J. Korean Math. Soc. **38** (2001), No. 1, pp. 49–59

MODIFIED CONDITIONAL YEH-WIENER INTEGRAL WITH VECTOR-VALUED CONDITIONING FUNCTION

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ABSTRACT. In this paper we introduce the modified conditional Yeh-Wiener integral. To do so, we first treat the modified Yeh-Wiener integral. And then we obtain the simple formula for the modified conditional Yeh-Wiener integral and evaluate the modified conditional Yeh-Wiener integral for certain functional using the simple formula obtained. Here we consider the functional on a set of continuous functions which are defined on various regions, for example, triangular, parabolic and circular regions.

1. Introduction

The Wiener space of functions of two variables is the collection of continous function $\{f(x,y)\}$ on the unit square $[0,1]\times[0,1]$ satisfying $f(x,y)=0$ for $xy=0$. Integration on this space was first introduced by T. Kitagawa ([7]). Yeh ([14]) treated the integration of this space for more general functions and made a firm logical foundation of this space. We call this space as a Yeh-Wiener measure space and integral as a Yeh-Wiener integral.

In [16,17], Yeh introduced the conditional expectation and conditional Wiener integral and evaluated the conditional Wiener integral for real-valued conditioning function using the inversion formulae. Chang and the author treated the conditional Wiener integral for vector-valued conditioning function ([5]). In [6], Chung and Ahn considered the conditional Yeh-Wiener integral for real-valued conditioning function.

Park and Skoug ([8,9]) introduced the simple formula for conditional Wiener integral and for conditional Yeh-Wiener integral. Chang, Chung, Ahn and the author ([5,6,17]) used the Yeh's inversion formulae to evaluate the conditional Wiener integral and the conditional Yeh-Wiener integral,

Received July 14, 1999.
2000 Mathematics Subject Classification: 60J65, 28C20.
Key words and phrases: Modified conditional Yeh-Wiener integral, modified conditional Yeh-Wiener integral, simple formula for modified conditional Yeh-Wiener integral.

but the Yeh's inversion formulae are very complicated to evaluate the conditional Wiener and Yeh-Wiener integral. Using the simple formula for conditional Yeh-Wiener integral, Park and Skoug treated the conditional Yeh-Wiener integral for sample path-valued, multiple path-valued, generalized sample path-valued, boundary-valued conditioning functions ([10, 11, 12, 13]).

The purpose of this paper is to introduce the modified conditional Yeh-Wiener integral. To do so, we first treat the modified Yeh-Wiener integral. And then we obtain the simple formula for the modified conditional Yeh-Wiener integral and finally we evaluate the modified conditional Yeh-Wiener integral for certain functional using the simple formula obtained. In [9], Park and Skoug treated the conditional Yeh-Wiener integral for the functional on a set of continous functions which are defined only on a rectangle. But, in this paper, we consider the set of continous functions on various regions, for example, triangular, parabolic and circular regions.

2. Modified conditional Yeh-Wiener integral

Let $g(x)$ be a strictly decreasing function on $[0,S]$ such that $g(S)=0$ and $g(0)=T$ and let $\Omega=\{(x,y)\mid 0\le x\le S\ ,\ 0\le y\le g(x)\}$. And let $C(\Omega)$ denote the space of all real-valued continous functions $f(x,y)$ on a triangle Ω such that $f(x,0)=f(0,y)=0$ for every (x,y) in Ω.

For each partition $\tau=\{(x_i,y_j)\mid i,j=1,2,\cdots,n\}$ of Ω with $0=x_0<x_1<\cdots<x_n=S$ and $y_i=g(x_{n-i}), i=0,1,2,\cdots,n$, define $X_\tau : C(\Omega)\to R^{\frac{n(n-1)}{2}}$ by $X_\tau(f)=(f(x_1,y_1),\cdots,f(x_1,y_{n-1}),f(x_2,y_1),\cdots,f(x_2,y_{n-2}),f(x_3,y_1),\cdots,f(x_{n-1},y_1))$. Let $\mathcal{B}^{\frac{n(n-1)}{2}}$ be the $\sigma-$algebra of Borel sets in $R^{\frac{n(n-1)}{2}}$. And let E be a Borel measurable set in $\mathcal{B}^{\frac{n(n-1)}{2}}$ and let a set of the type

