J. Korean Math. Soc. **38** (2001), No. 1, pp. 61–76

# CONDITIONAL FOURIER-FEYNMAN TRANSFORMS AND CONDITIONAL CONVOLUTION PRODUCTS

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ABSTRACT. In this paper we define the concept of a conditional Fourier-Feynman transform and a conditional convolution product and obtain several interesting relationships between them. In particular we show that the conditional transform of the conditional convolution product is the product of conditional transforms, and that the conditional convolution product of conditional transforms is the conditional transform of the product of the functionals.

## 1. Introduction

Let $C_0[0,T]$ denote one-parameter Wiener space; that is the space of all $\mathbb{R}$-valued continuous functions $x$ on $[0,T]$ with $x(0)=0$. In defining various analytic Feynman integrals one usually starts, for $\lambda > 0$, with the Wiener integral

$$E[F] = \int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}x)m(dx) \tag{1.1}$$

and then extends analytically in $\lambda$ to the right-half complex plane. Throughout this paper our starting point is the (generalized) Wiener integral

$$E_x\big[F(\lambda^{-\frac{1}{2}}z(x,\cdot))\big] = \int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}z(x,\cdot))m(dx) \tag{1.2}$$

where $z$ is the Gaussian process

$$z(x,t) = \int_0^t h(s)dx(s) \tag{1.3}$$

Received July 14, 1999.
2000 Mathematics Subject Classification: 28C20, 60J65.
Key words and phrases: analytic Feynman integral, Wiener integral, conditional Fourier-Feynman transform, conditional convolution product.

with $h$ in $L_2[0,T]$, and where $\int_0^T h(s)dx(s) = \langle h, x\rangle$ denotes the Paley-Wiener-Zygmund stochastic integral. Of course if $h(t) \equiv 1$ on $[0,T]$, then $z(x,t) = x(t)$ and so the (generalized) Wiener integral in (1.2) reduces to the Wiener integral in (1.1). We will simply refer to the integral in (1.2) as a Wiener integral.

The concept of an $L_1$ analytic Fourier-Feynman transform (FFT) was introduced by Brue in [1]. In [2], Cameron and Storvick introduced an $L_2$ analytic FFT. In [14], Johnson and Skoug developed an $L_p$ analytic FFT for $1 \leq p \leq 2$ which extended the results in [1,2] and gave various relationships between the $L_1$ and $L_2$ theories. In [10], Huffman, Park and Skoug defined a convolution product (CP) for functionals on Wiener space and in [11,12] obtained various results involving and relating the FFT and the CP. In [13], they worked with (generalized) FFT's and (generalized) CP's using ideas and results from [8]. In this paper we define the concept of a (generalized) conditional FFT (CFFT) and a (generalized) conditional CP(CCP) and obtain several interesting relationships between them. In particular we show that the conditional transform of the conditional convolution product is the product of conditional transforms. We also show that the conditional convolution product of conditional transforms is the conditional transform of the product of the functionals. In this paper, for notational simplicity, we decided to work with $p = 1$; however many of our results also hold for $1 < p \leq 2$. In particular all of our results in section 4 hold for $1 \leq p \leq 2$.

## 2. Definitions and preliminaries

Let $h \in L_2[0,T]$ with $\|h\| > 0$, let $z(x,t)$ be given by (1.3) and let

$$a(t) = \int_0^t h^2(u)du. \tag{2.1}$$

Then $z$ is a Gaussian process with mean zero and covariance function

$$E[z(x,s)z(x,t)] = a(\min\{s,t\}).$$

In addition, $z(\cdot,t)$ is stochastically continuous in $t$ on $[0,T]$.

Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let $m$ denote Wiener measure. A subset $B$ of $C_0[0,T]$ is said to be scale-invariant measurable [5,15] provided $\rho B \in \mathcal{M}$ for all $\rho > 0$, and a scale-invariant measurable set $N$ is said to be scale-invariant null provided $m(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). If two functionals $F$ and $G$ are equal s-a.e., we write $F \approx G$.

For a rather detailed discussion of scale-invariant measurability and its relation with other topics see [15]. In [20], Segal gives an interesting discussion of the relation between scale change in Wiener space and certain questions in quantum field theory.

In [2,14] all of the functionals $F$ on Wiener space and all the $\mathbb{C}$-valued functions $f$ on $\mathbb{R}^n$ were assumed to be Borel measurable. But, as was pointed out in [15, p. 170], the concept of scale-invariant measurability in Wiener space and Lebesgue measurability in $\mathbb{R}^n$ is precisely correct for the analytic FFT theory and the analytic Feynman integration theory. Thus, throughout this paper, we assume that every functional $F$ we consider is s-a.e. defined, is scale-invariant measurable, and for each $\lambda > 0$, $F(\lambda^{-\frac{1}{2}}z(x,\cdot))$ is Wiener integrable in $x$ on $C_0[0,T]$.

