Bull. Korean Math. Soc. **38** (2001), No. 3, pp. 505–516

SOME QUASILINEAR HYPERBOLIC EQUATIONS AND YOSIDA APPROXIMATIONS

JONG YEOUL PARK, IL HYO JUNG, AND YONG HAN KANG

ABSTRACT. We show the existence and uniqueness of solutions for the Cauchy problem for nonlinear evolution equations with the strong damping:

$$u''(t) - M(|\nabla u(t)|^2)\Delta u(t) - \delta\Delta u'(t) = f(t).$$

As an application, a Kirchhoff model with viscosity is given.

1. Introduction

We shall be concerned with abstract Cauchy initial value problems in a Hilbert space H for nonlinear evolution equations of Kirchhoff type

$$\text{(CP)} \qquad \begin{aligned} u''(t) - M(|\nabla u(t)|^2)\Delta u(t) - \delta\Delta u'(t) = f(t),\ t \in (0,T), \\ u(0) = u_0,\ u'(0) = u_1, \end{aligned}$$

where the function $M(\cdot)$, which satisfies convenient assumptions, is given, k is a fixed positive integer, $\delta > 0$ is a constant and $T > 0$ is arbitrary and fixed.
The motivation which the problem (CP) has attracted the attention of several researchers(see [1-4, 6-11] and references therein) is of its intimate connection with mathematical physics for describing small amplitude vibrations of an elastic string(see [4]).

When $\delta = 0$, Dickey [2] and Pohožaev [11] have shown the existence and uniqueness of local solutions to the problem (CP) by using

Received April 1, 2000.
2000 Mathematics Subject Classification: 35A05, 35L70.
Key words and phrases: Kirchhoff model, Yosida approximations, quasilinear hyperbolic equations.
This work was supported by the Brain Korea 21 Project.

a Galerkin procedure, respectively(see also [6]). Cavalcanti et al. [1] have shown the existence of global solutions and exponential decay to the problem (CP) by using a Galerkin procedure(see also [7, 8]).

The purpose of this paper is to show the existence and uniqueness of solutions to the problem (CP) without smallness condition of the initial data and under the presence of the strong damping term $\delta\triangle u'(t)(\delta > 0)$. The proof of the solvability of the problem (CP) is carried out by the Yosida approximation method(cf. [4]). Note that this technique for the proof of the existence theorem plays a central role in deriving some *a priori estimates* of solutions to the problem (CP).
The contents of this paper are as follows. In Section 2, we give the abstract setting and main results. In Section 3, we mention some useful facts about Yosida approximation of a nonnegative selfadjoint operator and give the proof of our main results. Section 4 is devoted to an application for our abstract results.

2. Preliminaries

Let H be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. Assume that A is a densely defined nonnegative selfadjoint linear operator in H. Then the square root of A, $A^{1/2}$ may be computed via the elementary spectral calculus and is known to be a nonnegative selfadjoint operator itself(see [12]) and also $D(A) \subset D(A^{1/2})$, where $D(A^{\alpha})$ is the domain of $A^{\alpha}(\alpha = 1, \frac{1}{2})$. Moreover suppose that the injection from $D(A^{1/2})$ from H is compact. Note that $A = -\triangle$ is a nonnegative selfadjoint operator with compact resolvent in some Sobolev space.

Generalizing the problem (CP), we now shall consider nonlinear evolution equations of the form:

$$u''(t)+M(|A^{1/2}u(t)|^2)Au(t) + \delta Au'(t) = f(t) \text{ in } H \tag{2.1}$$

$$u(t) \in D(A) \text{ for any } t \in [0,T) \tag{2.2}$$

$$u(0) = u_0 \in D(A),\ u'(0) = u_1 \in D(A^{1/2}). \tag{2.3}$$

Throughout what follows we will denote V and W by

$$V := D(A) \text{ and } W := D(A^{1/2}), \text{ respectively.}$$

We assume the following on $M(t)$, and $f(t)$:

(A.1) Let $M(t)$ be a $C^1[0,\infty)$ function satisfying $M(t) \geq m_0$ $(m_0 > 0)$;
(A.2) $f \in L^1(0,T;H)$.