$$(2.1)\qquad I=\{f\in C(\Omega)\mid X_\tau(f)\in E\}$$

be given. The measure m of such a set is given by

$$(2.2)\quad m(I)=\overbrace{\int\cdots\int_E}^{\frac{n(n-1)}{2}} W(x_1\ ,\ \cdots\ ,\ x_n\ ,\ y_1\ ,\ \cdots\ ,\ y_n)\,du_{1,1}\cdots du_{n-1,1}$$

where

$$
\begin{aligned}
(2.3)\quad & W(x_1, \cdots, x_n, y_1, \cdots, y_n) \\
&= (2\pi)^{-\frac{n(n-1)}{4}} \left[{x_1}^{n-1} (x_2 - x_1)^{n-2} \cdots (x_{n-1} - x_{n-2}) \right]^{-\frac{1}{2}} \\
& \left[{y_1}^{n-1} (y_2 - y_1)^{n-2} \cdots (y_{n-1} - y_{n-2}) \right]^{-\frac{1}{2}} \\
& exp\Big\{ -\frac{1}{2} \Big[\sum_{j=1}^{n-1} \frac{(u_{1,j} - u_{1,j-1})^2}{x_1 (y_j - y_{j-1})} + \sum_{j=1}^{n-2} \frac{(u_{2,j} - u_{2,j-1} - u_{1,j} + u_{1,j-1})^2}{(x_2 - x_1)(y_j - y_{j-1})} \\
& + \cdots + \frac{(u_{n-1,1} - u_{n-2,1})^2}{(x_{n-1} - x_{n-2}) y_1} \Big] \Big\}
\end{aligned}
$$

with $u_{j,0} = 0, j = 1, 2, \cdots, n-1$. Let $\mathcal{I}$ be the collection of subsets of the type (2.1). Then it can be shown that $\mathcal{I}$ is an interval class or semi-algebra of subsets of $C(\Omega)$ and the set function m defined by (2.2) is a measure defined on the interval class $\mathcal{I}$ and the factor W in (2.3) is chosen as to make $C(\Omega) = 1$ ([14]). The measure m can be extended to a measure on the Caratheodory extension of interval class in the usual way. With this Caratheodory extension measurable functionals on $C(\Omega)$ may be defined and their integration on $C(\Omega)$ can be considered.

It is well known ([15]) that if $G(u_{1,1}, \cdots, u_{n-1,1})$ is a Lebesgue measurable function on $R^{\frac{n(n-1)}{2}}$ and if $F : C(\Omega) \to R$ is defined by $F(f) = G(f(x_1, y_1), \cdots, f(x_{n-1}, y_1))$, then

$$
\begin{aligned}
(2.4)\quad & \int_{C(\Omega)} F(f) dm(f) \\
&= \int_{R^{\frac{n(n-1)}{2}}} G(u_{1,1}, \cdots, u_{n-1,1}) W(x_1, \cdots, x_n, y_1, \cdots, y_n) \\
& \qquad du_{1,1} \cdots du_{n-1,1}.
\end{aligned}
$$

Here we call $E(F) = \int_{C(\Omega)} F(f)\, dm(f)$ as a modified Yeh-Wiener integral if it exists. Using (2.4), we easily obtain that a process $\{f(x,y), (x,y) \in \Omega\}$ has mean $E(f(x,y)) = \int_{C(\Omega)} f(x,y)\, dm(f) = 0$ and covariance $E[f(x,y) f(u,v)] = \min\{x,u\}\min\{y,v\}$. Here we call the process $\{f(x,y), (x,y) \in \Omega\}$ as the modified Yeh-Wiener process.

