes H}(\alpha \otimes \Delta_H) = (\varepsilon_B \otimes \varepsilon_H)(1_B \otimes 1_H)$.
In this case, we say that H is a *α-Hopf algebra.*

In summation notation, (1) and (2) say for all $a \in B$ and $h \in H$,

$$\begin{aligned}(1) \qquad & m_H(\mathcal{S} \otimes 1_H)m_{H\otimes H}(\alpha \otimes \Delta_H)(a \otimes h) \\ &= \sum \mathcal{S}\big(\alpha_1(a)h_1\big) \otimes \alpha_2(a)h_2 \\ &= \varepsilon_B(a)\varepsilon_H(h)(1_B \otimes 1_H)\end{aligned}$$

and

$$(2) \qquad m_H(1_H \otimes \mathcal{S})m_{H\otimes H}(\alpha \otimes \Delta_H)(a \otimes h)$$
$$=\sum \alpha_1(a)h_1 \otimes \mathcal{S}\big(\alpha_2(a)h_2\big)$$
$$=\varepsilon_B(a)\varepsilon_H(h)(1_B \otimes 1_H),$$

respectively.

EXAMPLE 3.2. Let H be a Hopf algebra with an antipode $\mathcal{S}$. Suppose that $\alpha : k \to H \otimes H$ is a trivial linear map. Then we can regard $\mathcal{S}$ as a α-antipode of H.

PROPOSITION 3.3. *Let $B \bowtie^{\alpha} H$ be a bialgebra. Suppose that H is a α-Hopf algebra with α-antipode $\mathcal{S}_H$ and $\mathcal{S}_B$ is a convolution invertible element of 1_B in End(B). Then $B \bowtie^{\alpha} H$ is a Hopf algebra with antipode $\mathcal{S}$ defined by for all $a \in B$ and $h \in H$,*

$$\mathcal{S}(a \bowtie h) = \sum \Big(1 \bowtie \mathcal{S}_H\big(a_{(-1)}h\big)\Big)\big(\mathcal{S}_B(a_{(0)}) \bowtie 1\big).$$

Proof. We compute for all $a \in B$ and $h \in H$,

$$(1_{B\bowtie^{\alpha}H} * \mathcal{S})(a \bowtie h)$$
$$=m_{B\#H}(1 \otimes \mathcal{S})\Delta_{B\rtimes_{\alpha}H}(a \bowtie h)$$
$$=\sum m_{B\#H}\Big(\big(a_1 \bowtie a_{2(-1)}\alpha_1(a_3)h_1\big) \otimes \mathcal{S}\big(a_{2(0)} \bowtie \alpha_2(a_3)h_2\big)\Big)$$
$$=\sum \Big(\big(a_1 \bowtie a_{2(-1)}\alpha_1(a_3)h_1\big)\big(1 \bowtie \mathcal{S}_H(a_{2(0)(-1)}\alpha_2(a_3)h_2)\big)\Big)$$
$$\big(\mathcal{S}_B(a_{2(0)(0)}) \bowtie 1\big)$$
$$=\sum \Big[a_1 \bowtie \big(a_{2(-1)}\alpha_1(a_3)h_1\big)\mathcal{S}_H\big(a_{2(0)(-1)}\alpha_2(a_3)h_2\big)\Big]$$
$$\big(\mathcal{S}_B(a_{2(0)}) \bowtie 1\big)$$
$$=\sum (a_1 \bowtie 1)\big(\mathcal{S}_B(a_2) \bowtie 1\big)\varepsilon_H(h)$$
$$=\sum \big(a_1\mathcal{S}_B(a_2) \bowtie 1\big)\varepsilon_H(h)$$
$$=\varepsilon_B(a)\varepsilon_H(h)(1_B \bowtie 1_H)$$

the third equality using that it is an associative algebra, the fourth equality using that the weak coaction, the fifth equality using that $\mathcal{S}_H$ is a α-antipode of H and the seventh equality using that $\mathcal{S}_B$ is the convolution invertible of 1_B. The other case is analogue to this case. Therefore, we have $\mathcal{S}$ is an antipode for $B \bowtie^{\alpha} H$. This completes the proof of the proposition. □

PROPOSITION 3.4. *Suppose that $B \bowtie^{\alpha} H$ is a Hopf algebra with antipode $\mathcal{S}$. Then H is a α-Hopf algebra with the α-antipode S and the identity 1_B has an invertible in the convolution algebra $Hom_k(B, B)$.*

Proof. It is straightforward from similar calculations to those of [11, Proposition 2]. This completes the proof of the proposition. □

