Bull. Korean Math. Soc. **38** (2001), No. 3, pp. 517–526

CONTINUOUS MULTISCALE ANALYSIS ON THE HEISENBERG GROUP

JIANXUN HE

ABSTRACT. In this paper, we study the continuous wavelet transform on the Heisenberg group $\mathbf{H}^n$, and describe the related continuous multiscale analysis. By using the wavelet packet transform we obtain a reconstruction formula on $L^2(\mathbf{H}^n)$.

1. Introduction

Wavelet analysis has many applications in pure and applied mathematics. In [3], H. G. Stark studied the continuous wavelet transform and continuous multiscale analysis on the space $L^2(\mathbf{R})$. It is a useful tool for analyzing signal. When one considers the problems of radial function space on Heisenberg group, the fundamental manifold can be regarded as the Laguerre hypergroup $\mathbf{K} = [0, +\infty) \times \mathbf{R}$. M. M. Nessibi and K. Trimèche [2] gave the generalized wavelet transform on the Laguerre hypergroup $\mathbf{K}$. By using generalized wavelet they obtained the inversion formula of the Radon transform. In this paper, we deal with the continuous multiscale analysis and wavelet packet on the Heisenberg group $\mathbf{H}^n$. The properties of the Radon transform on $\mathbf{H}^n$ were exploited by R. S. Strichartz [5]. It is believed that the wavelet transform in this paper can be applied to discuss the Radon transform and others on the Heisenberg group. In the sequel we will consider this problem.

This paper is organized as follows. In this section we summarize the main results of harmonic analysis on the Heisenberg group $\mathbf{H}^n$. We will discuss the continuous wavelet transform on $L^2(\mathbf{H}^n)$ in the second

Received April 20, 2000. Revised January 12, 2001.

2000 Mathematics Subject Classification: Primary 43A85; Secondary 44A15.

Key words and phrases: Heisenberg group, wavelet transform, continuous multiscale analysis.

The work for this paper was supported by the National Natural Science Foundation of China (No.10071039) and the Natural Science Foundation of Jiangsu Education Commission.

section. The continuous multiscale analysis will be studied in the third section. And in the last section we shall give a reconstruction formula for wavelet packet transform on $L^2(\mathbf{H}^n)$.

Let $\mathbf{H}^n$ be the $(2n+1)$-dimensional Heisenberg group $\mathbf{H}^n = \mathbf{C}^n \times \mathbf{R}$ with the multiplication law

$$(z,t)(z',t') = (z+z', t+t'+2\mathrm{Im} z\overline{z'}). \tag{1.1}$$

Let $\mathbf{SU}(n+1,1) = \mathbf{ANK}$ be the Iwasawa decomposition, where

$$\mathbf{N} = \left\{ n(z,t) = \begin{pmatrix} I_n & iz^t & -iz^t \\ i\overline{z} & 1-\frac{|z|^2-it}{2} & \frac{|z|^2-it}{2} \\ i\overline{z} & -\frac{|z|^2-it}{2} & 1+\frac{|z|^2-it}{2} \end{pmatrix} : z \in \mathbf{C}^n, t \in \mathbf{R} \right\},$$

$$\mathbf{A} = \left\{ a(\zeta) = \begin{pmatrix} I_n & 0 & 0 \\ 0 & \cosh\zeta & \sinh\zeta \\ 0 & \sinh\zeta & \cosh\zeta \end{pmatrix} : \zeta \in \mathbf{R} \right\},$$

$$\mathbf{K} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : A \in \mathbf{U}(n+1), \det A \cdot D = 1 \right\},$$

z^t denotes the transpose of z. Consider the semi-direct product $\mathbf{P}$ of $\mathbf{A}$ and $\mathbf{N}$ which is given as follows:

$$\mathbf{P} = \{(z,t,\rho) : (z,t) \in \mathbf{H}^n, \rho \in \mathbf{R}^+\}.$$

$\mathbf{P}$ can be regarded as a solvable subgroup of $\mathbf{SU}(n+1,1)$ equipped with the group law:

$$(z,t,\rho)(z',t',\rho') = (z+\sqrt{\rho}z', t+\rho t' + 2\sqrt{\rho}\mathrm{Im} z\overline{z'}, \rho\rho'). \tag{1.2}$$