Let F be a real-valued integrable function on $C(\Omega)$ and let P_{X_τ} be the probability distribution of X_τ defined by $P_{X_\tau}(B) = m(X_\tau^{-1}(B))$ for B in

$\mathcal{B}^{\frac{n(n-1)}{2}}$. Then, by the definition of conditional expectation ([16]), for each function F in $L_1(C(\Omega))$,

$$(2.5)\qquad \int_{X_\tau^{-1}(B)} F(f)\, dm(f) = \int_B E(F(f)|X_\tau(f) = \vec{\xi})\, dP_{X_\tau}(\vec{\xi})$$

for B in $\mathcal{B}^{\frac{n(n-1)}{2}}$ and $E(F(f)|X_\tau(f) = \vec{\xi})$ is a Borel measurable function of $\vec{\xi}$ which is unique up to Borel null sets in $R^{\frac{n(n-1)}{2}}$. Here we call $E(F|X_\tau)(\vec{\xi}) = E(F(f)|X_\tau(f) = \vec{\xi})$ as a modified conditional Yeh-Wiener integral of F given by X_τ.

3. Simple formula for modified conditional Yeh-Wiener integral

For each partition $\tau = \tau_n$ of Ω and $f \in C(\Omega)$, define the modified quasi-polyhedric function $[f]$ on Ω by

(3.1)

$$\begin{aligned}[][f](x,y) = f(x_{i-1}, y_{j-1}) &+ \frac{x - x_{i-1}}{\Delta_i x}(f(x_i, y_{j-1}) - f(x_{i-1}, y_{j-1})) \\ &+ \frac{y - y_{j-1}}{\Delta_j y}(f(x_{i-1}, y_j) - f(x_{i-1}, y_{j-1})) \\ &+ \frac{(x - x_{i-1})(y - y_{j-1})}{\Delta_i x \Delta_j y}\Delta_{ij}\ f(x,y)\end{aligned}$$

on each $\Omega_{ij} \equiv (x_{i-1}, x_i] \times (y_{j-1}, y_j]$ in Ω, $i,j = 1,2,\cdots,n$, where $\Delta_i x = x_i - x_{i-1}, \Delta_j y = y_i - y_{j-1}$ and $\Delta_{ij} f(x,y) = f(x_i,y_j) - f(x_{i-1},y_j) - f(x_i, y_{j-1}) + f(x_{i-1}, y_{j-1})$, and

$$\begin{aligned}(3.2)\quad [f](x,y) = f(x_{i-1}, y_{n-i}) &+ \frac{x - x_{i-1}}{\Delta_i x}(f(x_i, y_{n-i}) - f(x_{i-1}, y_{n-i})) \\ &+ \frac{y - y_{n-i}}{\Delta_{n-i+1} y}(f(x_{i-1}, y_{n-i+1}) - f(x_{i-1}, y_{n-i}))\end{aligned}$$

on each $\Omega_i \equiv \{(x,y) \in \Omega \mid x_{i-1} < x \le x_i\ ,\ y_{n-i} < y \le g(x)\}, i = 1,2,\cdots,n$, and $[f](x,y) = 0$ if $xy = 0$.

Similarly, for each $\vec{\eta} = (\eta_{1,1}, \cdots, \eta_{1,n-1}, \eta_{2.1}, \cdots, \eta_{n-1,1}) \in R^{\frac{n(n-1)}{2}}$, define the modified quasi-polyhedric function $[\vec{\eta}]$ on Ω by

$$\begin{aligned}(3.3)\qquad [\vec{\eta}](x,y) = \eta_{i-1,j-1} &+ \frac{x - x_{i-1}}{\Delta_i x}(\eta_{i,j-1} - \eta_{i-1,j-1}) \\ &+ \frac{y - y_{j-1}}{\Delta_j y}(\eta_{i-1,j} - \eta_{i-1,j-1}) \\ &+ \frac{(x - x_{i-1})(y - y_{j-1})}{\Delta_i x \Delta_j y}\Delta_{ij}\vec{\eta}\end{aligned}$$

on each Ω_{ij}, where $\Delta_{ij}\vec{\eta} = \eta_{i,j} - \eta_{i-1,j} - \eta_{i,j-1} + \eta_{i-1,j-1}$, and

$$(3.4) \qquad [\vec{\eta}](x,y) = \eta_{i-1,n-i} + \frac{x - x_{i-1}}{\Delta_i x}(\eta_{i,n-i} - \eta_{i-1,n-i}) + \frac{y - y_{n-i}}{\Delta_{n-i+1} y}(\eta_{i-1,n-i+1} - \eta_{i-1,n-i})$$

on each Ω_i, $\eta_{i,0} = \eta_{0,j} = 0$ and $[\vec{\eta}](x,y) = 0$ if $xy = 0$.