First we state the definition of the (generalized) analytic Feynman integral of $F$ [8,13]. Let $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \text{Re } \lambda > 0\}$ and let $\tilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} : \lambda \neq 0 \text{ and Re } \lambda \geq 0\}$. Let $J(\lambda) = E[F(\lambda^{-\frac{1}{2}}z(x,\cdot))]$. If there exists a function $J^*(\lambda)$ analytic in $\lambda$ on $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is called the analytic Wiener integral of $F$ with parameter $\lambda$, and for $\lambda$ in $\mathbb{C}_+$ we write

$$E^{\text{anw}_\lambda}[F(z(x,\cdot))] = J^*(\lambda). \tag{2.2}$$

Let real $q \neq 0$ be given. Then we define the analytic Feynman integral of $F$ with parameter $q$ by ($\lambda \in \mathbb{C}_+$)

$$E^{\text{anf}_q}[F(z(x,\cdot))] = \lim_{\lambda \to -iq} E^{\text{anw}_\lambda}[F(z(x,\cdot))] \tag{2.3}$$

if the limit exists.

Next we state the definitions of the (generalized) $L_1$ analytic FFT and the (generalized) CP given in [13]. For $\lambda \in \mathbb{C}_+$ and $y \in C_0[0,T]$, let

$$T_\lambda(F)(y) = E_x^{\text{anw}_\lambda}[F(y + z(x,\cdot))]. \tag{2.4}$$

We define the $L_1$ analytic FFT, $T_q^{(1)}(F)$ of $F$, by the formula ($\lambda \in \mathbb{C}_+$)

$$T_q^{(1)}(F)(y) = \lim_{\lambda \to -iq} T_\lambda(F)(y) \tag{2.5}$$

if it exists. We note that $T_q^{(1)}(F)$ is only defined for s-a.e. $y \in C_0[0,T]$. Also if $T_q^{(1)}(F)$ exists and if $G \approx F$, then $T_q^{(1)}(G)$ exists and $T_q^{(1)}(G) \approx T_q^{(1)}(F)$.

We define the (generalized) convolution product $(F*G)_\lambda$ (if it exists) by

(2.6)
$$(F*G)_\lambda(y) = \begin{cases} E_x^{\text{anw}_\lambda}\left[F\left(\frac{y+z(x,\cdot)}{\sqrt{2}}\right)G\left(\frac{y-z(x,\cdot)}{\sqrt{2}}\right)\right], & \lambda\in\mathbb{C}_+ \\ E_x^{\text{anf}_q}\left[F\left(\frac{y+z(x,\cdot)}{\sqrt{2}}\right)G\left(\frac{y-z(x,\cdot)}{\sqrt{2}}\right)\right], & \lambda=-iq. \end{cases}$$

REMARK 1.

i) For all $\lambda\in\tilde{\mathbb{C}}_+$, $(F*G)_\lambda=(G*F)_\lambda$.

ii) When $\lambda=-iq$, we often denote $(F*G)_\lambda$ by $(F*G)_q$.

iii) For $\lambda>0$,

$$E_x^{\text{anw}_\lambda}[F(y+z(x,\cdot))] = E_x[F(y+\lambda^{-\frac{1}{2}}z(x,\cdot))]. \tag{2.7}$$

## 3. Conditional transforms and convolutions

For some related work involving conditional Wiener integrals see [7,9,17,22]. Throughout this paper we will always condition by

$$X(x)=z(x,T). \tag{3.1}$$

For $\lambda>0$ and $\eta\in\mathbb{R}$ let

$$J_\lambda(\eta)=E(F(\lambda^{-\frac{1}{2}}z(x,\cdot))|\lambda^{-\frac{1}{2}}z(x,T)=\eta) \tag{3.2}$$

denote the (generalized) conditional Wiener integral of $F(\lambda^{-\frac{1}{2}}z(x,\cdot))$ given $\lambda^{-\frac{1}{2}}z(x,T)$ [8]. If for almost all $\eta\in\mathbb{R}$, there exists a function $J_\lambda^*(\eta)$, analytic in $\lambda$ on $\mathbb{C}_+$ such that $J_\lambda^*(\eta)=J_\lambda(\eta)$ for $\lambda>0$ then $J_\lambda^*(\eta)$ is defined to be the conditional analytic Wiener integral of $F(z(x,\cdot))$ given $z(x,T)$ with parameter $\lambda$ and for $\lambda\in\mathbb{C}_+$ we write

$$J_\lambda^*(\eta)=E^{\text{anw}_\lambda}(F(z(x,\cdot))|z(x,T)=\eta). \tag{3.3}$$

If for fixed real $q\neq0$, $\lim_{\lambda\to-iq}J_\lambda^*(\eta)$ exists for almost all $\eta\in\mathbb{R}$, we denote the value of this limit by

$$E^{\text{anf}_q}(F(z(x,\cdot))|z(x,T)=\eta) \tag{3.4}$$

and call it the (generalized) conditional analytic Feynman integral of $F$ given $X$ with parameter $q$.