REMARK 3.5. Assume that B and H are finite dimensional algebra and Hopf algebra, respectively and that $B \bowtie^{\alpha} H$ is a Hopf algebra with antipode $\mathcal{S}$. Then the linear dual $(B \bowtie^{\alpha} H)^* \cong B^* \bowtie_{\sigma} H^*$ is a Hopf algebra which is the crossed product with the cocycle

$$\sigma : H^* \otimes H^* \cong \left(H \otimes H\right)^* \xrightarrow{\alpha^*} B^*$$

as an algebra and the smash coproduct as a coalgebra with the antipode

$$\mathcal{S}' : B^* \otimes H^* \cong \left(B \otimes H\right)^* \xrightarrow{\mathcal{S}^*} \left(B \otimes H\right)^* \cong B^* \otimes H^*.$$

That is, it is exactly the structure of [7].

4. The characterizing of the Hopf algebra $B \bowtie^{\alpha} H$

In this section we show that the weak coaction admissible mapping system characterize the new Hopf algebra $B \bowtie^{\alpha} H$ which was constructed in Theorem 2.7.

DEFINITION 4.1. Let H be a bialgebra and C a coalgebra. Suppose that $\alpha : C \to H \otimes H$ is a linear map. Then (C, ρ_l, α) is called a *left* (resp. *right*) *(H, α)-comodule* if there is a map $\rho_l : C \to H \otimes C$ (resp. $\rho_r : C \to C \otimes H$) such that the following conditions hold:

(1) $(1 \otimes \rho_l)\rho_l = (\omega \otimes 1)(1 \otimes \Delta_H \otimes 1)(1 \otimes \rho_l)\Delta_C$

(resp. $(\rho_r \otimes 1)\rho_r = (1 \otimes \omega)(1^2 \otimes \Delta_H)(1 \otimes \rho_r)\Delta_C$),

(2) $(\varepsilon_H \otimes 1)\rho_l = 1$ (resp. $(1 \otimes \varepsilon_H)\rho_r = 1$)

where $\omega : C \otimes H \otimes H \to H \otimes H$ is the linear map via $\omega(c \otimes h \otimes l) = \sum \alpha_1(c)h \otimes \alpha_2(c)l$ for all $c \in C$ and $h, l \in H$.

EXAMPLE 4.2. Let H be a bialgebra. If $\alpha : H \to H \otimes H$ is a trivial. Then (H, Δ_H, α) is a left (H, α)-comodule. This is the same as the comodule structure map on H obtained by the comultiplication of H as comodule structure.

EXAMPLE 4.3. Let C be a coalgebra. Assume that C is a left H-comodule with the structure map $\rho : C \to H \otimes C$ and $\alpha : C \to H \otimes H$ is trivial. Then (C, ρ, α) is a left (H, α)-comodule.

DEFINITIONS 4.4.

(1) Let C and D be coalgebras. A linear ma $f : C \to D$ is called a *weak coalgebra map* if $\varepsilon_D f = \varepsilon_C$.
(2) Let C be a coalgebra and $\rho_l : C \to H \otimes C$ and $\rho_r : C \to C \otimes H$ be a left coaction and a right coaction on C, respectively. Then B is called a *weak H-bicomodule* if the following identities hold:
   (i) $(\varepsilon_H \otimes 1)\rho_l = 1$ and $(1 \otimes \varepsilon_H)\rho_r = 1$,
   (ii) $(1 \otimes \rho_r)\rho_l = (\rho_l \otimes 1)\rho_r$.

DEFINITION 4.5. Let $B \bowtie^\alpha H$ be a bialgebra and assume that A is a bialgebra over k. Then $B \leftrightarrows_j^p A \rightleftarrows_i^\pi H$ is called a *weak coaction admissible mapping system* if the following conditions hold:

(a) $p \cdot j = 1_B$ and $\pi \cdot i = 1_H$,
(b) π is a bialgebra map, i is an algebra map and a weak coalgebra map, p is a coalgebra map and j is an algebra map,
(c) p is a H-bimodule map (A is given the weak H-bimodule structure via pullback along i and B is given the trivial right H-module structure),
(d) $j(B)$ is a sub-H-bicomodule of A and $p|_{j(B)}$ is a weak bicomodule map (A is given the weak H-bicomodule structure via pushout along π and B is given the trivial right H-comodule structure) and there is a map $\widetilde{\alpha} : B \bowtie^\alpha H \to H \otimes H$ such that A is a left (right) $(H, \widetilde{\alpha})$-comodule,
(e) $(j \cdot p) * (i \cdot \pi) = 1$.