We note that both $[f]$ and $[\vec{\eta}]$ belong to $C(\Omega)$ for each f in $C(\Omega)$ and each $\vec{\eta}$ in $R^{\frac{n(n-1)}{2}}$. And $[f](x_i,y_j) = f(x_i,y_j)$ and $[\vec{\eta}](x_i,y_j) = \eta_{i,j}$ for all i and j. On each Ω_{ij} ana Ω_i, each $[f](x,y)$ and $[\vec{\eta}](x,y)$ is a linear function of one variable for each value of the other variable.

The following theorem plays a key role in this paper.

THEOREM 3.1. *If $\{f(x,y),(x,y) \in \Omega\}$ is the modified Yeh-Wiener process, then the process $\{f(x,y) - [f](x,y),(x,y) \in \Omega\}$ and $X_\tau(f) = (f(x_1,y_1)\ ,\cdots, f(x_1,y_{n-1}), f(x_2,y_1),\cdots, f(x_{n-1},y_1))$ are (stochastically) independent.*

Proof. Let (x_p,y_q) be in τ. Since $E[f(x,y)f(u,v)] = (x \wedge u)(y \wedge v)$, it is easy to show that $E(f(x_p,y_q)(f(x,y) - [f](x,y))) = 0$ for (x,y) in Ω_{ij}. By (3.2), we have

$$(3.5) \qquad \begin{aligned} f(x,y) - [f](x,y) = {} & f(x,y) - f(x_{i-1},y_{n-i}) \\ & - \frac{x - x_{i-1}}{\Delta_i x}(f(x_i,y_{n-i}) - f(x_{i-1},y_{n-i})) \\ & - \frac{y - y_{n-i}}{\Delta_{n-i+1} y}(f(x_{i-1},y_{n-i+1}) - f(x_{i-1},y_{n-i})) \end{aligned}$$

for (x,y) in Ω_i. Thus it suffices to show that (3.5) is independent of $f(x_p,y_q)$.

For each Ω_i and (x_p,y_q) in τ, we have three cases:

$$(3.6) \qquad \begin{array}{lll} (i) & x_p \le x_{i-1}, & y_q \le y_{n-i} \\ (ii) & x_p \le x_{i-1}, & y_{n-i+1} \le y_q \\ (iii) & x_i \le x_p, & y_q \le y_{n-i}. \end{array}$$

For each case in (3.6), we can obtain $E\{f(x_p,y_q)[f(x,y) - [f](x,y)]\} = 0$ for (x,y) in Ω_i using (3.5). Because both $f(x_p,y_q)$ and $f(x,y) - [f](x,y)$ are Gaussian and uncorrelated, we may conclude that they are independent.□

From Theorem 3.1 and the definition of $[f]$ on Ω, we obtain that the two processes $\{f(x,y)-[f](x,y),(x,y)\in\Omega\}$ and $\{[f](x,y),(x,y)\in\Omega\}$ are independent. Using a similar techniqnue as in the proof of Theorem 2 in [9] we have the following theorem which also plays an important role to obtain a simple formula for modified conditional Yeh-Wiener integral.

THEOREM 3.2. *Let $F\in L_1(C(\Omega),m)$. Then for every $B\in\mathcal{B}^{\frac{n(n-1)}{2}}$,*

$$\int_{X_\tau^{-1}(B)} F(f)\,dm(f) = \int_B E[F(f-[f]+[\vec{\eta}])]\,dP_{X_\tau}(\vec{\eta}). \tag{3.7}$$

From (2.5) and (3.7), we may conclude that for F in $L_1(C(\Omega),m)$, $E(F(f)|X_\tau(f)=\vec{\eta})$ and $E[F(f-[f]+[\vec{\eta}])]$ are equal for a.e. $\vec{\eta}$ in $R^{\frac{n(n-1)}{2}}$. But while the former is Borel measurable by definition, the latter may only be Lebesgue measurable.

