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DECOMPOSITION OF SOME CENTRAL SEPARABLE ALGEBRAS

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Abstract. If an Azumaya algebra A is a homomorphic image of a finite group ring RG where G is a direct product of subgroups then A can be decomposed into subalgebras A_i which are homomorphic images of subgroup rings of RG. This result is extended to projective Schur algebras, and in this case behaviors of 2-cocycles will play major role. Moreover considering the situation that A is represented by Azumaya group ring RG, we study relationships between the representing groups for A and A_i.

1. Introduction

Let R denote a commutative ring. The Brauer group $B(R)$ is the group of similar classes $[A]$ consisting of an Azumaya (i.e., central separable) R-algebra A over R (refer to [4, (2.5)]). An Azumaya R-algebra which is a homomorphic image of a group ring RG for some finite group G is called the Schur algebra. The set of similar classes of Schur algebras forms the Schur subgroup $S(R)$ of $B(R)$. In [5], two subgroups of $S(R)$ were introduced; one is $S'(R)$ consisting of elements in $S(R)$ that are represented by cyclotomic algebras $(R(\varepsilon_n)/R, \alpha)$ with 2-cocycle α on $\mathrm{Gal}(R(\varepsilon_n)/R)$ having values in $\langle \varepsilon_n \rangle$ for $n > 0$. The other is $S''(R)$ consisting of elements in $S(R)$ with a representative which is a homomorphic image of separable group algebra RG. The group ring RG is separable if and only if $|G|$ is unit of R. The $S'(R)$ and $S''(R)$ need not equal, however if $R = k$ a field then $S''(k) = S'(k) = S(k)$ due to Brauer-Witt theorem [10].

The Schur k-algebra was generalized by Lorenz and Opolka (1978) that a finite dimensional central simple k-algebra which is a homomorphic image of a twisted group algebra kG^{α} for some finite group G and some $\bar{\alpha} \in H^2(G, k^*)$ is called the projective Schur algebra over k. The set of similar classes of projective Schur algebras forms the projective Schur group $PS(k)$.

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In the paper we study decompositions of some Azumaya algebras such as Schur and projective Schur algebras. In section 2, we prove that if A is an Azumaya algebra which is an epimorphic image of RG and if $G = G_1 \times G_2$ then A is decomposed into Azumaya subalgebras A_i where A_i is represented by RG_i $(i = 1, 2)$. The similar result can be obtained with respect to projective Schur algebras, and in this case behaviors of 2-cocycles corresponding to representing groups play important roles. In section 3, regarding RG itself as an Azumaya group ring which represents A, we study interrelationships between the representing Azumaya group rings for A and A_i.

Throughout the paper, R will always denote a connected commutative ring. Let $[A] \in B(R)$ denote a similar class of finite dimensional Azumaya R-algebra A, and for $A' \in [A]$ we denote $A' \sim A$. Let $u(R)$ be the set of units of R, k^* be the multiplicative subgroup of a field k and ε_n $(n > 0)$ be a primitive n-th root of unity. For a field extension L/k, we denote $H^2(\mathrm{Gal}(L/k), L^*)$ by $H^2(L/k)$.

2. Schur and projective Schur algebras

For Galois extensions of commutative ring R, we may refer to [1] or [4]. Let A be any R-algebra and B be any subalgebra of A. Let $A^B = \{a \in A \mid ab = ba \text{ for any } b \in B\}$. Then A^B is an R-subalgebra of A which commutes with B. We remark the following lemma for convenience.

LEMMA 1 [4, (2.4.3)]. *Let A be an Azumaya R-algebra.*

(1) *If B is an Azumaya R-subalgebra of A then so is A^B. Moreover $(A)^{A^B} = B$ and $B \otimes A^B \cong A$; $b \otimes a \mapsto ba$ for $b \in B, a \in A^B$.*
(2) *Suppose B and C are subalgebras with $B \otimes C \cong A$, $b \otimes c \mapsto bc$ ($b \in B, c \in C$). Then B, C are Azumaya algebras with $A^B = C$, $A^C = B$.*

Thus for a given Azumaya algebra A, Azumaya subalgebras of A occur in pairs, each of the pair is the commutator subalgebra of the other and whose tensor product is isomorphic to A.