For the later use, we set

$$\overline{M}(r) = \int_0^r M(s)ds, \tag{2.4}$$

$$E(t) = \frac{1}{2}\Big[|u'(t)|^2 + \overline{M}(|A^{1/2}u(t)|^2)\Big]. \tag{2.5}$$

Note that by assumption $(A.1)$,

$$\overline{M}(r) \geq m_0 r \quad \text{on} \quad [0, \infty). \tag{2.6}$$

DEFINITION 2.1. A function $u(t) : [0, T) \to H$ is called a solution to the problem (2.1)-(2.3) on $[0, T)$ if

(i) $u \in L^\infty(0, T; V) \cap BC([0, T); W)$, $u' \in L^2(0, T; V) \cap L^\infty(0, T; W) \cap BC([0, T); H)$, $u'' \in L^2(0, T; H)$;
(ii) u satisfies (2.1) on $[0, T]$;
(iii) $u(0) = u_0$ and $u'(0) = u_1$.

Here $BC([0, T); H)$ denotes the set of all H-valued bounded continuous functions on $[0, T)$.

3. Global existence and uniqueness result

At first, we mention some useful facts about Yosida approximations of nonnegative selfadjoint operators.

Define the Yosida approximation A_λ of A by

$$A_\lambda = \lambda^{-1}(I - J_\lambda) = AJ_\lambda \quad \text{for} \quad \lambda > 0, \tag{3.1}$$

where $J_\lambda = (I + \lambda A)^{-1}$ and I is the identity on H. Then it is well known that $J_\lambda \to I$ strongly as $\lambda \to 0$ and

$$A_\lambda u \to Au \ (\lambda \to 0) \quad \text{for} \quad u \in D(A).$$

Next we consider the power of degree $\frac{1}{2}$ of A_λ :

$$A_\lambda^{1/2} = A^{1/2} J_\lambda^{1/2} \ (\lambda > 0), \tag{3.2}$$

where $J_\lambda^{1/2} = (I + \lambda A)^{-1/2}$. We obtain several basic properties of the operators $J_\lambda^{1/2}$ and $A_\lambda^{1/2}$(for proof, see [10]).

LEMMA 3.1. *Let $\lambda > 0$. Then*

(1) $||J_\lambda^{1/2}|| \le 1$ *and* $|A_\lambda^{1/2} u| \le |A^{1/2} u|$ *for* $u \in D(A^{1/2})$;

(2) $||A_\lambda^{1/2}|| \le \lambda^{-1/2}$;

(3) $|u - J_\lambda^{1/2} u| \le \lambda^{1/2} |A^{1/2} u|, u \in H$.

(4) $|u - J_\lambda^1 u| \le \lambda |Au|, u \in H$.

Here A_λ^γ is the power of degree γ of A_λ.

We next introduce the Bihari-type inequality without proof(see [5]).

LEMMA 3.2. *Let F and G be nonnegative continuous functions on $[0,T]$ $(T > 0)$. If*

$$[F(t)]^2 \le C + \int_0^t F(s)G(s)ds \quad on \quad [0,T],$$

then

$$F(t) \le C^{1/2} + \frac{1}{2}\int_0^t G(s)ds \quad on \quad [0,T],$$

where C is a positive constant.

From now on we shall let $\lambda > 0$ be any number and let A_λ be the Yosida approximation of A.