EXAMPLE 4.6. Assume that the dual cocycle $\alpha : B \to H \otimes H$ are trivial, that is, $\alpha(a) = \varepsilon_B(a)1_H$ for all $a \in B$. Then this concept is exactly that of [11].

THEOREM 4.7. *Suppose that $B \bowtie^\alpha H$ is a bialgebra. Then $B \leftrightarrows_{j_B}^{p_B} B \bowtie^\alpha H \rightleftarrows_{i_H}^{\pi_H} H$ is a weak coaction admissible mapping system.*

To complete the proof of this theorem, it is enough to show that the following three lemmas hold via some natural maps as the follows; for all $a \in B$ and $h, l \in H$,

$$p_B : B \bowtie^\alpha H \to B, \quad a \bowtie h \mapsto a\varepsilon(h),$$
$$\pi_H : B \bowtie^\alpha H \to H, \quad a \bowtie h \mapsto \varepsilon_B(a)h,$$
$$j_B : B \to B \bowtie^\alpha H, \quad a \mapsto a \bowtie 1,$$

and

$$i_H : H \to B \bowtie^\alpha H, \quad h \mapsto 1 \bowtie h.$$

Also we define a right and a left actions of H on $B \bowtie^{\alpha} H$ is given by

$$\psi_r : B \bowtie^{\alpha} H \otimes H \to B \bowtie^{\alpha} H, \quad a \bowtie h \otimes l \mapsto \sum a \bowtie hl$$

and

$$\psi_l : H \otimes B \bowtie^{\alpha} H \to B \bowtie^{\alpha} H, \quad h \otimes a \bowtie l \mapsto \sum (h_1 \cdot a) \bowtie h_2 l,$$

respectively.

Similarly we define a right and a left coactions of H on $B \bowtie^{\alpha} H$ by

$$\rho_r : B \bowtie^{\alpha} H \to B \bowtie^{\alpha} H \otimes H, \quad a \bowtie h \mapsto \sum a_1 \bowtie \alpha_1(a_2)h_1 \otimes \alpha_2(a_2)h_2$$

and

$$\rho_l : B \bowtie^{\alpha} H \to H \otimes B \bowtie^{\alpha} H,$$
$$a \bowtie h \mapsto \sum a_{1(-1)}\alpha_1(a_2)h_1 \otimes a_{1(0)} \bowtie \alpha_2(a_2)h_2,$$

respectively.

LEMMA 4.8. *Suppose that $B \bowtie^{\alpha} H$ is a bialgebra. Then $(B \bowtie^{\alpha} H, \rho_r, \widetilde{\alpha})$ is a right $(H, \widetilde{\alpha})$-comodule where $\widetilde{\alpha} : B \bowtie^{\alpha} H \to H \otimes H$ via $\widetilde{\alpha}(a \bowtie h) = \sum \alpha_1(a) \otimes \alpha_2(a)\varepsilon(h)$ for all $a \in B$.*

Proof. We compute for all $a \in B$ and $h \in H$,

$$\begin{aligned}
&(\rho_r \otimes 1)\rho_r(a \bowtie h)\\
=&\sum a_1 \bowtie \alpha_1(a_2)\alpha_1(a_3)_1 h_1 \otimes \alpha_2(a_2)\alpha_1(a_3)_2 h_2\\
&\quad \otimes \alpha_2(a_3)h_3\\
=&\sum a_1 \bowtie a_{2(-1)}\alpha_1(a_3)h_1 \otimes \alpha_1(a_{2(0)})\alpha_2(a_3)_1 h_2\\
&\quad \otimes \alpha_2(a_{2(0)})\alpha_2(a_3)_2 h_3\\
=&(1_{B\bowtie^{\alpha}H} \otimes \omega)(1^2_{B\bowtie^{\alpha}H} \otimes \Delta_H)(1_{B\bowtie^{\alpha}H} \otimes \rho_r)\Delta_{B\bowtie^{\alpha}H}(a \bowtie h)
\end{aligned}$$

the second equality using that the cocycle condition holds and the third equality using that the normal cocycle condition holds. This completes the proof of the lemma. □

LEMMA 4.9. *Suppose that $B \bowtie^{\alpha} H$ is a bialgebra. Then $(B \bowtie^{\alpha} H, \rho_l, \widetilde{\alpha})$ is a left $(H, \widetilde{\alpha})$-comodule where $\widetilde{\alpha} : B \bowtie^{\alpha} H \to H \otimes H$ via $\widetilde{\alpha}(a \bowtie h) = \sum \alpha_1(a) \otimes \alpha_2(a)\varepsilon(h)$ for all $a \in B$.*