THEOREM 2. *Suppose that $[A] \in S(R)$ and the Azumaya algebra A is represented by a finite group ring RG. If $G = G_1 \times G_2$ then A can be decomposed into $A_1 \otimes A_2$ where each A_i is represented by RG_i, thus $[A_i] \in S(R)$ for $i = 1, 2$. Furthermore if $[A] \in S''(R)$ then each $[A_i] \in S''(R)$.*

Proof. Let f be the surjective homomorphism $RG \to A$ and let $\{u_g \mid g \in G\}$ denote an R-basis for RG with multiplication $u_g u_x = u_{gx}$ for $g, x \in G$. Since $RG = R(G_1 \times G_2) \cong RG_1 \otimes_R RG_2$ as R-algebras [9], for $g = g_1 g_2 \in G$ $(g_i \in G_i)$, the R-basis element $u_g \in RG$ corresponds to $u_{g_1} \otimes u_{g_2}$ where

u_{g_i} is an R-basis of RG_i. Hence we may use the same notation f for the surjection $RG_1 \otimes RG_2 \to A$ defined by $f(u_{g_1} \otimes u_{g_2}) = f(u_g)$.

Let f_i be the restriction of f to RG_i and let $A_i = f_i(RG_i)$ for $i = 1, 2$. Then $f_1 \otimes f_2 : RG \to A_1 \otimes A_2$ maps $u_{g_1} \otimes u_{g_2}$ to $f_1(u_{g_1}) \otimes f_2(u_{g_2})$, this implies that A_1 and A_2 are R-subalgebras of A such that $A_1 \otimes A_2 \to A$, $a_1 \otimes a_2 \mapsto a_1a_2$ is an R-algebra isomorphism. Thus $f = f_1 \otimes f_2$ and A_i is an Azumaya R-algebra due to Lemma 1. Clearly $[A_i] \in S(R)$ for $i = 1, 2$.

In particular if $[A] \in S''(R)$ then $|G| \in u(R)$ hence there exists $m \in R$ such that $m|G| = 1_R$. Since $|G_i|$ divides $|G|$, $|G| = |G_i|t_i$ for some $t_i > 0$ and $1_R = t_im|G_i|$, thus $|G_i|$ is unit in R. It thus follows that $[A_i] \in S''(R)$. □

For finite groups G and H, if $\alpha \in Z^2(G, u(R))$ and $\beta \in Z^2(H, u(R))$ then $\alpha \times \beta$ defined by $\alpha \times \beta((g_1, h_1), (g_2, h_2)) = \alpha(g_1, g_2)\beta(h_1, h_2)$ with $g_i \in G$, $h_i \in H$ is an element in $Z^2(G \times H, u(R))$. In particular if $G = H$ then $\alpha\beta$ defined by $\alpha\beta(g_1, g_2) = \alpha(g_1, g_2)\beta(g_1, g_2)$ is contained in $Z^2(G, u(R))$.

THEOREM 3. *Let $[A] \in S'(R)$ and A be a cyclotomic algebra $(R(\varepsilon_n)/R, \alpha)$ where α has values in $\langle \varepsilon_n \rangle$. If n is divisible by pq with primes $p \neq q$ then A can be decomposed into $A_1 \otimes A_2$ and $[A_i] \in S'(R)$ for $i = 1, 2$.*