REMARK 2. In [18], Park and Skoug give a formula for expressing conditional Wiener integrals in terms of ordinary Wiener integrals; namely that for $\lambda > 0$,

$$\begin{aligned}(3.5)\qquad & E\big(F(\lambda^{-\frac{1}{2}}z(x,\cdot))|\lambda^{-\frac{1}{2}}z(x,T)=\eta\big) \\ &= E_x\left[F\left(\lambda^{-\frac{1}{2}}z(x,\cdot)-\lambda^{-\frac{1}{2}}z(x,T)\frac{a(\cdot)}{a(T)}+\frac{a(\cdot)\eta}{a(T)}\right)\right].\end{aligned}$$

Thus we have that

$$\begin{aligned}(3.6)\qquad & E^{\mathrm{anw}_\lambda}(F(z(x,\cdot))|z(x,T)=\eta) \\ &= E_x^{\mathrm{anw}_\lambda}\left[F\left(z(x,\cdot)-\frac{a(\cdot)}{a(T)}z(x,T)+\frac{a(\cdot)\eta}{a(T)}\right)\right]\end{aligned}$$

and

$$\begin{aligned}(3.7)\qquad & E^{\mathrm{anf}_q}(F(z(x,\cdot))|z(x,T)=\eta) \\ &= E_x^{\mathrm{anf}_q}\left[F\left(z(x,\cdot)-\frac{a(\cdot)}{a(T)}z(x,T)+\frac{a(\cdot)\eta}{a(T)}\right)\right]\end{aligned}$$

where in (3.6) and (3.7) the existence of either side implies the existence of the other side and their equality.

Next we define the (generalized) conditional FFT (CFFT) and the conditional convolution product (CCP). For $\lambda \in \mathbb{C}_+$, $\eta \in \mathbb{R}$ and $y \in C_0[0,T]$, let $T_\lambda(F|X)(y,\eta)$ denote the conditional analytic Wiener integral of $F(y+z(x,\cdot))$ given $X(x)=z(x,T)$; that is to say

$$\begin{aligned}(3.8)\qquad T_\lambda(F|X)(y,\eta) &= E^{\mathrm{anw}_\lambda}(F(y+z(x,\cdot))|z(x,T)=\eta) \\ &= E_x^{\mathrm{anw}_\lambda}\left[F\left(y+z(x,\cdot)-\frac{a(\cdot)}{a(T)}z(x,T)+\frac{a(\cdot)\eta}{a(T)}\right)\right].\end{aligned}$$

We define the CFFT, $T_q^{(1)}(F|X)(y,\eta)$ of $F$ by the formula

$$\begin{aligned}(3.9)\qquad T_q^{(1)}(F|X)(y,\eta) &= \lim_{\lambda\to -iq} T_\lambda(F|X)(y,\eta) \\ &= E_x^{anf_q}\left[F\left(y+z(x,\cdot)-\frac{a(\cdot)z(x,T)}{a(T)}+\frac{a(\cdot)\eta}{a(T)}\right)\right]\end{aligned}$$

if it exists, and we define the CCP $((F*G)_\lambda|X)(y,\eta)$ (if it exists) by the formula

(3.10)

$$\begin{aligned}
&\Big((F*G)_\lambda|X\Big)(y,\eta)\\
&=\begin{cases} E_x^{anw_\lambda}\left(F\left(\frac{y+z(x,\cdot)}{\sqrt{2}}\right)G\left(\frac{y-z(x,\cdot)}{\sqrt{2}}\right)\Big|z(x,T)=\eta\right), & \lambda\in\mathbb{C}_+\\ E_x^{anf_q}\left(F\left(\frac{y+z(x,\cdot)}{\sqrt{2}}\right)G\left(\frac{y-z(x,\cdot)}{\sqrt{2}}\right)\Big|z(x,T)=\eta\right), & \lambda=-iq\end{cases}\\
&=\begin{cases} E_x^{anw_\lambda}\Big[F\Big(\frac{y+z(x,\cdot)}{\sqrt{2}}-\frac{a(\cdot)z(x,T)}{a(T)\sqrt{2}}+\frac{a(\cdot)\eta}{a(T)\sqrt{2}}\Big)\\ \qquad\cdot G\Big(\frac{y-z(x,\cdot)}{\sqrt{2}}+\frac{a(\cdot)z(x,T)}{a(T)\sqrt{2}}-\frac{a(\cdot)\eta}{a(T)\sqrt{2}}\Big)\Big], & \lambda\in\mathbb{C}_+\\ E_x^{anf_q}\Big[F\Big(\frac{y+z(x,\cdot)}{\sqrt{2}}-\frac{a(\cdot)z(x,T)}{a(T)\sqrt{2}}+\frac{a(\cdot)\eta}{a(T)\sqrt{2}}\Big)\\ \qquad\cdot G\Big(\frac{y-z(x,\cdot)}{\sqrt{2}}+\frac{a(\cdot)z(x,T)}{a(T)\sqrt{2}}-\frac{a(\cdot)\eta}{a(T)\sqrt{2}}\Big)\Big], & \lambda=-iq\,.\end{cases}
\end{aligned}$$

Again if $\lambda=-iq$, we will denote $((F*G)_\lambda|X)(y,\eta)$ by $((F*G)_q)|X)(y,\eta)$.