We note that if $h(\vec{\eta})$ is Lebesgue measurable on $R^{\frac{n(n-1)}{2}}$, then there exists a Borel measurable function $\hat{h}(\vec{\eta})$, which is unique up to Borel null set, such that $\hat{h}(\vec{\eta})=h(\vec{\eta})$ a.e. on $R^{\frac{n(n-1)}{2}}$. Thus we define $\hat{E}[F(f-[f]+[\vec{\eta}])]$ by any Borel measurable function of $\vec{\eta}$ which is equal to $E[F(f-[f]+[\vec{\eta}])]$ for a.e. $\vec{\eta}$ in $R^{\frac{n(n-1)}{2}}$ for F in $L_1(C(\Omega))$.

Thus we have the following simple formula for the modified conditional Yeh-Wiener integral which is simple to apply in application.

THEOREM 3.3. *If F is in $L_1(C(\Omega),m)$, then*

$$E(F(f)\mid X_\tau(f)=\vec{\eta})=\hat{E}[F(f-[f]+[\vec{\eta}])]. \tag{3.8}$$

In particular, if F is Borel measurable, then

$$E(F(f)\mid X_\tau(f)=\vec{\eta})=E[F(f-[f]+[\vec{\eta}])]. \tag{3.9}$$

The equalities in (3.8) and (3.9) mean that both sides are Borel measurable function of $\vec{\eta}$ and they are equal except for Borel null sets.

4. Examples of modified conditional Yeh-Wiener integrals

In [9], Park and Skoug treated conditional Yeh-Wiener integral for the functional F on $C(\Omega)$ which is a set of continous function f on the rectangle $\Omega=[0,S]\times[0,T]$ satisfying $f(x,y)=0$ for $xy=0$. That is, they treated the function g on $[0,S]$ given by $g(x)=T$ for x in $[0,S]$.

In this section we treat the region Ω as the triangular, parabolic and circular region rather than rectangular region in [9].

EXAMPLE 1. Let Ω be a triangular region in the first quadrant given by $\Omega = \{(x,y) \mid 0 \le x \le S\ ,\ 0 \le y \le g(x)\}$ for $g(x) = -\frac{T}{S}x + T$. And let F on $C(\Omega)$ be given by $F(f) = \int_\Omega f(x,y)\,dx\,dy$. Then the modified conditional Yeh-Wiener integral of F given X_τ at $\vec{\eta}$ is

$$E(F|X_\tau)(\vec{\eta}) = \int_\Omega E\Big(f(x,y) - [f](x,y) + [\vec{\eta}](x,y)\Big)\,dx\,dy. \tag{4.1}$$

The equality in (4.1) comes from Theorem 3.2 and the Fubini theorem. Since $E(f) = E([f]) = 0$, we have

$$\begin{aligned} E(F|X_\tau)(\vec{\eta}) &= \int_\Omega [\vec{\eta}](x,y)\,dx\,dy \\ &= \sum_{i=1}^{n-1}\sum_{j=1}^{n-i}\int_{\Omega_{ij}} [\vec{\eta}](x,y)\,dx\,dy + \sum_{i=1}^{n}\int_{\Omega_i} [\vec{\eta}](x,y)\,dx\,dy. \end{aligned} \tag{4.2}$$

Here we easily obtain

$$\int_{\Omega_{ij}} [\vec{\eta}](x,y)\,dx\,dy = \frac{\Delta_i x \Delta_j y}{4}(\eta_{i-1,j-1} + \eta_{i-1,j} + \eta_{i,j-1} + \eta_{i,j}). \tag{4.3}$$

And let $\Omega_i = \{(x,y) \in \Omega \mid x_{i-1} < x \le x_i\ ,\ y_{j-1} < y \le -\frac{T}{S}x + T\}$, with $i + j - 1 = n$. On Ω_i, $[\vec{\eta}]$ can be represented by

$$\begin{aligned} [\vec{\eta}](x,y) = \eta_{i-1,j-1} &+ \frac{x - x_{i-1}}{\Delta_i x}(\eta_{i,j-1} - \eta_{i-1,j-1}) \\ &+ \frac{y - y_{j-1}}{\Delta_j y}(\eta_{i-1,j} - \eta_{i-1,j-1}). \end{aligned} \tag{4.4}$$