Approximate problems

First we consider the approximate problem of the following differential equation by applying the Yosida approximation :

$$u_\lambda''(t) + M(|A_\lambda^{1/2} u_\lambda(t)|^2) A_\lambda u_\lambda(t) + \delta A_\lambda u_\lambda'(t) = f(t), \tag{3.3}$$

$$u_\lambda(0) = u_0 \in V, \; u_\lambda'(0) = u_1 \in W. \tag{3.4}$$

By using the mean value theorem for $M(t)$, we can easily show that the mapping $u \to M(|A_\lambda^{1/2} u|^2) A_\lambda u$ is locally Lipschitz continuous for each λ. Then it is well known that the problem (3.3) and (3.4) has a unique local approximate solution $u_\lambda \in C^1([0, T_\lambda); H)$ on some interval $[0, T_\lambda)$ and moreover, $u_\lambda'(t)$ is absolutely continuous and (3.3) holds *a.e.*, on $[0, T_\lambda)$ (cf. [4]).

Now we shall see that $u_\lambda(t)$ can be extended to $[0, T)$. To see this, we need *a priori estimates* for the solution $u_\lambda(t)$.

A priori estimates

LEMMA 3.3. *Let $u_0 \in V$ and $u_1 \in W$. Then the following inequality holds on $[0, T_\lambda)$:*

$$\sup_{t\in[0,T_\lambda)} \Big\{ |u'_\lambda(t)|^2, m_0|A_\lambda^{1/2}u_\lambda(t)|^2, 2\delta \int_0^t |A_\lambda^{1/2}u'_\lambda(s)|^2 ds \Big\} \le B_0^2, \tag{3.5}$$

where B_0 is given by

$$B_0 = \sqrt{2}E(0)^{1/2} + \int_0^T |f(s)|ds, \tag{3.6}$$

and m_0 is the constant of assumption (A.2).

Proof. If we multiply (3.3) by $2u'_\lambda(t)$, then we obtain *a.e.* on $[0, T_\lambda)$,

$$\begin{aligned} &\frac{d}{dt}|u'_\lambda(t)|^2 + M(|A_\lambda^{1/2}u_\lambda(t)|^2)\frac{d}{dt}|A_\lambda^{1/2}u_\lambda(t)|^2 + 2\delta|A_\lambda^{1/2}u'_\lambda(t)|^2 \\ &= 2(f(t), u'_\lambda(t)). \end{aligned} \tag{3.7}$$

Integrating (3.7) on $(0,t), t \in [0, T_\lambda)$, we produce from (2.4)-(2.6) that

$$\begin{aligned} &|u'_\lambda(t)|^2 + m_0|A_\lambda^{1/2}u_\lambda(t)|^2 + 2\delta \int_0^t |A_\lambda^{1/2}u'_\lambda(s)|^2 ds \\ &\le 2E(0) + 2\int_0^t |f(t)||u'_\lambda(s)|ds. \end{aligned} \tag{3.8}$$

If we set

$$F(t) := \Big[|u'_\lambda(t)|^2 + m_0|A_\lambda^{1/2}u_\lambda(t)|^2 + 2\delta \int_0^t |A_\lambda^{1/2}u'_\lambda(s)|^2 ds \Big]^{1/2},$$

then we obtain by (3.8)

$$[F(t)]^2 \le 2E(0) + 2\int_0^t |f(s)|F(s)ds. \tag{3.9}$$

Therefore, our desired result follows from applying Lemma 3.2 to (3.9).□

We can easily show the extension of a local solution to the whole interval $[0, T)$ from Lemma 3.3.

Now we shall prove that $u_\lambda(t)$ and $u'_\lambda(t)$ are uniformly bounded in V and W, respectively.

We set $B_1 := \max\{|M(s)| : 0 \le s \le \frac{B_0^2}{m_0}\}$ and $B_2 := \max\{|M'(s)| : 0 \le s \le \frac{B_0^2}{m_0}\}$, where B_0 is the constant given by (3.6).