Proof. For any $x, y \in \mathrm{Gal}(R(\varepsilon_n)/R)$, the order of $\alpha(x, y)$ divides n because $\alpha(x, y) \in \langle \varepsilon_n \rangle$. With the prime divisor p of n, write $\alpha(x, y) = \alpha(x, y)_p\alpha(x, y)_{p'}$ and $n = n_pn_{p'}$ where $\alpha(x, y)_p$ [resp. n_p] is the p-part and $\alpha(x, y)_{p'}$ [resp. $n_{p'}$] is the p' part of $\alpha(x, y)$ [resp. n]. In fact, $\alpha(x, y)_p$ and $\alpha(x, y)_{p'}$ are powers of $\alpha(x, y)$ such that the order of $\alpha(x, y)_p$ is a power of p while the order of $\alpha(x, y)_{p'}$ is prime to p. Since $pq|n$ for $p \neq q$, $n_{p'} \neq 1$ and $\alpha(x, y)_{p'} \neq 1$. Thus it follows that $\langle \varepsilon_n \rangle = \langle \varepsilon_{n_p} \rangle \times \langle \varepsilon_{n_{p'}} \rangle$ hence $\alpha(x, y)_p \in \langle \varepsilon_{n_p} \rangle$ and $\alpha(x, y)_{p'} \in \langle \varepsilon_{n_{p'}} \rangle$.

Let $\alpha_1(x, y) = \alpha(x, y)_p$ and $\alpha_2(x, y) = \alpha(x, y)_{p'}$. Then it is easy to see that

$$\alpha_1(x, y)\alpha_1(xy, z) \cdot \alpha_2(x, y)\alpha_2(xy, z) = \alpha_1(x, yz)x\alpha_1(y, z) \cdot \alpha_2(x, yz)x\alpha_2(y, z)$$

for any $x, y, z \in \mathrm{Gal}(R(\varepsilon_n)/R)$. Thus due to the uniqueness of p, p'-part, it follows that α_1, $\alpha_2 \in Z^2(R(\varepsilon_n)/R, u(R(\varepsilon_n)))$ on which the natural Galois action is defined, and the values of α_1, α_2 are contained in $\langle \varepsilon_{n_p} \rangle$ and $\langle \varepsilon_{n_{p'}} \rangle$ respectively. Consequently $\alpha = \alpha_1\alpha_2$ and it follows from [4, (4.2.13)] (or [7, (29.9)]) that $A = (R(\varepsilon_n)/R, \alpha)$ is similar to $(R(\varepsilon_n)/R, \alpha_1) \otimes (R(\varepsilon_n)/R, \alpha_2)$. Now let $(R(\varepsilon_n)/R, \alpha_i) = A_i$. Then $A = A_1 \otimes A_2$ and $[A_i] \in S'(R)$. □

The converse of Theorem 2 follows immediately that if an R-algebra A is decomposed into $A_1 \otimes A_2$ where A_i are Schur R-algebras $(i = 1, 2)$ then $[A]$ belongs to $S(R)$. In particular if $[A_i] \in S''(R)$ for $i = 1, 2$ then $[A] \in S''(R)$. For the same question with respect to $S'(R)$, we have the next theorem.

THEOREM 4. *Let $A = A_1 \otimes A_2$ with $[A_i] \in S'(R)$ $(i = 1,2)$. Then $[A] \in S'(R)$. Moreover if $A_i = (R(\varepsilon_{n_i})/R, \alpha_i)$ for a 2-cocycle α_i and if $(n_1, n_2) = 1$ then A is a cyclotomic R-algebra with respect to the 2-cocycle $\alpha_1 \times \alpha_2$.*

Proof. We denote the inflation map $H^2(R(\varepsilon_{n_i})/R) \to H^2(R(\varepsilon_{n_1}, \varepsilon_{n_2})/R)$ by inf_i for $i = 1, 2$, and we consider $\mathrm{inf}_i\alpha_i$ defined by

$$(\mathrm{inf}_i\alpha_i)(\theta_{i1}, \theta_{i2}) = \alpha_i(\bar{\theta}_{i1}, \bar{\theta}_{i2}),$$

where $\bar{\theta}_{ij} = \theta_{ij}\mathrm{Gal}(R(\varepsilon_{n_1}, \varepsilon_{n_2})/R(\varepsilon_i))$. Then following [7, (29.16)]), it is easy to see that $(R(\varepsilon_{n_i})/R, \alpha_i)$ is similar to $(R(\varepsilon_{n_1}, \varepsilon_{n_2})\ /R,\ \mathrm{inf}_i\alpha_i)$, and it thus follows that

$$A_1 \otimes A_2 = (R(\varepsilon_{n_1})/R, \alpha_1) \otimes (R(\varepsilon_{n_2})/R, \alpha_2) \sim (R(\varepsilon_{n_1}, \varepsilon_{n_2})/R, \mathrm{inf}_1\alpha_1\mathrm{inf}_2\alpha_2).$$

Thus A is similar to $(R(\varepsilon_{n_1}, \varepsilon_{n_2})/R, \beta)$ for $\beta = \mathrm{inf}_1\alpha_1\mathrm{inf}_2\alpha_2$, hence $[A] \in S'(R)$.