LEMMA 1. *If $((F*G)_q|X)$ exists then $((G*F)_q|X)$ exists and*

$$((G*F)_q|X)(y,\eta)=((F*G)_q|X)(y,-\eta). \tag{3.11}$$

*Proof.* Clearly it suffices to show that

$$((G*F)_\lambda|X)(y,\eta)=((F*G)_\lambda)|X)(y,-\eta)$$

for all $\lambda>0$. But this follows from (3.10), (2.7), and the fact that if $H(\lambda^{-\frac{1}{2}}z(x,\cdot))$ is Wiener integrable in $x$ on $C_0[0,T]$, then

$$E\big[H\big(-\lambda^{-\frac{1}{2}}z(x,\cdot)\big)\big]=E\big[H\big(\lambda^{-\frac{1}{2}}z(-x,\cdot)\big)\big]=E\big[H\big(\lambda^{-\frac{1}{2}}z(x,\cdot)\big)\big].$$ □

THEOREM 1. *Assume that $T_q^{(1)}(F)$, $T_q^{(1)}(G)$ and $T_q^{(1)}((F*G)_q)$ all exist at $q\in\mathbb{R}-\{0\}$. Then*

$$T_q^{(1)}((F*G)_q)(y)=T_q^{(1)}(F)(y/\sqrt{2})T_q^{(1)}(G)(y/\sqrt{2}) \tag{3.12}$$

*for s-a.e. $y\in C_0[0,T]$.*

*Proof.* In view of (2.5), (2.4) and (2.7) it will suffice to show that $T_\lambda((F*G)_\lambda)(y) = T_\lambda(F)(y/\sqrt{2})T_\lambda(G)(y/\sqrt{2})$ for $\lambda > 0$. But for all $\lambda > 0$,

$$\begin{aligned} T_\lambda((F*G)_\lambda)(y) =& E_x\Bigg[E_w\Bigg[F\Bigg(\frac{y+\lambda^{-\frac{1}{2}}z(x,\cdot)+\lambda^{-\frac{1}{2}}z(w,\cdot)}{\sqrt{2}}\Bigg) \\ &\cdot G\Bigg(\frac{y+\lambda^{-\frac{1}{2}}z(x,\cdot)-\lambda^{-\frac{1}{2}}z(w,\cdot)}{\sqrt{2}}\Bigg)\Bigg]\Bigg]. \end{aligned}$$

But, $\frac{z(x,\cdot)+z(w,\cdot)}{\sqrt{2}}$ and $\frac{z(x,\cdot)-z(w,\cdot)}{\sqrt{2}}$ are independent Gaussian processes, and each is equivalent to $z(x,\cdot)$. Hence

$$\begin{aligned} &T_\lambda((F*G)_\lambda)(y) \\ =&E_x\Bigg[F\Bigg(\frac{y}{\sqrt{2}}+\lambda^{-\frac{1}{2}}z(x,\cdot)\Bigg)\Bigg]E_x\Bigg[G\Bigg(\frac{y}{\sqrt{2}}+\lambda^{-\frac{1}{2}}z(x,\cdot)\Bigg)\Bigg] \\ =&T_\lambda(F)(y/\sqrt{2})T_\lambda(G)(y/\sqrt{2}) \end{aligned}$$

which concludes the proof of Theorem 1. □

We are now ready to establish one of our main results; namely that the conditional transform of the conditional convolution product is the product of the conditional transforms.

THEOREM 2. *Assume that* $T_q^{(1)}$ $(((F*G)_q|X)(\cdot,\eta_1)|X)$ $(y,\eta_2)$, $T_q^{(1)}$ $(F|X)$ *and* $T_q^{(1)}(G|X)$ *all exist at* $q \in \mathbb{R}-\{0\}$. *Then*

(3.13)

$$\begin{aligned} &T_q^{(1)}(((F*G)_q|X)(\cdot,\eta_1)|X)(y,\eta_2) \\ &\quad = T_q^{(1)}(F|X)\Bigg(y/\sqrt{2},\frac{\eta_2+\eta_1}{\sqrt{2}}\Bigg)T_q^{(1)}(G|X)\Bigg(y/\sqrt{2},\frac{\eta_2-\eta_1}{\sqrt{2}}\Bigg) \end{aligned}$$

*for s-a.e.* $y \in C_0[0,T]$.

*Proof.* Again, as noted in the proof of Theorem 1, we only need to consider the case where $\lambda > 0$. But using (2.7), (3.8) and (3.10), we observe that for all $\lambda > 0$,

$$\begin{aligned}
&T_\lambda(((F*G)_\lambda|X)(\cdot,\eta_1)|X)(y,\eta_2)\\
=&E_x\biggl(E_w\biggl(F\Bigl(\frac{y+\lambda^{-\frac12}z(x,\cdot)+\lambda^{-\frac12}z(w,\cdot)}{\sqrt2}\Bigr)\\
&\cdot G\Bigl(\frac{y+\lambda^{-\frac12}z(x,\cdot)-\lambda^{-\frac12}z(w,\cdot)}{\sqrt2}\Bigr)\Big|\lambda^{-\frac12}z(w,T)=\eta_1\biggr)\Big|\lambda^{-\frac12}z(x,T)=\eta_2\biggr)\\
=&E_x\biggl[E_w\biggl[F\Bigl(\frac{y}{\sqrt2}+\frac{1}{\sqrt{2\lambda}}\Bigl(z(x,\cdot)-\frac{a(\cdot)z(x,T)}{a(T)}+z(w,\cdot)-\frac{a(\cdot)z(w,T)}{a(T)}\Bigr)\\
&\qquad+\frac{a(\cdot)}{a(T)}(\eta_2+\eta_1)\Bigr)\\
&\cdot G\Bigl(\frac{y}{\sqrt2}+\frac{1}{\sqrt{2\lambda}}\Bigl(z(x,\cdot)-\frac{a(\cdot)z(x,T)}{a(T)}-z(w,\cdot)+\frac{a(\cdot)z(w,T)}{a(T)}\Bigr)\\
&\qquad+\frac{a(\cdot)}{a(T)}(\eta_2-\eta_1)\Bigr)\biggr]\biggr].
\end{aligned}$$