$d\mu_l(z,t,\rho) = \frac{dzdtd\rho}{\rho^{n+2}}$ and $d\mu_r(z,t,\rho) = \frac{dzdtd\rho}{\rho}$ are the left and right invariant measures of $\mathbf{P}$ respectively. The square integrable unitary representation of $\mathbf{P}$ on $L^2(\mathbf{H}^n)$ is defined by

$$U(z,t,\rho)f(z',t') = \rho^{-\frac{n+1}{2}} f\Big(\frac{z'-z}{\sqrt{\rho}}, \frac{t'-t-2\mathrm{Im} z\overline{z'}}{\rho}\Big). \tag{1.3}$$

Let $\mathbf{Z}_+ = \{0,1,2\cdots\}$, $\alpha = (\alpha_1, \alpha_2, \cdots \alpha_n) \in (\mathbf{Z}_+)^n$. The Fock space $\mathcal{H}$ is the space of holomorphic functions F on $\mathbf{C}^n$ such that

$$\|F\|_{\mathcal{H}}^2 = \Big(\frac{2}{\pi}\Big)^2 \int_{\mathbf{C}^n} |F(\zeta)|^2 e^{-2|\zeta|^2} d\zeta < +\infty.$$

$\{E_\alpha(z) = \frac{(\sqrt{2}\zeta)^\alpha}{\sqrt{\alpha!}} : \alpha \in (\mathbf{Z}_+)^n\}$ is an orthonormal basis of the Hilbert space $\mathcal{H}$. For $\lambda \in \mathbf{R}$, $\lambda \neq 0$, let π_λ be the Bargmann-Fock representation of $\mathbf{H}^n$ which acts on $\mathcal{H}$ by

$$(1.4)\quad \pi_\lambda(z,t)F(\zeta) = \begin{cases} e^{-i\lambda t - \lambda|z|^2 + 2\sqrt{\lambda}\zeta\overline{z}} F(\zeta - \sqrt{\lambda}z), & \text{if } \lambda > 0, \\ e^{-i\lambda t + \lambda|z|^2 - 2\sqrt{|\lambda|}\zeta z} F(\zeta + \sqrt{|\lambda|}\overline{z}), & \text{if } \lambda < 0. \end{cases}$$

The group Fourier transform of a function $f \in L^1(\mathbf{H}^n)$ is defined by

$$(1.5)\qquad \widehat{f}(\lambda) = \int_{\mathbf{H}^n} f(z,t)\pi_\lambda(z,t)dzdt.$$

For $f, g \in L^2(\mathbf{H}^n)$, the Parseval formula is

$$(1.6)\qquad \langle f, g\rangle_{L^2(\mathbf{H}^n)} = \frac{2^{n-1}}{\pi^{n+1}} \int_{-\infty}^{+\infty} \operatorname{tr}\left(\widehat{g}(\lambda)^* \widehat{f}(\lambda)\right) |\lambda|^n d\lambda,$$

where $\widehat{g}(\lambda)^*$ denotes the adjoint of $\widehat{g}(\lambda)$. Let $\|\cdot\|_{HS}$ be the Hilbert-Schmidt norm of operators, then Plancherel formula is given by

$$(1.7)\qquad \|f\|_{L^2(\mathbf{H}^n)} = \left\{ \frac{2^{n-1}}{\pi^{n+1}} \int_{-\infty}^{+\infty} \|\widehat{f}(\lambda)\|_{HS}^2 |\lambda|^n d\lambda \right\}^{\frac{1}{2}}.$$

And the inversion of the Fourier transform is

$$(1.8)\qquad f(z,t) = \frac{2^{n-1}}{\pi^{n+1}} \int_{-\infty}^{+\infty} \operatorname{tr}\left(\pi_\lambda(z,t)^* \widehat{f}(\lambda)\right) |\lambda|^n d\lambda.$$

The further details can be looked up in the references [1] and [4].