We now suppose that $(n_1, n_2) = 1$. Then for any $\theta_i \in \mathrm{Gal}(R(\varepsilon_{n_1}, \varepsilon_{n_2})/R)$, θ_i can be written as (σ_i, τ_i) where $\sigma_i \in \mathrm{Gal}\ (R(\varepsilon_{n_1})/R)$ and $\tau_i \in \mathrm{Gal}(R(\varepsilon_{n_2})\ /R)$. Thus $\mathrm{inf}_1\alpha_1(\theta_1, \theta_2) = \alpha_1(\sigma_1, \sigma_2)$ and $\mathrm{inf}_2\alpha_2(\theta_1, \theta_2) = \alpha_2(\tau_1, \tau_2)$, this shows that

$$\mathrm{inf}_1\alpha_1\mathrm{inf}_2\alpha_2(\theta_1, \theta_2) = \alpha_1 \times \alpha_2((\sigma_1, \tau_1), (\sigma_2, \tau_2)),$$
$$\text{and } \beta = \mathrm{inf}_1\alpha_1\mathrm{inf}_2\alpha_2 = \alpha_1 \times \alpha_2.$$

This completes the proof. □

Let α be a 2-cocycle in $Z^2(G, u(R))$ with trivial G-action on R and let $\{u_g |\ g \in G\}$, $u_1 = 1$ denote an R-basis for the twisted group ring RG^α with multiplication $(ru_x)(su_y) = rs\alpha(x, y)u_{xy}$ and $\alpha(x, 1) = \alpha(1, x) = 1$ for all $r, s \in R$, $x, y \in G$. An Azumaya R-algebra A is called a projective Schur R-algebra if it is a homomorphic image of RG^α for finite group G, and the set of similar classes of projective Schur algebras forms a group $PS(R)$.

THEOREM 5. *Suppose that $[A] \in PS(R)$ and A is represented by a twisted group ring RG^α. Assume $G = G_1 \times G_2$ with $(|G_1|, |G_2|) = 1$. Then A can be decomposed into $A_1 \otimes A_2$ where A_i is represented by $RG_i^{\alpha_i}$ for a 2-cocycle $\alpha_i \in Z^2(G_i, u(R))$, thus $[A_i] \in PS(R)$ for $i = 1, 2$.*

Proof. Let f be the surjective homomorphism $RG^\alpha \to A$, and let $\alpha_i \in Z^2(G_i, u(R))$ be the restrictions of α to G_i for $i = 1, 2$. Since $(|G_1|, |G_2|) = 1$, it follows from [6, (2.3.14)] that the pairing $\phi_\alpha : G_1 \times G_2 \to u(R)$ defined by $\phi_\alpha(a, b) = \alpha(a, b)\alpha(b, a)^{-1}$ for $a \in G_1, b \in G_2$ is trivial, thus $\alpha(a, b) = \alpha(b, a)$ and $H^2(G_1 \times G_2, u(R))$ is isomorphic to $H^2(G_1, u(R)) \times H^2(G_2, u(R))$ that makes α correspond to (α_1, α_2).