PROPOSITION 3.1. *Let $u_0 \in V$ and $u_1 \in W$. If $f(t) \in L^2(0,T;W)$, then there exists a positive constant M_1, which does not depend on λ such that*

$$\sup_{t\in[0,T)} \left\{ |A_\lambda u_\lambda(t)|, |A_\lambda^{1/2} u'_\lambda(t)|, \int_0^T |A_\lambda u'_\lambda(t)|^2 dt \right\} \le M_1.$$

Proof. If we multiply (3.3) by $2A_\lambda u'_\lambda(t)$, then we obtain *a.e.*, on $[0,T)$
(3.10)

$$\begin{aligned} &\frac{d}{dt}\{|A_\lambda^{1/2}u'_\lambda(t)|^2 + M(|A_\lambda^{1/2}u_\lambda(t)|^2)|A_\lambda u_\lambda(t)|^2\} + 2\delta|A_\lambda u'_\lambda(t)|^2 \\ &= 2(f(t), A_\lambda u'_\lambda(t)) + |A_\lambda u_\lambda(t)|^2 \frac{d}{dt} M(|A_\lambda^{1/2}u_\lambda(t)|^2). \end{aligned}$$

Integrating (3.10) on $(0,t)$, $t \in [0,T]$, we have
(3.11)

$$\begin{aligned} &|A_\lambda^{1/2}u'_\lambda(t)|^2 + M(|A_\lambda^{1/2}u_\lambda(t)|^2)|A_\lambda u_\lambda(t)|^2 + 2\delta \int_0^t |A_\lambda u'_\lambda(s)|^2 ds \\ &= |A^{1/2}u_1|^2 + M(|A^{1/2}u_0|^2)|Au_0|^2 + 2\int_0^t (A_\lambda^{1/2} f(s), A_\lambda^{1/2}u'_\lambda(s))ds \\ &\quad + 2\int_0^t M'(|A_\lambda^{1/2}u_\lambda(s)|^2)(A_\lambda^{1/2}u'_\lambda(s), A_\lambda^{1/2}u_\lambda(s))|A_\lambda u_\lambda(s)|^2 ds. \end{aligned}$$

Using the Schwarz inequality, (A.2) and (3.5), we produce by (3.11)

$$\begin{aligned} &|A_\lambda^{1/2}u'_\lambda(t)|^2 + m_0|A_\lambda u_\lambda(t)|^2 + 2\delta \int_0^t |A_\lambda u'_\lambda(s)|^2 ds \\ &\le |A^{1/2}u_1|^2 + B_1|Au_0|^2 + 2\int_0^t |A_\lambda^{1/2} f(s)||A_\lambda^{1/2}u'_\lambda(s)|ds \\ &\quad + \frac{2B_0B_2}{\sqrt{m_0}} \int_0^t |A_\lambda^{1/2}u'_\lambda(s)||A_\lambda u_\lambda(s)|^2 ds \\ &\le L(u_0,u_1,f,\delta) + \frac{2B_0B_2}{\sqrt{m_0}} \int_0^t |A_\lambda^{1/2}u'_\lambda(s)||A_\lambda u_\lambda(s)|^2 ds, \end{aligned} \tag{3.12}$$

where $L(u_0,u_1,f,\delta) = |A^{1/2}u_1|^2 + B_1|Au_0|^2 + \int_0^t |A_\lambda^{1/2} f(s)|^2 ds + \frac{B_0^2}{2\delta}$. Thus, applying Gronwall's inequality to (3.12) and using Lemma 3.3, we obtain the desired estimates. □

PROPOSITION 3.2. *Let $u_0 \in V$ and $u_1 \in W$. Then there exists a positive constant M_2, which does not depend on λ such that*

$$\int_0^t |u''_\lambda(s)|^2 ds \le M_2 \quad on \quad [0,T).$$

Proof. If we multiply (3.3) by $u''_\lambda(t)$, then we have *a.e.* on $[0,T)$

$$|u''_\lambda(t)|^2 + (M(|A_\lambda^{1/2}u_\lambda(t)|^2)A_\lambda u_\lambda(t) + \delta A_\lambda u'_\lambda(t) - f(t), u''_\lambda(t)) = 0.$$