2. Continuous wavelet transform

For $h \in L^2(\mathbf{H}^n), h \neq 0$, if it satisfies the admissibility condition:

$$(2.1)\qquad \frac{1}{\|h\|_{L^2(\mathbf{H}^n)}^2} \int_{\mathbf{P}} |\langle h, U(z,t,\rho)h\rangle_{L^2(\mathbf{H}^n)}|^2 \frac{dzdtd\rho}{\rho^{n+2}} < +\infty,$$

then we say that h is an admissible wavelet. We now discuss the admissibility condition. Since

$$\int_{\mathbf{H}^n} \langle h, U(z,t,\rho)h\rangle_{L^2(\mathbf{H}^n)} \pi_\lambda(z,t)dzdt = \rho^{\frac{n+1}{2}} \widehat{h}(\lambda)\widehat{h}(\rho\lambda)^*,$$

by the Plancherel formula, we have

$$\begin{aligned}&\int_{\mathbf{P}} |\langle h, U(z,t,\rho)h\rangle_{L^2(\mathbf{H}^n)}|^2 \frac{dzdtd\rho}{\rho^{n+2}}\\ =&\frac{2^{n-1}}{\pi^{n+1}}\int_0^{+\infty}\left(\int_{-\infty}^{+\infty}\|\widehat{h}(\lambda)\widehat{h}(\rho\lambda)^*\|_{HS}^2|\lambda|^n d\lambda\right)\frac{d\rho}{\rho}\\ =&\frac{2^{n-1}}{\pi^{n+1}}\int_0^{+\infty}\left(\int_{-\infty}^{+\infty}\mathrm{tr}(\widehat{h}(\lambda)^*\widehat{h}(\lambda)\widehat{h}(\rho\lambda)^*\widehat{h}(\rho\lambda))|\lambda|^n d\lambda\right)\frac{d\rho}{\rho}.\end{aligned}$$

We note that

$$\mathrm{tr}\left(\widehat{h}(\lambda)^*\widehat{h}(\lambda)\widehat{h}(\rho\lambda)^*\widehat{h}(\rho\lambda)\right)=\sum_\alpha\langle\widehat{h}(\rho\lambda)^*\widehat{h}(\rho\lambda)E_\alpha,\widehat{h}(\lambda)^*\widehat{h}(\lambda)E_\alpha\rangle_{\mathcal{H}}.$$

We now assume that for any $\alpha,\beta\in(\mathbf{Z}_+)^n$,

$$\begin{aligned}&\left\langle\int_0^{+\infty}\widehat{h}(\rho\lambda)^*\widehat{h}(\rho\lambda)\frac{d\rho}{\rho}E_\alpha,E_\alpha\right\rangle_{\mathcal{H}}\\ =&\left\langle\int_0^{+\infty}\widehat{h}(\rho\lambda)^*\widehat{h}(\rho\lambda)\frac{d\rho}{\rho}E_\beta,E_\beta\right\rangle_{\mathcal{H}},\end{aligned}$$

and we write that $C_h=\langle\int_0^\infty\widehat{h}(\rho\lambda)^*\widehat{h}(\rho\lambda)\frac{d\rho}{\rho}E_\alpha,E_\alpha\rangle_{\mathcal{H}}$. Then we obtain