Using the expression in (4.4), we get

(4.5)

$$\begin{aligned} &\int_{\Omega_i} [\vec{\eta}](x,y)\,dy\,dx \\ &= \int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{-\frac{T}{S}x+T} \{\eta_{i-1,j-1} + \frac{x - x_{i-1}}{\Delta_i x}(\eta_{i,j-1} - \eta_{i-1,j-1}) \\ &+ \frac{y - y_{j-1}}{\Delta_j y}(\eta_{i-1,j} - \eta_{i-1,j-1})\}\,dy\,dx \\ &= \frac{\Delta_i x \Delta_j y}{6}(\eta_{i-1,j-1} + \eta_{i-1,j} + \eta_{i,j-1}). \end{aligned}$$

The last equality in (4.5) comes from the fact that $y_j = -\frac{T}{S}x_{i-1} + T$, $y_{j-1} = -\frac{T}{S}x_i + T$ and $\Delta_j y = \frac{T}{S}\Delta_i x$ for $i,j = 0,1,2,\cdots,n$. Since $i+j-1=n$, we have

$$(4.6)\qquad \begin{aligned} &E(F \mid X_\tau)(\vec{\eta}) \\ &= \sum_{i=1}^{n-1}\sum_{j=1}^{n-i} \frac{\Delta_i x \Delta_j y}{4}(\eta_{i-1,j-1} + \eta_{i-1,j} + \eta_{i,j-1} + \eta_{i,j}) \\ &+ \sum_{i=1}^{n} \frac{\Delta_i x \Delta_{n-i+1} y}{6}(\eta_{i-1,n-i} + \eta_{i-1,n-i+1} + \eta_{i,n-i}). \end{aligned}$$

for $\vec{\eta}$ in $R^{\frac{n(n-1)}{2}}$.

EXAMPLE 2. Let Ω be the parabolic region in the first quadrant given by $\Omega = \{(x,y) \mid 0 \le x \le S \ , \ 0 \le y \le g(x)\}$ for $g(x) = -\frac{T}{S^2}x^2 + T$. And let F on $C(\Omega)$ be given by $F(f) = \int_\Omega f(x,y)\, dx\, dy$. Then, from (4.1) and (4.2), we have

$$(4.7)\qquad \begin{aligned} &E(F|X_\tau)(\vec{\eta}) \\ &= \sum_{i=1}^{n-1}\sum_{j=1}^{n-i} \int_{\Omega_{ij}} [\vec{\eta}](x,y)\, dx\ dy + \sum_{i=1}^{n} \int_{\Omega_i} [\vec{\eta}](x,y)\, dx\ dy. \end{aligned}$$

Let $\Omega_i = \{(x,y) \in \Omega \mid x_{i-1} < x \le x_i \ , \ y_{j-1} < y \le -\frac{T}{S^2}x^2 + T\}$, with $i+j-1=n$ for $i = 0,1,2,\cdots,n$. From (4.4), we obtain

$$(4.8)\qquad \begin{aligned} &\int_{\Omega_i} [\vec{\eta}](x,y)\ dxdy \\ &= \Big\{ \frac{T}{3S^2}\eta_{i-1,j-1}(2x_i + x_{i-1}) + \frac{T}{12S^2}(\eta_{i,j-1} - \eta_{i-1,j-1})(3x_i + x_{i-1}) \\ &+ \frac{T}{30S^2}(\eta_{i-1,j} - \eta_{i-1,j-1})(8x_i^2 + 9x_i x_{i-1} + 3x_{i-1}^2)\frac{\Delta_i x}{\Delta_j y}\Big\}(\Delta_i x)^2 \\ &= \frac{T}{60S^2}\Big\{20\eta_{i-1,j-1}(2x_i + x_{i-1}) + 5(\eta_{i,j-1} - \eta_{i-1,j-1})(3x_i + x_{i-1}) \\ &+ \frac{2S^2}{T}(\eta_{i-1,j} - \eta_{i-1,j-1})R(x)\Big\}(\Delta_i x)^2 \end{aligned}$$