Proof. We compute for all $a \in B$ and $h \in H$,

$$
\begin{aligned}
&(1 \otimes \rho_l)\rho_l(a \bowtie h) \\
=&\sum a_{1(-1)}\alpha_1(a_2)h_1 \otimes a_{1(0)1(-1)}\alpha_1(a_{1(0)2})\alpha_2(a_2)_1h_2 \\
&\quad \otimes a_{1(0)1(0)} \bowtie \alpha_2(a_{1(0)2})\alpha_2(a_2)_2h_3 \\
=&\sum a_{1(-1)}a_{2(-1)}\alpha_1(a_3)h_1 \otimes a_{1(0)(-1)}\alpha_1(a_{2(0)})\alpha_2(a_3)_1h_2 \\
&\quad \otimes a_{1(0)(0)} \bowtie \alpha_2(a_{2(0)})\alpha_2(a_3)_2h_3 \\
=&\sum a_{1(-1)}\alpha_1(a_2)\alpha_1(a_3)_1 \otimes a_{1(0)(-1)}\alpha_2(a_2)\alpha_1(a_3)_2h_2 \\
&\quad \otimes a_{1(0)(0)} \bowtie \alpha_2(a_3)h_3 \\
=&(\omega \otimes 1)(1 \otimes \Delta_H \otimes 1)(1 \otimes \rho_l)\Delta_{B \bowtie^\alpha H}(a \bowtie h)
\end{aligned}
$$

the second equality using that H coact weakly on B, the third equality using that the cocycle condition holds and the fourth equality using that the normal cocycle condition holds. This completes the proof of the lemma. □

LEMMA 4.10. *Suppose that $B \bowtie^\alpha H$ is a bialgebra. Then $(B \bowtie^\alpha H, \rho_l, \rho_r)$ is a weak H-bicomodule.*

Proof. We compute for all $a \in B$ and $h \in H$,

$$
\begin{aligned}
&(1 \otimes \rho_r)\rho_l(a \bowtie h) \\
=&\sum a_{1(-1)}\alpha_1(a_2)h_1 \otimes a_{1(0)1} \\
&\quad \bowtie \alpha_1(a_{1(0)2})\alpha_2(a_2)_1h_2 \otimes \alpha_2(a_{1(0)2})\alpha_2(a_2)_2h_3 \\
=&\sum a_{1(-1)}a_{2(-1)}\alpha_1(a_3)h_1 \otimes a_{1(0)} \bowtie \alpha_1(a_{2(0)})\alpha_2(a_3)_1h_2 \\
&\quad \otimes \alpha_2(a_{2(0)})\alpha_2(a_3)_2h_3 \\
=&\sum a_{1(-1)}\alpha_1(a_2)\alpha_1(a_3)_1h_1 \otimes a_{1(0)} \bowtie \alpha_2(a_2)\alpha_1(a_3)_2h_2 \\
&\quad \otimes \alpha_2(a_3)h_3 \\
=&(\rho_l \otimes 1)\rho_r(a \bowtie h)
\end{aligned}
$$

the second equality using that H coacts weakly on B and the third equality using that the cocycle condition holds. This completes the proof of the lemma. □

THEOREM 4.11. *Let $B \leftrightarrows^p_j A \rightleftarrows^\pi_i H$ be a weak coaction admissible mapping system. Then there exists only one bialgebra isomorphism*

$f : B \bowtie^{\alpha} H \to A$ such that

$$\begin{array}{ccccc}
B & \underset{p_B}{\overset{j_B}{\rightleftarrows}} & B \bowtie^{\alpha} H & \underset{i_H}{\overset{\pi_H}{\rightleftarrows}} & H \\
\| & & \downarrow f & & \| \\
B & \underset{p}{\overset{j}{\rightleftarrows}} & A & \underset{i}{\overset{\pi}{\rightleftarrows}} & H
\end{array}$$

are commute.

Proof. We define $f : B \bowtie^{\alpha} H \to A$ by $a \bowtie h \mapsto j(a)i(h)$ and $g : A \to B \bowtie^{\alpha} H$ via $a \mapsto \sum p(a_1) \bowtie \pi(a_2)$ for all $a \in B$ and $h \in H$. Then it is easy check from similar calculations to those of [11, Theorem 2(2)] that f and g are bialgebra maps and inverse to each other. This completes the proof of the theorem. □

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