$$\begin{aligned}&\int_{\mathbf{P}} |\langle h, U(z,t,\rho)h\rangle_{L^2(\mathbf{H}^n)}|^2 \frac{dzdtd\rho}{\rho^{n+2}}\\ =&\frac{2^{n-1}}{\pi^{n+1}}\int_0^{+\infty}\left(\int_{-\infty}^{+\infty}\sum_\alpha\langle\widehat{h}(\rho\lambda)^*\widehat{h}(\rho\lambda)E_\alpha,\widehat{h}(\lambda)^*\widehat{h}(\lambda)E_\alpha\rangle_{\mathcal{H}}|\lambda|^n d\lambda\right)\frac{d\rho}{\rho}\\ =&\frac{2^{n-1}}{\pi^{n+1}}\int_{-\infty}^{+\infty}\sum_\alpha\langle\int_0^{+\infty}\widehat{h}(\rho\lambda)^*\widehat{h}(\rho\lambda)\frac{d\rho}{\rho}E_\alpha,\widehat{h}(\lambda)^*\widehat{h}(\lambda)E_\alpha\rangle_{\mathcal{H}}|\lambda|^n d\lambda\\ =&\left(\langle\int_0^{+\infty}\widehat{h}(\rho\lambda)^*\widehat{h}(\rho\lambda)\frac{d\rho}{\rho}E_\alpha,E_\alpha\rangle_{\mathcal{H}}\right)\left(\frac{2^{n-1}}{\pi^{n+1}}\int_{-\infty}^{+\infty}\mathrm{tr}(\widehat{h}(\lambda)^*\widehat{h}(\lambda))|\lambda|^n d\lambda\right)\\ =&C_h\|h\|_{L^2(\mathbf{H}^n)}^2.\end{aligned}$$

Write $\mathbf{R}^-=-\mathbf{R}^+$. Then we have the following:

THEOREM 1. *Let $h \in L^2(\mathbf{H}^n)$, $\sigma = +$ or $-$. Then h is an admissible wavelet if there exists a constant $C_h \in \mathbf{R}^+$, such that for any $\alpha \in (\mathbf{Z}_+)^n$,*

$$C_h = \left\langle \int_{\mathbf{R}^\sigma} \widehat{h}(\lambda)^* \widehat{h}(\lambda) \frac{d\lambda}{|\lambda|} E_\alpha, E_\alpha \right\rangle_{\mathcal{H}}. \tag{2.2}$$

We now give some examples of admissible wavelets. Let $\nu > -1, m \in \mathbf{Z}_+$, $L_m^{(\nu)}$ the Laguerre polynomials of degree m and order ν defined by

$$L_m^{(\nu)}(s) = \sum_{\mu=0}^{m} \binom{m+\nu}{m-\nu} \frac{(-s)^\mu}{\mu!} = \frac{1}{m!} e^s s^{-\nu} \left(\frac{d}{ds}\right)^m \left(e^{-s} s^{m+\nu}\right).$$

They satisfy the following orthogonal relations:

$$\int_0^{+\infty} e^{-s} s^\nu L_m^{(\nu)}(s) L_k^{(\nu)}(s) ds = \Gamma(\nu+1) \binom{\nu+m}{m} \delta_{mk}, \tag{2.3}$$

([6]). Let p_m be the even function defined by

$$p_m(s) = \Gamma(\nu+1)^{-\frac{1}{2}} \binom{\nu+m}{m}^{-\frac{1}{2}} (2s)^{\frac{\nu+1}{2}} e^{-s} L_m^{(\nu)}(s), \quad s \in \mathbf{R}^+.$$

Define a wavelet function $h_m(z,t)$ in terms of the Fourier transform by $\widehat{h}_m(\lambda) = p_m(\lambda) I$, where I denotes the identity operator. Clearly, for $\sigma = +$ or $-$, $m \in \mathbf{Z}_+$, $\alpha \in (\mathbf{Z}_+)^n$, from (2.3) it is easy to compute that

$$C_h = \left\langle \int_{\mathbf{R}^\sigma} \widehat{h}_m(\lambda)^* \widehat{h}_m(\lambda) \frac{d\lambda}{|\lambda|} E_\alpha, E_\alpha \right\rangle_{\mathcal{H}} = 1.$$