Using the Schwarz inequality and integrating over $(0,t), t \in [0,T)$, we obtain,

$$\int_0^t |u''_\lambda(s)|^2 ds \le 2B_1 \int_0^t |A_\lambda u_\lambda(s)|^2 ds + 2\delta \int_0^t |A_\lambda u'_\lambda(s)|^2 ds + \int_0^t |f(s)|^2 ds.$$

Therefore, our result is an immediate consequence of Proposition 3.1.□

<u>*Passage to the limit*</u>

In this section we establish the uniform convergence of solutions to the problem (2.1)-(2.3) on finite intervals of arbitrary length as $\lambda \to 0$. In what follows we will let $\{\lambda_n\}_n$ be a sequence such that $\lambda_n > 0$ $(n \in \mathbb{N})$ and $\lambda_n \to 0$ as $n \to \infty$.

LEMMA 3.4. *If for any $\lambda > 0$, $u_\lambda(\cdot)$ is a solution to the problem (3.3) and (3.4), then there exist a subsequence $\{u_{\lambda_n}(\cdot)\}_n$ of $\{u_\lambda(\cdot)\}_\lambda$ and $u(\cdot) \in BC([0,T);H)$ such that*

$$u_{\lambda_n}(\cdot) \to u(\cdot) \quad in \quad C([0,T];H) \quad as \quad n \to \infty. \tag{3.13}$$

Moreover, there is a subsequence $\{u_{\mu_n}(\cdot)\}_n$ of $\{u_{\lambda_n}(\cdot)\}_n$ and $u'(\cdot) \in BC([0,T);H)$ such that

$$u'_{\mu_n}(\cdot) \to u'(\cdot) \quad in \quad C([0,T];H) \quad as \quad n \to \infty. \tag{3.14}$$

Here the convergence is uniform with respect to $t \in [0,T]$.

Proof. At first we show that for any $t \in [0,T)$, $\{J_\lambda^{1/2} u_\lambda(t)\}_\lambda$ is precompact in H. In fact, we have for any $t \in [0,T)$,

$$\begin{aligned} &|J_\lambda^{1/2} u_\lambda(t) - J_\mu^{1/2} u_\mu(t)|_W \\ &= |J_\lambda^{1/2} u_\lambda(t) - J_\mu^{1/2} u_\mu(t)| + |A_\lambda^{1/2} u_\lambda(t) - A_\mu^{1/2} u_\mu(t)| \\ &\le |u_\lambda(t)| + |u_\mu(t)| + |A_\lambda^{1/2} u_\lambda(t)| + |A_\mu^{1/2} u_\mu(t)|, \end{aligned} \tag{3.15}$$

where $\lambda > 0$ and $\mu > 0$ are arbitrary. From Lemma 3.3 and the definition of $A_\lambda^{1/2}$, (3.15) implies that for any $t \in [0,T)$, $\{J_\lambda^{1/2} u_\lambda(t)\}_\lambda$ is bounded in W. Thus we can see from the compactness that for any $t \in [0,T)$, $\{J_\lambda^{1/2} u_\lambda(t)\}_\lambda$ is precompact in H. Moreover, from (1) of Lemma 3.1 and (3.5), we can easily observe that $\{J_\lambda^{1/2} u_\lambda(\cdot)\}_\lambda$ is equicontinuous. Hence, applying the Arzela-Ascoli theorem to $\{J_\lambda^{1/2} u_\lambda(\cdot)\}_\lambda$ in $C([0,T];H)$, we can find a subsequence $\{J_{\lambda_n}^{1/2} u_{\lambda_n}(\cdot)\}_\lambda$ and $u(\cdot) \in BC([0,T);H)$ such that

$$J_{\lambda_n}^{1/2} u_{\lambda_n}(\cdot) \to u(\cdot) \ \text{ in } \ C([0,T];H) \ \text{ as } \ n \to \infty. \tag{3.16}$$

Noting that $|u_{\lambda_n}(t) - J_{\lambda_n}^{1/2} u_{\lambda_n}(t)| \le \lambda_n^{1/2} |A_{\lambda_n}^{1/2} u_{\lambda_n}(t)|$ (see (3) of Lemma 3.1, we can observe that $u_{\lambda_n}(\cdot) \to u(\cdot)$ in $C([0,T];H)$ as $n \to \infty$.