Now, $z(x,\cdot)-\frac{a(\cdot)z(x,T)}{a(T)}+z(w,\cdot)-\frac{a(\cdot)z(w,T)}{a(T)}$ and $z(x,\cdot)-\frac{a(\cdot)z(x,T)}{a(T)}-z(w,\cdot)+\frac{a(\cdot)z(w,T)}{a(T)}$ are independent processes as can be seen by checking their covariance functions. Hence the expectation of $FG$ equals the product of the expectations and so using (3.10) and (3.8) we see that

$$\begin{aligned}
&T_\lambda(((F*G)_\lambda|X)(\cdot,\eta_1)|X)(y,\eta_2)\\
=&E_x\biggl(E_w\biggl(F\Bigl(\frac{y+\lambda^{-\frac12}z(x,\cdot)+\lambda^{-\frac12}z(w,\cdot)}{\sqrt2}\Bigr)\Big|\lambda^{-\frac12}z(w,T)=\eta_1\biggr)\\
&\qquad\Big|\lambda^{-\frac12}z(x,T)=\eta_2\biggr)\\
&\cdot E_x\biggl(E_w\biggl(G\Bigl(\frac{y+\lambda^{-\frac12}z(x,\cdot)-\lambda^{-\frac12}z(w,\cdot)}{\sqrt2}\Bigr)\Big|\lambda^{-\frac12}z(w,T)=\eta_1\biggr)\\
&\qquad\Big|\lambda^{-\frac12}z(x,T)=\eta_2\biggr).
\end{aligned}$$

Now $\frac{z(x,\cdot)+z(w,\cdot)}{\sqrt2}$ is equivalent to $z(x,\cdot)$ and so is $\frac{z(x,\cdot)-z(w,\cdot)}{\sqrt2}$. Hence for $\lambda > 0$,

$$\begin{aligned}
&T_\lambda(((F*G)_\lambda|X)(\cdot,\eta_1)|X)(y,\eta_2)\\
&= E_x\Big(F\big(\frac{y}{\sqrt{2}}+\lambda^{-\frac{1}{2}}z(x,\cdot)\big)\Big|\lambda^{-\frac{1}{2}}z(x,T)=\frac{\eta_2+\eta_1}{\sqrt{2}}\Big)\\
&\quad\cdot E_x(G(\frac{y}{\sqrt{2}}+\lambda^{-\frac{1}{2}}z(x,\cdot))\Big|\lambda^{-\frac{1}{2}}z(x,T)=\frac{\eta_2-\eta_1}{\sqrt{2}}\Big)\\
&= T_\lambda(F|X)\Big(\frac{y}{\sqrt{2}},\frac{\eta_2+\eta_1}{\sqrt{2}}\Big)T_\lambda(G|X)\Big(\frac{y}{\sqrt{2}},\frac{\eta_2-\eta_1}{\sqrt{2}}\Big).
\end{aligned}$$

□

## 4. Conditional transforms and convolutions for the Banach algebra $\mathcal{S}$

The Banach algebra $\mathcal{S}$ was introduced by Cameron and Storvick in [3] and consists of functionals expressible in the form

$$F(x)=\int_{L_2[0,T]}\exp\Big\{i\int_0^T v(t)dx(t)\Big\}df(v) \tag{4.1}$$

for s-a.e. $x\in C_0[0,T]$ where $f$ is an element of $M(L_2[0,T])$, the space of all $\mathbb{C}$-valued countably additive finite Borel measures on $L_2[0,T]$. Further work on $\mathcal{S}$ shows that it contains many functionals of interest in Feynman integration theory [4, 6, 16, 19, 21].

From [13, p. 23] we have that for each $u\in L_2[0,T]$ and each $h\in L_\infty[0,T]$,

$$\int_0^T u(s)dz(x,s)=\int_0^T u(s)h(s)dx(s) \tag{4.2}$$

for s-a.e $x\in C_0[0,T]$, and if $F\in\mathcal{S}$, then $H(x)=F(z(x,\cdot))$ belongs to $\mathcal{S}$. Thus, throughout this section we require $h$ to be in $L_\infty[0,T]$, rather than simply in $L_2[0,T]$.

In this section we use the following well-known Wiener integration formula several times. For $\lambda>0$ and $u\in L_2[0,T]$,

$$E\Big[\exp\Big\{\frac{i}{\sqrt{\lambda}}\langle u,x\rangle\Big\}\Big]=\exp\Big\{-\frac{\|u\|^2}{2\lambda}\Big\} \tag{4.3}$$

where $\langle u,x\rangle$ denotes the Paley-Wiener-Zygmund integral $\int_0^T u(t)dx(t)$.