Hence h_m is an admissible wavelet. The explicit expression of $h_m(z,t)$ can be obtained from the inversion formula (1.8). Let AW denote the set of admissible wavelets. Then for $h \in AW$, the continuous wavelet transform is defined by

$$(W_h f)(z,t,\rho) = \langle f, U(z,t,\rho) h \rangle_{L^2(\mathbf{H}^n)}. \tag{2.4}$$

Similarly as in the proof of Theorem 1, we have

$$\langle W_h f, W_h g \rangle_{L^2(\mathbf{P}, d\mu_l)} = C_h \langle f, g \rangle_{L^2(\mathbf{H}^n)}. \tag{2.5}$$

And the following reconstruction formula holds in the weak sense:

$$f(z',t') = C_h^{-1} \int_0^{+\infty} \int_{\mathbf{H}^n} (W_h f)(z,t,\rho) U(z,t,\rho) h(z',t') \frac{dz dt d\rho}{\rho^{n+2}}. \tag{2.6}$$

3. Continuous multiscale analysis

A continuous multiscale analysis on $\mathbf{H}^n$ is defined by a net $\{V_\rho\}_{\rho\in\mathbf{R}^+}$ of closed subspaces of $L^2(\mathbf{H}^n)$ satisfying

1) (inclusion property) If $\rho_1 \le \rho_2$, then $V_{\rho_2} \subset V_{\rho_1}$.

2) $\lim_{\rho\to 0} V_\rho = L^2(\mathbf{H}^n)$, and $\lim_{\rho\to+\infty} V_\rho = \{0\}$.

3) (rescaling property) $f \in V_{\rho_1}$ if and only if $f_{\frac{\rho_1}{\rho_2}} \in V_{\rho_2}$, where

$$f_{\frac{\rho_1}{\rho_2}}(z,t) = f(\sqrt{\frac{\rho_1}{\rho_2}}z, \frac{\rho_1}{\rho_2}t).$$

4) (translation invariance of the subspaces V_ρ) If $f \in V_\rho$, then for any $(z',t') \in \mathbf{H}^n$, $f(z-z', t-t'-2\mathrm{Im}z\overline{z'}) \in V_\rho$.

For any $\rho \in \mathbf{R}^+$, we define the subset $\mathbf{P}_\rho$ of $\mathbf{P}$ by

$$\mathbf{P}_\rho = \{(z,t,\rho') : (z,t) \in \mathbf{H}^n, \rho' \in [\rho, +\infty)\}. \tag{3.1}$$

Let Θ denote any measurable subset of $\mathbf{R}-\{0\}$ whose Lebesgue measure vanishes. Set

$$V_\rho = \{f \in L^2(\mathbf{H}^n) : \langle f, U(z,t,\rho')h\rangle_{L^2(\mathbf{H}^n)} = 0 \text{ a.e. on } \mathbf{P}-\mathbf{P}_\rho\}. \tag{3.2}$$

Then we get

LEMMA 1. *$f \in V_\rho$ if and only if* $\mathrm{supp}\widehat{f} \cap \{\frac{1}{\rho'}\mathrm{supp}\widehat{h}\} = \Theta$ *for all* $\rho' \in (0,\rho)$.

Proof. We note that

$$\int_{\mathbf{H}^n} \langle f, U(z,t,\rho')h\rangle_{L^2(\mathbf{H}^n)} \pi_\lambda(z,t)dzdt = \rho'^{\frac{n+1}{2}} \widehat{f}(\lambda)\widehat{h}(\rho'\lambda)^*.$$

It is easy to see that

$$\langle f, U(z,t,\rho')h\rangle_{L^2(\mathbf{H}^n)} = 0$$

a.e. on $\mathbf{P}-\mathbf{P}_\rho$ if and only if $\widehat{f}(\lambda)\widehat{h}(\rho'\lambda)^* = 0$ almost everywhere for $\lambda \in \mathbf{R}-\{0\}$. This is equivalent to the condition that $\mathrm{supp}\widehat{f}\cap\{\frac{1}{\rho'}\mathrm{supp}\widehat{h}\}$ is of measure zero, namely

$$\mathrm{supp}\widehat{f} \cap \{\frac{1}{\rho'}\mathrm{supp}\widehat{h}\} = \Theta, \quad \text{for all } \rho' \in (0,\rho). \tag{3.3}$$