Noting that $J_{\lambda_n}^{1/2} u'_{\lambda_n}(\cdot)$ and $A_\lambda^{1/2} u_{\lambda_n}(\cdot)$ belong to $BC([0,T);H)$, we can also prove (3.14) in the same way as in the proof of (3.13). □

LEMMA 3.5. *Let $u(\cdot)$, $\{\lambda_n\}_n$, and $\{\mu_n\}_n$ be as in Lemma 3.4. Then $u(\cdot) \in L^\infty(0,T;V)$, $u'(\cdot) \in L^2(0,T;V)$, $u'(\cdot) \in L^\infty(0,T;W)$ and*

$$Au(t) = \textit{weak} \lim_{n\to\infty} A_{\lambda_n} u_{\lambda_n}(t) \ \textit{ in } \ H, \tag{3.17}$$

$$A^{1/2} u'(t) = \textit{weak} \lim_{n\to\infty} A_{\mu_n}^{1/2} u'_{\mu_n}(t) \ \textit{ in } \ H, \tag{3.18}$$

$$Au'(t) = \textit{weak} \lim_{n\to\infty} A_{\gamma_n} u'_{\gamma_n}(t) \ \textit{ in } \ L^2(0,T;H), \tag{3.19}$$

where $\{u_{\gamma_n}(\cdot)\}_n$ is a subsequence of $\{u_{\lambda_n}(\cdot)\}_n$. Moreover, $u(\cdot) \in BC([0,T);W)$ and

$$\begin{aligned} &M(|A^{1/2} u(t)|^2) Au(t) \\ &= \textit{weak} \lim_{n\to\infty} M(|A_{\lambda_n}^{1/2} u_{\lambda_n}(t)|^2) A_{\lambda_n} u_{\lambda_n}(t). \end{aligned} \tag{3.20}$$

Here the convergence is uniform with respect to $t \in [0,T]$.

Proof. We note that A^{α} is weakly closed and $D(A^{\alpha})$ is dense in H $(\alpha = 1, 1/2)$. From Proposition 3.1, we can observe that $A_{\lambda}u_{\lambda_n}(\cdot)$ and $A_{\mu_n}u'_{\mu_n}(\cdot)$ belong to $BC([0,T); H)$.

Thus (3.17), (3.18), and (3.19) follow from (3.13) and (3.14). We also have

$$
\text{(3.21)} \qquad
\begin{aligned}
|Au(t)| &\le \lim_{n\to\infty}\inf |A_{\lambda_n}u_{\lambda_n}(t)| \le M_1,\\
|A^{1/2}u'(t)| &\le \lim_{n\to\infty}\inf |A^{1/2}_{\mu_n}u'_{\mu_n}(t)| \le M_1,\\
\int_0^T |Au'(t)|^2 dt &\le \lim_{n\to\infty}\inf \int_0^T |A_{\gamma_n}u'_{\gamma_n}(t)|^2 dt \le M_1.
\end{aligned}
$$

These imply that

$$
u(\cdot) \in L^{\infty}(0,T;V), u'(\cdot) \in L^2(0,T;V) u'(\cdot) \in L^{\infty}(0,T;W).
$$

In order to prove (3.20), we first show that $u(\cdot) \in BC([0,T); W)$ and

$$
\text{(3.22)} \qquad A^{1/2}_{\lambda_n}u_{\lambda_n}(\cdot) \to A^{1/2}u(\cdot) \text{ in } C([0,T]; H) \text{ as } n \to \infty.
$$