For any $g = g_1g_2 \in G$ $(g_i \in G_i)$, we define

$$\psi : RG^{\alpha_1\times\alpha_2} \to RG^{\alpha}, \quad \psi(w_g) = \alpha(g_1, g_2)u_g$$

where w_g, u_g are bases for $RG^{\alpha_1\times\alpha_2}$ and RG^{α} respectively. For $x = x_1x_2 \in G$, since

$$\begin{aligned}
\psi(w_g)\psi(w_x) \quad &= \alpha(g_1,g_2)\alpha(x_1,x_2)\alpha(g_1g_2,x_1x_2)u_{gx}\\
&= \alpha(g_1,g_2x_1x_2)\alpha(g_2,x_1x_2)\alpha(x_1,x_2)u_{gx}\\
&= \alpha(g_1,x_1g_2x_2)\alpha(g_2x_1,x_2)\alpha(g_2,x_1)u_{gx}\\
&= \alpha(g_1,x_1)\alpha(g_2,x_2)\alpha(g_1x_1,g_2x_2)u_{gx}\\
&= \alpha_1(g_1,x_1)\alpha_2(g_2,x_2)\alpha(g_1x_1,g_2x_2)u_{gx}\\
&= (\alpha_1\times\alpha_2)(g,x)\alpha(g_1x_1,g_2x_2)u_{gx}\\
&= \psi((\alpha_1\times\alpha_2)(g,x)w_{gx}) = \psi(w_gw_x),
\end{aligned}$$

it follows that ψ is an isomorphism. Moreover due to [6, (5.1.1)], we have $RG^{\alpha} \cong RG^{\alpha_1\times\alpha_2} = R(G_1\times G_2)^{\alpha_1\times\alpha_2} \cong RG_1^{\alpha_1}\otimes RG_2^{\alpha_2}$. Using the same notation f, write $f : RG_1^{\alpha_1}\otimes RG_2^{\alpha_2} \to A$ abusively and let f_i be the restrictions of f to $RG_i^{\alpha_i}$ and $A_i = f_i(RG_i^{\alpha_i})$. Then $A_1\otimes A_2 = f(RG_1^{\alpha_1}\otimes RG_2^{\alpha_2}) \cong A$ which maps $a_1\otimes a_2 \in A_1\otimes A_2$ to $a_1a_2 \in A$. This implies that A is a homomorphic image of $R(G_1\times G_2)^{\alpha_1\times\alpha_2}$, and A_1 and A_2 are Azumaya algebras because of Lemma 1. Hence it follows that $[A_i] \in PS(R)$. $\square$

Moreover, the next corollary follows immediately.

COROLLARY 6. *If $A = A_1\otimes A_2$ with $[A_i] \in PS(R)$ and if A_i is an image of $RG_i^{\alpha_i}$ for $i = 1, 2$, then $[A] \in PS(R)$ and A is the image of $R(G_1\times G_2)^{\alpha_1\times\alpha_2}$.*

3. Azumaya algebras

Consider projective Schur algebras which are epimorphic images of twisted group rings RG^{α} for $\alpha \in Z^2(G, u(R))$ that are separable algebras, i.e., $|G| \in u(R)$. Then the set $PS''(R)$

$$PS''(R) = \{[A] \in PS(R)|\ RG^{\alpha} \to A, |G| \in u(R),\ \bar{\alpha} \in H^2(G, u(R))\}$$

forms a subgroup of $PS(R)$ ([9, 1.2(7)]), and $S''(R) < PS''(R)$.

We restrict our attention to the situation that $[A] \in PS''(R)$ where A is a homomorphic image of RG^{α} such that $G = G_1\times G_2$. Then by Theorem 5 we have that $A = A_1\otimes A_2$ where A_i is represented by $RG_i^{\alpha_i}$ for $\alpha_i = \mathrm{res}\alpha$, and moreover $[A_i] \in PS''(R)$. In [9], it was studied the situation that an Azumaya algebra A that is represented by an epimorphic image of RG^{α} may also be obtained as the image of such a group ring which is moreover itself an Azumaya algebra.

In this section we find relationships between the Azumaya twisted group rings RG^{α} and $RG_i^{\alpha_i}$ that represents A and A_i respectively. We use Lemma 1 to discuss commutator subalgebras of each other. As an application we study a situation that when a Schur algebra is represented by certain twisted group ring which is itself an Azumaya algebra.