This completes the proof. □

THEOREM 2. *$\{V_\rho\}_{\rho\in\mathbf{R}^+}$ is a continuous multiscale analysis if and only if there exists $\eta > 0$, such that $[-\eta,\eta]\cap \mathrm{supp}\widehat{h} = \Theta$.*

Proof. Let $\{f_n\}$ be some Cauchy sequence in V_ρ which converges to $f \in L^2(\mathbf{H}^n)$ in the norm sense. Then

$$(W_h f_n)(z,t,\rho') = \langle f_n, U(z,t,\rho'\rangle_{L^2(\mathbf{H}^n)} \to \langle f, U(z,t,\rho')h\rangle_{L^2(\mathbf{H}^n)}. \tag{3.4}$$

Since $(W_h f_n)(z,t,\rho') = 0$ a.e. on $\mathbf{P}-\mathbf{P}_\rho$, we know that $(W_h f)(z,t,\rho') = 0$ a.e. on $\mathbf{P}-\mathbf{P}_\rho$, and so $f \in V_\rho$. Thus V_ρ is closed. We note that

$$(W_h f_{\frac{\rho_1}{\rho_2}})(z,t,\rho) = \Big\langle f, U\Big(\sqrt{\frac{\rho_2}{\rho_1}}z, \frac{\rho_2}{\rho_1}t, \frac{\rho_2}{\rho_1}\rho\Big)h\Big\rangle_{L^2(\mathbf{H}^n)}, \tag{3.5}$$

and

$$\begin{aligned}&\langle T_{(z',t')}f, U(z,t,\rho)h\rangle_{L^2(\mathbf{H}^n)}\\ =&\langle f, U(z+z', t+t'+2\mathrm{Im}z\overline{z'}, \rho)h\rangle_{L^2(\mathbf{H}^n)}.\end{aligned} \tag{3.6}$$

where $T_{(z',t')}f(z,t) = f(z-z', t-t'-2\mathrm{Im}z\overline{z'})$. Then $f \in V_{\rho_1}$ if and only if $f_{\frac{\rho_1}{\rho_2}} \in V_{\rho_2}$, and $f \in V_\rho$ if and only if $T(z',t')f \in V_\rho$. The necessary condition of this theorem can be proved by the similar way as that of Theorem 3.2 in [4], and we know that there exists $\eta > 0$, such that

$$\mathrm{supp}\widehat{h} \subset \{\lambda \in \mathbf{R} : |\lambda| > \eta\}. \tag{3.7}$$

Let β be defined as the largest positive number η such that (3.7) holds, here we also call that β is the lower limit frequency. Hence for all $\rho \in (0,+\infty)$,

$$\bigcup_{0<\alpha<\rho}\Big(\frac{1}{\alpha}\mathrm{supp}\widehat{h}\Big) = \{\lambda \in \mathbf{R} : |\lambda| > \frac{\beta}{\rho}\}.$$

From Lemma 1 we know that $f \in V_\rho$ if and only if

$$\mathrm{supp}\widehat{f} \cap \Big(\bigcup_{0<\alpha<\rho}\frac{1}{\alpha}\mathrm{supp}\widehat{h}\Big) = \Theta.$$

Thus we have

$$\mathrm{supp}\widehat{f} \subset \{\lambda \in \mathbf{R} : |\lambda| \le \frac{\beta}{\rho}\}.$$

And we obtain

$$\lim_{\rho\to 0} V_\rho = L^2(\mathbf{H}^n) \quad \text{and} \quad \lim_{\rho\to+\infty} V_\rho = \{0\}. \tag{3.8}$$

The proof of Theorem 2 is complete.