In our first lemma we show that $T_q^{(1)}(F|X)$ exists for all $F\in\mathcal{S}$.

LEMMA 2. *Let $F \in \mathcal{S}$ be given by (4.1). Then for all $q \in \mathbb{R} - \{0\}$,*

$$T_q^{(1)}(F|X)(y,\eta)$$

$$= \int_{L_2[0,T]} \exp\left\{ i\langle u,y\rangle + i\eta b - \frac{i}{2q}\int_0^T h^2(t)[u(t)-b]^2 dt \right\} df(u)$$

$$= \int_{L_2[0,T]} \exp\left\{ i\langle u,y\rangle + i\eta b - \frac{i\|uh\|^2}{2q} + \frac{ib^2 a(T)}{2q} \right\} df(u) \tag{4.4}$$

*for s-a.e. $y \in C_0[0,T]$ where $\langle u,y\rangle = \int_0^t u(s)dy(s)$ and $b = \frac{1}{a(T)} \int_0^T u(t)h^2(t)dt$.*

*Proof.* Using (3.8), (2.7), the Fubini theorem, and (4.3), we see that for $\lambda > 0$,

$$T_\lambda(F|X)(y,\eta)$$

$$= E_x\left[ \int_{L_2[0,T]} \exp\left\{ i\int_0^T u(t)d\left( y(t) + \lambda^{-\frac{1}{2}} z(x,t) - \frac{\lambda^{-\frac{1}{2}} a(t) z(x,T)}{a(T)} + \frac{a(t)\eta}{a(T)} \right) \right\} df(u) \right]$$

$$= \int_{L_2[0,T]} \exp\left\{ i\int_0^T u(t)dy(t) + \frac{i\eta}{a(T)} \int_0^T u(t)h^2(t)dt \right\}$$

$$\cdot E_x\left[ \exp\left\{ \frac{i}{\sqrt{\lambda}} \int_0^T u(t)h(t)dx(t) - \frac{i}{\sqrt{\lambda}a(T)} \int_0^T u(t)h^2(t)dt \int_0^T h(t)dx(t) \right\} \right] df(u)$$

$$= \int_{L_2[0,T]} \exp\left\{ i\langle u,y\rangle + i\eta b \right\} \cdot E_x\left[ \exp\left\{ \frac{i}{\sqrt{\lambda}} \int_0^T h(t)[u(t)-b]dx(t) \right\} \right] df(u)$$

$$= \int_{L_2[0,T]} \exp\left\{ i\langle u,y\rangle + i\eta b - \frac{1}{2\lambda}\int_0^T h^2(t)[u(t)-b]^2 dt \right\} df(u).$$

But the last expression above is an analytic function of $\lambda$ in $\mathbb{C}_+$ and is a bounded continuous function of $\lambda$ for all $\lambda \in \tilde{\mathbb{C}}_+$ since $f$ is a finite Borel measure. Hence $T_q^{(1)}(F|X)$ exists and is given by (4.4). □

In our next lemma, for $F$ and $G$ in $\mathcal{S}$, we obtain a formula for the conditional convolution product $((F * G)_q|X)(y,\eta)$.

LEMMA 3. *Let $F$ and $G$ be elements of $\mathcal{S}$ with corresponding finite Borel measures $f$ and $g$ in $M(L_2[0,T])$. Then for all $q \in \mathbb{R} - \{0\}$ and s-a.e. $y \in C_0[0,T]$,*

$$
\begin{aligned}
&((F * G)_q|X)(y,\eta) \\
&= \int_{L_2^2[0,T]} \exp\left\{ \frac{i}{\sqrt{2}}\langle u+v, y\rangle + \frac{i\eta(b-c)}{\sqrt{2}} \right\} \\
&\quad \cdot \exp\left\{ -\frac{i}{4q}\int_0^T h^2(t)[u(t)-v(t)-(b-c)]^2 dt \right\} df(u)dg(v)
\end{aligned} \tag{4.5}
$$

*where $b = \frac{1}{a(T)}\int_0^T u(t)h^2(t)dt$ and $c = \frac{1}{a(T)}\int_0^T v(t)h^2(t)dt$.*

*Proof.* Using (3.10), (2.7), the Fubini theorem, and (4.3), and proceeding as in the proof of Lemma 2 above, we obtain for all $\lambda > 0$ that the expression $((F * G)_\lambda|X)(y,\eta)$ equals the expression

(4.6)

$$
\begin{aligned}
&\int_{L_2^2[0,T]} \exp\left\{ \frac{i}{\sqrt{2}}\langle u+v, y\rangle + \frac{i\eta}{\sqrt{\lambda}a(T)}\int_0^T h^2(t)[u(t)-v(t)]dt \right\} \\
&\quad \cdot E_x\left[ \exp\left\{ \frac{i}{\sqrt{2\lambda}}\int_0^T h(t)[(u(t)-b)-(v(t)-c)]dx(t) \right\} \right] df(u)dg(v) \\
&= \int_{L_2^2[0,T]} \exp\left\{ \frac{i}{\sqrt{2}}\langle u+v, y\rangle + \frac{i\eta(b-c)}{\sqrt{2}} \right. \\
&\quad \left. - \frac{1}{4\lambda}\int_0^T h^2(t)[(u(t)-v(t))-(b-c)]^2 dt \right\} df(u)dg(v).
\end{aligned}
$$

But (4.5) now follows directly from (4.6) since the last expression in (4.6) above is an analytic function of $\lambda$ in $\mathbb{C}_+$ and is a bounded continuous function of $\lambda$ throughout $\tilde{\mathbb{C}}_+$. □

Our next lemma shows that the conclusions of Theorem 2 hold if $F$ and $G$ are in $\mathcal{S}$.