For $\alpha \in Z^2(G, u(R))$, an element $x \in G$ is said to be α-regular if $\alpha(g,x) = \alpha(x,g)$ for all $g \in C_G(x)$ the centralizer of x. The set $Z(G)_\alpha = \{x \in Z(G) |\ x \text{ is } \alpha\text{-regular}\}$ is called a root group of G with respect to α, and this group plays an important role for RG^{α} to be central. In fact, a necessary condition for RG^{α} to be central is that $Z(G)_\alpha$ is trivial. For an abelian group, this condition is also sufficient.

LEMMA 7 ([9, Theorem 2.2]). *For $[A] \in PS(R)$, we may assume that it is given by an epimorphism $RG^{\alpha} \to A$ where $Z(G)_\alpha = 1$. Hence if $[A] \in PS''(R)$ then RG^{α} itself is an Azumaya algebra.*

THEOREM 8. *Let $G = G_1 \times G_2$ with $(|G_1|, |G_2|) = 1$. If $\alpha \in Z^2(G, u(R))$ and $\alpha_i \in Z^2(G_i, u(R))$ is the restriction of α, then α is cohomologous to $\alpha_1 \times \alpha_2$ so that the corresponding root groups are equal. Moreover we have the following.*

(1) *Let $x = x_1x_2 \in G$ with $x_i \in G_i$ $(i = 1, 2)$. Then x is α-regular if and only if x_i is α_i-regular for $i = 1, 2$.*
(2) *$Z(G)_\alpha = Z(G_1)_{\alpha_1} \times Z(G_2)_{\alpha_2}$ and $G/Z(G)_\alpha \cong G_1/Z(G_1)_{\alpha_1} \times G_2/Z(G_2)_{\alpha_2}$.*

Proof. Define a map $t : G \to u(R)$ by $g \mapsto \alpha(g_1, g_2)$ for $g = g_1g_2$. Then

$$\begin{aligned}\alpha(g,x)t(g)t(x) &= \alpha(g_1, g_2x_1x_2)\alpha(g_2, x_1x_2)\alpha(x_1, x_2)\\ &= \alpha(g_1, x_1g_2x_2)\alpha(x_1g_2, x_2)\alpha(x_1, g_2) = t(gx)(\alpha_1 \times \alpha_2)(g, x),\end{aligned}$$

where the second equality holds because of trivial pairing of G_1 and G_2 [6, (2.3.14)]. Thus α is cohomologous to $\alpha_1 \times \alpha_2$, and the α-regularity is equal to the $\alpha_1 \times \alpha_2$-regularity by [6, (3.6.1)], so that $Z(G)_\alpha$ corresponds to $Z(G)_{\alpha_1 \times \alpha_2}$.

For $x = x_1x_2$, if x is α-regular and if $a \in C_{G_1}(x_1)$ then $xa = x_1ax_2 = ax_1x_2 = ax$, thus $\alpha(x,a) = \alpha(a,x)$. Moreover we have that

$$\alpha_1(x_1, a)\alpha(x, x_2^{-1}) = \alpha(xx_2^{-1}, a)\alpha(x, x_2^{-1}) = \alpha_1(a, x_1)\alpha(x, x_2^{-1}),$$

hence x_1 is α_1-regular, and similarly we get x_2 is α_2-regular.

Conversely, assume that x_i is α_i-regular and choose any $g \in G$ such that $xg = gx$. Then $x_ig_i = g_ix_i$ and $\alpha_i(x_i, g_i) = \alpha_i(g_i, x_i)$ for $i = 1, 2$. Thus $(\alpha_1 \times \alpha_2)(x, g) = \alpha_1(x_1, g_1)\alpha_2(x_2, g_2) = (\alpha_1 \times \alpha_2)(g, x)$, which proves that x is $(\alpha_1 \times \alpha_2)$-regular, so that x is α-regular, this proves (1).