□

4. Wavelet packet

Let $\mathbf{Z}$ denote the set of all integers, and let $\{\rho_l\}_{l\in\mathbf{Z}}$ be a strictly decreasing sequence such that

$$\lim_{l\to+\infty}\rho_l=0 \quad\text{and}\quad \lim_{l\to-\infty}\rho_l=+\infty.$$

For $h\in AW$, $\lambda\in\mathbf{R}-\{0\}$. Since for $\alpha\in(\mathbf{Z}_+)^n$,

$$\langle\widehat{g}(\rho\lambda)^*\widehat{g}(\rho\lambda)E_\alpha,E_\alpha\rangle_{\mathcal{H}}=\langle\widehat{g}(\rho\lambda)E_\alpha,\widehat{g}(\rho\lambda)E_\alpha\rangle_{\mathcal{H}}\geq 0,$$

it is not difficult to see that $\int_{\rho_{l+1}}^{\rho_l}\widehat{h}(\rho\lambda)^*\widehat{h}(\rho\lambda)\frac{d\rho}{\rho}$ is a positive linear bounded operator on Fock space $\mathcal{H}$. Assume that for all $\alpha,\beta\in(\mathbf{Z}_+)^n$,

$$\left\langle\int_{\rho_{l+1}}^{\rho_l}\widehat{h}(\rho\lambda)^*\widehat{h}(\rho\lambda)\frac{d\rho}{\rho}E_\alpha,E_\alpha\right\rangle_{\mathcal{H}}=\left\langle\int_{\rho_{l+1}}^{\rho_l}\widehat{h}(\rho\lambda)^*\widehat{h}(\rho\lambda)\frac{d\rho}{\rho}E_\beta,E_\beta\right\rangle_{\mathcal{H}}.$$

Define

$$H^l(\lambda)=\left\{\int_{\rho_{l+1}}^{\rho_l}\widehat{h}(\rho\lambda)^*\widehat{h}(\rho\lambda)\frac{d\rho}{\rho}\right\}^{\frac{1}{2}}. \tag{4.1}$$

Clearly, $H^l(\lambda)$ is a self-conjugate operator. Let $H^l(z',t')\in L^2(\mathbf{H}^n)$, whose Fourier transform is $\widehat{H}^l(\lambda)$. Write

$$H^l_{(z,t)}=H^l(z'-z,t'-t-2\mathrm{Im}z'\overline{z}).$$

Then for $f,g\in L^2(\mathbf{H}^n)$, we get

$$\begin{aligned}
&\int_{\mathbf{H}^n}\langle f,H^l_{(z,t)}\rangle_{L^2(\mathbf{H}^n)}\overline{\langle g,H^l_{(z,t)}\rangle}_{L^2(\mathbf{H}^n)}dzdt\\
=&\frac{2^{n-1}}{\pi^{n+1}}\int_{\mathbf{R}}\mathrm{tr}(\widehat{g}(\lambda)^*\widehat{f}(\lambda)\widehat{H}^l(\lambda)^*\widehat{H}^l(\lambda))|\lambda|^n d\lambda\\
=&\frac{2^{n-1}}{\pi^{n+1}}\int_{\mathbf{R}}\sum_{\alpha\in(\mathbf{Z}_+)^n}\langle\widehat{H}^l(\lambda)^*\widehat{H}^l(\lambda)E_\alpha,\widehat{g}(\lambda)^*\widehat{f}(\lambda)E_\alpha\rangle_{\mathcal{H}}|\lambda|^n d\lambda\\
=&\left(\langle\int_{\rho_{l+1}}^{\rho_l}\widehat{h}(\rho\lambda)^*\widehat{h}(\rho\lambda)\frac{d\rho}{\rho}E_\alpha,E_\alpha\rangle_{\mathcal{H}}\right)\left(\frac{2^{n-1}}{\pi^{n+1}}\int_{\mathbf{R}}\mathrm{tr}(\widehat{g}(\lambda)^*\widehat{f}(\lambda))|\lambda|^n d\lambda\right).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\sum_{l=-\infty}^{+\infty}\int_{\mathbf{H}^n}\langle f,H^l_{(z,t)}\rangle_{L^2(\mathbf{H}^n)}\overline{\langle g,H^l_{(z,t)}\rangle}_{L^2(\mathbf{H}^n)}dzdt\\&=\langle\int_0^{+\infty}\widehat{h}(\rho\lambda)^*\widehat{h}(\rho\lambda)\frac{d\rho}{\rho}E_\alpha,E_\alpha\rangle_{\mathcal{H}}\langle f,g\rangle_{L^2(\mathbf{H}^n)}\\&=C_h\langle f,g\rangle_{L^2(\mathbf{H}^n)}.\end{aligned}$$