From the definition of $A^{1/2}_{\lambda_n}$ and using the Schwarz inequality, we observe that

$$
\begin{aligned}
&|A^{1/2}_{\lambda_n}u_{\lambda_n}(t) - A^{1/2}u(t)|^2\\
&= |A^{1/2}_{\lambda_n}u_{\lambda_n}(t)|^2 - 2(Au(t), J^{1/2}_{\lambda_n}u_{\lambda_n}(t)) + |A^{1/2}u(t)|^2\\
&= |A^{1/2}_{\lambda_n}u_{\lambda_n}(t)|^2 - |A^{1/2}u(t)|^2 + 2(Au(t), u(t) - J^{1/2}_{\lambda_n}u_{\lambda_n}(t))\\
&\le |A^{1/2}_{\lambda_n}u_{\lambda_n}(t)|^2 - |A^{1/2}u(t)|^2 + 2|Au(t)||u(t) - J^{1/2}_{\lambda_n}u_{\lambda_n}(t)|.
\end{aligned}
$$

Thus it suffices by (3.16) and (3.21) to show that

$$
\text{(3.23)} \qquad (A_{\lambda_n}u_{\lambda_n}(t), u_{\lambda_n}(t)) \to (Au(t), u(t)) \text{ in } C[0,T] \text{ as } n \to \infty.
$$

Indeed, from (4) of Lemma 3.1 and using the Schwarz inequality, we have

$$
\begin{aligned}
&|(A_{\lambda_n}u_{\lambda_n}(t), u_{\lambda_n}(t)) - (Au(t), u(t))|\\
&= |(A_{\lambda_n}u_{\lambda_n}(t) - A_{\lambda_n}u(t), u_{\lambda_n}(t))\\
&\quad + (J_{\lambda_n}u(t), Au_{\lambda_n}(t)) - (Au(t), u(t))|\\
&= |(A_{\lambda_n}u_{\lambda_n}(t), u_{\lambda_n}(t) - u(t)) + (Au(t), J_{\lambda_n}u(t) - u(t))\\
&\quad + (A_{\lambda_n}u(t), u_{\lambda_n}(t) - u(t))|\\
&\le \lambda_n|Au(t)|^2 + (|A_{\lambda_n}u_{\lambda_n}(t)| + |A_{\lambda_n}u(t)|)|u_{\lambda_n}(t) - u(t)|.
\end{aligned}
$$

So (3.23) follows from (3.13) and (3.21). Hence we obtain (3.22) and we also have

$$|A^{1/2}u(t)| = \lim_{n\to\infty} |A_{\lambda_n}^{1/2} u_{\lambda_n}(t)| \le \frac{B_0}{\sqrt{m_0}},$$

where B_0 is the constant given in (3.6), that is, $u(\cdot) \in BC([0,T);W)$. Finally, by using the mean value theorem for $M(\cdot)$, our final assertion immediately follows from (3.5), (3.17), and (3.23). □

We are now in a position to show that $u(\cdot)$, given by Lemma 3.4, is a solution to the problem (2.1)–(2.3).

PROPOSITION 3.3. *Let $u(\cdot)$ and $\{\mu_n\}_n$ be as in Lemma 3.4. Then $u''(\cdot) \in L^2(0,T;H)$ and*

$$u''(t) + M(|A^{1/2}u(t)|^2)Au(t) + \delta Au'(t) = f(t) \quad a.e. \ in \ \ H.$$

Proof. We can observe that $u'(t)$ is Lipschitz continuous and so is absolutely continuous. Hence from Proposition 3.2, we can easily show that $u''(t) \in L^2(0,T;H)$ exists *a.e.*. We also see from (3.3), (3.14), and (3.20) that

$$f(t) - \delta u'(t) - M(|A^{1/2}u(t)|^2)Au(t) = weak \ \lim_{n\to\infty} u''_{\mu_n}(t). \tag{3.24}$$