LEMMA 4. *Let $F$ and $G$ be as in Lemma 3. Then for all real $q \neq 0$,*

$$
\begin{aligned}
&T_q^{(1)}(((F * G)_q|X)(\cdot,\eta_1)|X)(y,\eta_2) \\
&= T_q^{(1)}(F|X)\left(y/\sqrt{2}, \frac{\eta_2+\eta_1}{\sqrt{2}}\right) T_q^{(1)}(G|X)\left(y/\sqrt{2}, \frac{\eta_2-\eta_1}{\sqrt{2}}\right)
\end{aligned} \tag{4.7}
$$

*for s-a.e. $y \in C_0[0,T]$.*

*Proof (outline).* Proceeding as in the proofs of the above lemmas, a direct calculation shows that the left-hand side of (4.7) equals the expression

$$\int_{L_2^2[0,T]} \exp\left\{ \frac{i}{\sqrt{2}}\langle u+v, y\rangle + \frac{ib(\eta_2+\eta_1)}{\sqrt{2}} + \frac{ic(\eta_2-\eta_1)}{\sqrt{2}} \right\}$$
$$\cdot \exp\left\{ -\frac{i}{4q}\int_0^T h^2(t)[u(t)-v(t)-(b-c)]^2 dt \right\}$$
$$\cdot \exp\left\{ -\frac{i}{4q}\int_0^T h^2(t)[u(t)+v(t)-(b+c)]^2 dt \right\} df(u)dg(v),$$

which, upon simplification, equals the right-hand side of (4.7). □

In our next theorem we obtain an expression for the conditional convolution product of conditional transforms.

THEOREM 3. *Let $F$ and $G$ be as in Lemma 3 and let $q \in \mathbb{R}-\{0\}$. Then for $s$ - a.e. $y \in C_0[0,T]$,*

$$\begin{aligned}
&\left(\left(T_q^{(1)}(F|X)(\cdot,\eta_1) * T_q^{(1)}(G|X)(\cdot,\eta_2)\right)_{-q} |X\right)(y,\eta_3) \\
&= \int_{L_2^2[0,T]} \exp\left\{ \frac{i}{\sqrt{2}}\langle u+v, y\rangle + i(\eta_1 b + \eta_2 c) \right. \\
&\qquad\qquad \left. + \frac{i\eta_3}{\sqrt{2}}(b-c) + \frac{ia(T)(b+c)^2}{4q} \right\} \\
&\qquad \cdot \exp\left\{ -\frac{i}{4q}\int_0^T h^2(t)[u(t)+v(t)]^2 dt \right\} df(u)dg(v).
\end{aligned} \tag{4.8}$$

*Proof.* Using (3.10), (4.4), the Fubini theorem and (4.3), we obtain that for $\lambda > 0$,

$$\Big(\Big(T_q^{(1)}(F|X)(\cdot,\eta_1) * T_q^{(1)}(G|X)(\cdot,\eta_2)\Big)_\lambda |X\Big)(y,\eta_3)$$
$$= E_x\bigg[T_q^{(1)}(F|X)\Big(\frac{y}{\sqrt{2}} + \frac{z(x,\cdot)}{\sqrt{2\lambda}} - \frac{a(\cdot)z(x,T)}{a(T)\sqrt{2\lambda}} + \frac{a(\cdot)\eta_3}{a(T)\sqrt{2}},\eta_1\Big)$$
$$\cdot T_q^{(1)}(G|X)\Big(\frac{y}{\sqrt{2}} - \frac{z(x,\cdot)}{\sqrt{2\lambda}} + \frac{a(\cdot)z(x,T)}{a(T)\sqrt{2\lambda}} - \frac{a(\cdot)\eta_3}{a(T)\sqrt{2}},\eta_2\Big)\bigg]$$
$$= \int_{L_2^2[0,T]} \exp\Big\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle + ib\eta_1 + ic\eta_2 + \frac{i\eta_3(b-c)}{\sqrt{2}}\Big\}$$
$$\cdot \exp\Big\{\frac{ia(T)}{2q}(b^2+c^2) - \frac{i}{2q}\Big(\|uh\|^2 + \|vh\|^2\Big)\Big\}$$
$$\cdot E_x\bigg[\exp\Big\{\frac{i}{\sqrt{2\lambda}}\int_0^T h(t)[u(t)-v(t)-(b-c)]dx(t)\Big\}\bigg]df(u)dg(v)$$
$$= \int L_2^2[0,T] \exp\Big\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle + ib\eta_1 + ic\eta_2 + \frac{i\eta_3(b-c)}{\sqrt{2}}\Big\}$$
$$\cdot \exp\Big\{\frac{ia(T)}{2q}(b^2+c^2) - \frac{i}{2q}\Big(\|uh\|^2 + \|vh\|^2\Big)\Big\}$$
$$\cdot \exp\Big\{-\frac{1}{4\lambda}\int_0^T h^2(t)[u(t)-v(t)-(b-c)]^2dt\Big\}df(u)dg(v)\ .$$