If $ab \in Z(G_1)_{\alpha_1} \times Z(G_2)_{\alpha_2}$ for $a \in Z(G_1)_{\alpha_1}$ and $b \in Z(G_2)_{\alpha_2}$, then we have $a \in Z(G_1), b \in Z(G_2)$ and for any $g_1 \in G_1$ and $x_2 \in G_2$, $\alpha_1(a, g_1) = \alpha_1(g_1, a)$ and $\alpha_2(b, x_2) = \alpha_2(x_2, b)$. Thus $ab \in Z(G_1) \times Z(G_2) = Z(G)$. Furthermore ab is α-regular because

$$(\alpha_1 \times \alpha_2)(ab, l) = \alpha_1(a, l_1)\alpha_2(b, l_2) = \alpha_1(l_1, a)\alpha_2(l_2, b) = (\alpha_1 \times \alpha_2)(l, ab)$$

for any $l = l_1 l_2 \in G$ with $l_i \in G_i$. Hence $ab \in Z(G)_{\alpha_1 \times \alpha_2} = Z(G)_\alpha$.

On the other hand, if $g \in Z(G)_{\alpha_1 \times \alpha_2}$ then $g \in Z(G)$ and g is $(\alpha_1 \times \alpha_2)$-regular. Clearly $g = g_1 g_2 \in Z(G_1) \times Z(G_2)$, g is α-regular and g_i is α_i-regular due to (1) hence it concludes that $g = g_1 g_2 \in Z(G_1)_{\alpha_1} \times Z(G_2)_{\alpha_2}$. The remaining of (2) is clear. □

THEOREM 9. *Let $[A] \in PS''(R)$ where A is represented by RG^α. If $G = G_1 \times G_2$ such that $(|G_1|, |G_2|) = 1$ then $A = A_1 \otimes A_2$ where A and A_i are represented by Azumaya twisted group rings RN^β and $RN_i^{\beta_i}$ respectively for some finite groups N and N_i. Moreover $(RN^\beta)^{RN_1^{\beta_1}} = RN_2^{\beta_2}$ and $(RN^\beta)^{RN_2^{\beta_2}} = RN_1^{\beta_1}$.*

Proof. It was proved in Theorem 5 that $RG^\alpha \cong R(G_1 \times G_2)^{\alpha_1 \times \alpha_2} \cong RG_1^{\alpha_1} \otimes RG_2^{\alpha_2}$ where α_i is the restriction of α to G_i $(i = 1, 2)$. And $A = A_1 \otimes A_2$ where each A_i $(i = 1, 2)$ is given by epimorphism $RG_i^{\alpha_i} \to A_i$. Moreover since $[A] \in PS''(R)$, each $[A_i]$ belongs to $PS''(R)$ for $i = 1, 2$. Thus due to Lemma 7 we may consider that the algebras A and A_i are epimorphic images of some twisted group rings which are Azumaya algebras.

In fact since $[A] \in PS''(R)$ we may assume that the representing twisted group ring RG^α for A is separable, i.e., $|G| \in u(R)$. Thus if $Z(G)_\alpha$ is trivial then RG^α itself is central so that Azumaya. In case that $Z(G)_\alpha$ is not trivial, we consider the quotient group $G/Z(G)_\alpha$ following the idea in [9, (2.2)]. Then there is a 2-cocycle $\beta \in Z^2(G/Z(G)_\alpha, u(R))$ such that A is given by epimorphism $R(G/Z(G)_\alpha)^\beta \to A$ by Lemma 7. Replacing the representing pair (G, α) by $(G/Z(G)_\alpha, \beta)$ and continuing this process, we get a representation with trivial root group with respect to β. If we denote $G/Z(G)_\alpha$ by N then A is an epimorphic image of RN^β, $Z(N)_\beta = 1$ and RN^β is an Azumaya algebra.

Since $G = G_1 \times G_2$ with $(|G_1|, |G_2|) = 1$, it follows from Theorem 8 that $G/Z(G)_\alpha \cong G_1/Z(G_1)_{\alpha_1} \times G_2/Z(G_2)_{\alpha_2}$. Thus if we let $N_i = G_i/Z(G_i)_{\alpha_i}$ then $N = N_1 \times N_2$ with $(|N_1|, |N_2|) = 1$. Regarding N_i as a subgroup of N, let β_i be the restriction of β to N_i. Then it follows from Theorem 8 that β is cohomologous to $\beta_1 \times \beta_2$ and $Z(N)_\beta = Z(N_1)_{\beta_1} \times Z(N_2)_{\beta_2}$.