Put $w_n(t) := \dfrac{u'_{\mu_n}(t) - u'_{\mu_n}(s)}{t-s}$ on $t \in [0,T]$. Here $s(\neq t)$ is arbitrary but fixed on $[0,T]$. Then clearly, $\lim_{t\to s} w_n(t) = u''_{\mu_n}(s)$ *a.e.* on $(0,T)$ and by virtue of (3.15), $\lim_{n\to\infty} w_n(t) = \dfrac{u'(t)-u'(s)}{t-s}$, uniformly on $[0,T]$. Hence we obtain by the continuity of $(\cdot,\cdot)$,

$$\lim_{n\to\infty}(u''_{\mu_n}(s), v) = \lim_{n\to\infty}(\lim_{t\to s} w_n(t), v) = (u''(s), v), \ \ v \in H. \tag{3.25}$$

So, our assertion follows from (3.24) and (3.25). □

Uniqueness

We need the following lemma for uniqueness, which is proved in [10].

LEMMA 3.6. *Let u and v be solutions to the problem (2.1)–(2.3). If $w(t) \in C^1([0,T),V)$ is a solution of*

$$w''(t)+M(|A^{1/2}u(t)|^2)Aw(t) + \delta Aw'(t) = F(u(t), v(t))$$
$$w(0) = 0, w'(0) = 0,$$

where $|F(u(t),v(t))| \leq M_4|A^{1/2}w(t)|$ for all $t \in [0,T)$ and some constant $M_4 > 0$. Then $w(t) \equiv 0$ for $t \in [0,T)$.

From Lemma 3.6, we can easily derive that the solution to the problem (2.1)–(2.3) is unique.

From lemmas and propositions above, we obtain the following main result :

THEOREM 3.1. *Let all assumptions (A.1) and (A.2) be satisfied and $(u_0, u_1, f) \in V \times W \times L^2(0,T;W)$. Then there exists a unique solution $u(t)$ on $[0,T]$ to the problem (2.1)–(2.3).*

4. An application for a Kirchhoff model

Let Ω be a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$. We consider the initial boundary value problem with Dirichlet boundary conditions of the form

$$u''(x,t) - (\alpha + \int_\Omega |\nabla u(x,t)|^2 dx)\triangle u(x,t) - \delta\triangle u'(x,t) = f(x,t),$$

(4.1) $\qquad x \in \Omega,\ t \in [0,T),$

(4.2) $\qquad u(x,t) = 0,\ x \in \partial\Omega,\ t \in [0,T),$

(4.3) $\qquad u(x,0) = u_0(x), u'(x,0) = u_1(x),\ x \in \Omega,$

where $\triangle$ and ∇ are the Laplacian and the gradient in $\mathbb{R}^n$, respectively, and α and δ are positive constants.
Let $H = L^2(\Omega)$ be the Hilbert space with inner product (,) and norm $||\cdot||$. Define a linear operator A in H by

$$Au = -\triangle u \ \text{ with } \ D(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

Here $H^\gamma(\Omega)$ is the usual Sobolev space of order γ and $H_0^\gamma(\Omega)$ is the closure of C_0^∞ in $H^\gamma(\Omega)$. It is well known that when $A = -\triangle$, $D(A^{1/2}) = H_0^1(\Omega)$, and $||A^{1/2}u|| = ||\nabla u||, u \in D(A^{1/2})$. By pointwise evaluation $u(x,t) = (u(t))(x)$, the problem (4.1)–(4.3) can be written in an abstract form (2.1)–(2.3).

Form Theorem 2.1, we obtain the following:

THEOREM 4.1. *Let* $(u_0, u_1, f) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times L^2(0,T; H_0^1(\Omega))$. *Then there exists a unique solution* $u(t)$ *on* $[0,T)$ *to the problem* (4.1)–(4.3) *such that*

$$u \in L^\infty(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \cap BC([0,T); H_0^1(\Omega)),$$
$$u' \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^\infty(0,T; H_0^1(\Omega) \cap BC([0,T); H),$$
$$u'' \in L^2(0,T; H).$$

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DEPARTMENT OF MATHEMATICS, PUSAN NATIONAL UNIVERSITY, PUSAN 609-735, KOREA

E-mail: jyepark@hyowon.pusan.ac.kr