But the last expression above is an analytic function of $\lambda$ in $\mathbb{C}_+$ and is a bounded continuous function of $\lambda$ in $\widetilde{\mathbb{C}_+}$, and so setting $\lambda = -i(-q) = iq$ yields (4.8) upon use of the equality

$$\frac{ia(T)}{2q}(b^2+c^2) + \frac{i}{4q}\int_0^T h^2(t)(b-c)^2dt$$
$$= \frac{ia(T)}{4q}(b+c)^2 + \frac{2i}{4q}\int_0^T h^2(t)(u(t)-v(t))(b-c)dt. \qquad \square$$

In our next theorem we obtain an expression for the conditional transform of a product of functionals in $\mathcal{S}$.

THEOREM 4. *Let $F$ and $G$ be as in Lemma 3 and let $q \in \mathbb{R} - \{0\}$. Then for s-a.e. $y \in C_0[0,T]$,*

$$T_q^{(1)}(F(\cdot/\sqrt{2})G(\cdot/\sqrt{2})|X)(y,\eta)$$
$$= \int_{L_2^2[0,T]} \exp\Big\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle + \frac{i\eta(b+c)}{\sqrt{2}} + \frac{ia(T)(b+c)^2}{4q}\Big\} \tag{4.9}$$
$$\cdot \exp\Big\{-\frac{i}{4q}\int_0^T h^2(t)[u(t)+v(t)]^2dt\Big\}df(u)dg(v).$$

*Proof.* Using (3.8), the Fubini theorem and (4.3), we obtain that for all $\lambda > 0$,

$$T_\lambda(F(\cdot/\sqrt{2})G(\cdot/\sqrt{2})|X)(y,\eta)$$
$$= \int_{L_2^2[0,T]} \exp\left\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle + \frac{i\eta(b+c)}{\sqrt{2}}\right\}$$
$$\cdot \exp\left\{-\frac{i}{4\lambda}\int_0^T h^2(t)[u(t)-v(t)-(b+c)]^2 dt\right\} df(u)dg(v).$$

Again letting $\lambda \to -iq$, and simplifying, yields (4.9). □

A close examination of the right-hand sides of (4.8) and (4.9) shows that they are equal if $\{\eta_1,\eta_2,\eta_3,\eta\}$ is in the solution set of the system

$$\begin{cases} \eta - \sqrt{2}\,\eta_1 - \eta_3 = 0 \\ \eta - \sqrt{2}\,\eta_2 + \eta_3 = 0. \end{cases} \tag{4.10}$$

THEOREM 5. *Let $F$ and $G$ be as in Lemma 3 and let $\{\eta_1,\eta_2,\eta_3,\eta\}$ satisfy the system (4.10). Then for all $q \in \mathbb{R} - \{0\}$,*

$$\begin{aligned} &((T_q^{(1)}(F|X)(\cdot,\eta_1)*T_q^{(1)}(G|X)(\cdot,\eta_2))_{-q}|X)(y,\eta_3) \\ &\qquad = T_q^{(1)}(F(\cdot/\sqrt{2})G(\cdot/\sqrt{2})|X)(y,\eta) \end{aligned} \tag{4.11}$$

*for s-a.e. $y \in C_0[0,T]$.*

Following are some interesting special cases of (4.11):

(a) $$\left(\left(T_q^{(1)}(F|X)(\cdot,\eta_1) * T_q^{(1)}(G|X)(\cdot,\eta_2)\right)_{-q}|X\right)\left(y,\frac{\eta_2-\eta_1}{\sqrt{2}}\right) = T_q^{(1)}(F(\cdot/\sqrt{2})G(\cdot/\sqrt{2})|X)\left(y,\frac{\eta_2+\eta_1}{\sqrt{2}}\right).$$

(b) $$\left(\left(T_q^{(1)}(F|X)\left(\cdot,\frac{\eta-\eta_3}{\sqrt{2}}\right) * T_q^{(1)}(G|X)\left(\cdot,\frac{\eta+\eta_3}{\sqrt{2}}\right)\right)_{-q}|X\right)(y,\eta_3) = T_q^{(1)}(F(\cdot/\sqrt{2})G(\cdot/\sqrt{2})|X)(y,\eta).$$

(c) $$((T_q^{(1)}(F|X)(\cdot,\eta) * T_q^{(1)}(G|X)(\cdot,2\eta))_{-q}|X)(y,\eta/\sqrt{2}) = T_q^{(1)}(F(\cdot/\sqrt{2})G(\cdot/\sqrt{2})|X)\left(y,\frac{3\eta}{\sqrt{2}}\right).$$

ACKNOWLEDGEMENT. The second author expresses his gratitude for the hospitality of Idaho State University where much of the research was carried out.

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