Because $Z(N)_\beta$ is trivial, so are $Z(N_i)_{\beta_i} = 1$ hence $RN_i^{\beta_i}$ is an Azumaya algebra. Moreover $RN_1^{\beta_1} \otimes RN_2^{\beta_2} \cong RN^\beta$ and the epimorphism $RN^\beta \to A$ gives rise to epimorphisms $RN_i^{\beta_i} \to A_i$, as is required.

As Azumaya algebras RN^β and $RN_i^{\beta_i}$ $(i = 1, 2)$, the isomorphism $RN_1^{\beta_1} \otimes RN_2^{\beta_2} \cong RN^\beta$ is defined by $u_{n_1} \otimes u_{n_2} \mapsto u_{n_1}u_{n_2} = \beta(n_1, n_2)u_n$ for $n = n_1n_2 \in N$ by Theorem 5. Hence due to Lemma 1, the commutator subalgebras are

$$(RN^\beta)^{RN_1^{\beta_1}} = RN_2^{\beta_2} \text{ and } (RN^\beta)^{RN_2^{\beta_2}} = RN_1^{\beta_1},$$

this completes the proof. □

Finally we study a relationship between Schur and projective Schur algebras. Clearly a Schur algebra is a projective Schur algebra with trivial 2-cocycle. Besides the trivial case, we may regard a Schur algebra as a projective Schur algebra with respect to non-trivial 2-cocycle α in the bijective correspondence between projective representations of a finite group G and ordinary representations of the covering group H of G. In this case α is determined by a group extension H by G, however the values of α may not contained in the ring R.

THEOREM 10. *Let $[A] \in S(R)$ and $f : RG \to A$ be an epimorphism with finite group G. Assume the center $Z(G) \neq 1$. Then there is a finite group H, 2-cocycle $\alpha \in Z^2(H, u(R))$ such that A is an epimorphic image of $RH^\alpha \to A$. Moreover if $G = G_1 \times G_2$ with $(|G_1|, |G_2|) = 1$ then we have the following.*

1. *$RH^\alpha \cong RH_1^{\alpha_1} \otimes RH_2^{\alpha_2}$ for subgroups H_i of H and restrictions α_i of α. Thus A_i is an epimorphic image of $RH_i^{\alpha_i}$ such that $A = A_1 \otimes A_2$.*
2. *Furthermore if $[A] \in S''(R)$ then from $RH^\alpha \to A$, we may assume RH^α is central, if necessary, by taking quotient $H/Z(H)_\alpha$ until $Z(H)_\alpha$ is trivial. Hence $RH_i^{\alpha_i}$ are assumed to be Azumaya.*

Proof. Consider a central group extension $E : 1 \to Z(G) \to G \to G/Z(G) \to 1$. Then due to the well known correspondence between ordinary representations of G and projective representations of $G/Z(G)$, there is an epimorphism on twisted group ring of $G/Z(G)$ over R with respect to a factor set λ of the extension E. The factor set $\lambda \in Z^2(G/Z(G), Z(G))$, however the values of λ need not belong to R.

Regarding λ as an element in $Z^2(G/Z(G), u(RZ(G)))$, it was proved in [9] that $RZ(G)(G/Z(G))^\lambda$ is isomorphic to RG, thus the same notation f can be used for the map $f : RZ(G)(G/Z(G))^\lambda \to A$.

For any $x \in Z(G)$, $f(x)$ is central in A, hence in $u(R)$. If we let $\alpha = f\lambda$ then α belongs to $Z^2(G/Z(G), u(R))$ thus we have a surjective homomorphism $R(G/Z(G))^\alpha \to A$.

By setting $G/Z(G) = H$, (1) follows immediately from Theorem 5. Moreover if $[A] \in S''(R)$ then the representing group algebra RG satisfies $|G| \in u(R)$, thus by regarding $[A]$ as in $PS''(R)$ (2) follows from Theorem 9. □

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