Especially,

$$\sum_{l=-\infty}^{+\infty}\int_{\mathbf{H}^n}|\langle f,H^l_{(z,t)}\rangle_{L^2(\mathbf{H}^n)}|^2dzdt=C_h\|f\|^2_{L^2(\mathbf{H}^n)}. \tag{4.2}$$

Here we call $\{H^l\}_{l\in\mathbf{Z}}$ a wavelet packet on $\mathbf{H}^n$. Let $L^2(\mathbf{Z}\times\mathbf{H}^n)$ be the family of measurable functions G on $\mathbf{Z}\times\mathbf{H}^n$ satisfying

$$\sum_{l=-\infty}^{+\infty}\int_{\mathbf{H}^n}|G(l,(z,t))|^2dzdt<+\infty.$$

For $f\in L^2(\mathbf{H}^n)$, the wavelet packet transform W_Hf is defined by

$$(W_Hf)(l,(z,t))=\langle f,H^l_{(z,t)}\rangle_{L^2(\mathbf{H}^n)}. \tag{4.3}$$

Then $(W_Hf)(l,(z,t))\in L^2(\mathbf{Z}\times\mathbf{H}^n)$. From the above discussion and (2.6) we obtain the following reconstruction formula:

THEOREM 3. *Let $\{H^l_{(z,t)}\}_{l\in\mathbf{Z}}$ be a wavelet packet on $\mathbf{H}^n$. Then for all $f\in L^2(\mathbf{H}^n)$,*

$$f(z',t')=C_h^{-1}\sum_{l=-\infty}^{+\infty}\int_{\mathbf{H}^n}(W_Hf)(l,(z,t))H^l_{(z,t)}dzdt. \tag{4.4}$$

Let

$$\hbar^l(\lambda)=\left(\sum_{-\infty}^{l}H^j(\lambda)^2\right)^{\frac12}=\left(\sum_{-\infty}^{l}\int_{\rho_{j+1}}^{\rho_j}\widehat{h}(\rho\lambda)^*\widehat{h}(\rho\lambda)\frac{d\rho}{\rho}\right)^{\frac12}.$$

Then for all $f\in L^2(\mathbf{H}^n)$, we have

$$f(z',t')=C_h^{-1}\lim_{l\to+\infty}\int_{\mathbf{H}^n}(W_\hbar f)(l,(z,t))\hbar^l_{(z,t)}dzdt. \tag{4.5}$$

References

[1] D. Geller, *Fourier Analysis on the Heisenberg Group I: Schwartz Space*, J. Funct. Anal. **36** (1980), 205–254.

[2] M. M. Nessibi and K. Trimèche, *Inversion of the Radon Transform on the Laguerre Hypergroup by Using Generalized Wavelets*, J. Math. Anal. Appl. **208** (1997), 337–363.

[3] H. G. Stark, *Continuous Wavelet Transform and Continuous Multiscale Analysis*, J. Math. Anal. Appl. **169** (1992), 179–196.

[4] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, 1993.

[5] R. S. Strichartz, *L^p Harmonic Analysis and Radon Transform on the Heisenberg Group*, J. Funct. Anal. **96** (1991), 350–406.

[6] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Colloq. Publications **23** (1939).

Department of Mathematics, Nanjing Normal University, Nanjing 210097, P. R. China

E-mail: jxhe@pine.njnu.edu.cn