where

$$R(x) = \frac{8x_i^2 + 9x_i x_{i-1} + 3x_{i-1}^2}{x_i + x_{i-1}}.$$

In (4.8), we used the fact that $y_j = -\frac{T}{S^2}x_{i-1}^2 + T$, $y_{j-1} = -\frac{T}{S^2}x_i^2 + T$ and $\Delta_j y = \frac{T}{S^2}(x_i + x_{i-1})\Delta_i x$. Thus we can evaluate the modified conditional Yeh-Wiener integral $E(F|X_\tau)(\vec{\eta})$ from (4.7), (4.3), (4.8) and $i+j-1=n$.

EXAMPLE 3. Let Ω be the circular region in the first quadrant given by $\Omega = \{(x,y) \mid 0 \le x \le T\ ,\ 0 \le y \le g(x)\}$ for $g(x) = \sqrt{T^2 - x^2}$. And let F on $C(\Omega)$ be given by $F(f) = \int_\Omega f(x,y)\ dxdy$. Then to evaluate the modified conditional Yeh-Wiener integral $E(F|X_\tau)(\vec{\eta})$, we first consider the set $\Omega_i = \{(x,y) \in \Omega \mid x_{i-1} < x \le x_i\ ,\ y_{j-1} < y \le \sqrt{T^2 - x^2}\}$ with $i+j-1=n$ for $i = 0,1,2,\cdots,n$. Here the area of Ω_i, $A(\Omega_i)$, can be easily obtained by

$$A(\Omega_i) = \frac{1}{2}(x_{i-1}y_{j-1} - x_{i-1}y_j + T^2 \sin^{-1}\frac{x_i y_j - x_{i-1}y_{j-1}}{T^2}). \tag{4.9}$$

From (4.4) and (4.9), we obtain

(4.10)
$$\begin{aligned}
&\int_{\Omega_i} [\vec{\eta}](x,y)\ dy\ dx \\
&= A(\Omega_i)\Big[\eta_{i-1,j-1} - \frac{x_{i-1}}{\Delta_i x}(\eta_{i,j-1} - \eta_{i-1,j-1}) - \frac{y_{j-1}}{\Delta_j y}(\eta_{i-1,j} - \eta_{i-1,j-1})\Big] \\
&+ \frac{1}{6}\Big[\frac{(\Delta_j y)^2}{\Delta_i x}(2y_j + y_{j-1})(\eta_{i,j-1} - \eta_{i-1,j-1}) \\
&+ \frac{(\Delta_i x)^2}{\Delta_j y}(2x_i + x_{i-1})(\eta_{i-1,j} - \eta_{i-1,j-1})\Big].
\end{aligned}$$

In (4.9) and (4.10), we used the fact that $y_j^2 = T^2 - x_{i-1}^2$, $y_{j-1}^2 = T^2 - x_i^2$ and $(x_i + x_{i-1})\Delta_i x = (y_j + y_{j-1})\Delta_j y$. From (4.2), (4.3), (4.9), (4.10) and $i+j-1=n$, we obtain the modified conditional Yeh-Wiener integral

(4.11)
$$\begin{aligned}
E(F \mid X_\tau)(\vec{\eta}) &= \sum_{i=1}^{n-1}\sum_{j=1}^{n-i} \frac{\Delta_i x \Delta_j y}{4}(\eta_{i-1,j-1} + \eta_{i-1,j} + \eta_{i,j-1} + \eta_{i,j}) \\
&+ \sum_{i=1}^{n}\Big\{A(\Omega_i)\Big[\eta_{i-1,n-i} - \frac{x_{i-1}}{\Delta_i x}(\eta_{i,n-i} - \eta_{i-1,n-i})
\end{aligned}$$

$$
\begin{aligned}
&- \frac{y_{n-i}}{\Delta_{n-i+1}y}(\eta_{i-1,n-i+1} - \eta_{i-1,n-i})\Big] \\
&+ \frac{1}{6}\Big[\frac{(\Delta_{n-i+1}y)^2}{\Delta_i x}(2y_{n-i+1} + y_{n-i})(\eta_{i,n-i} - \eta_{i-1,n-i}) \\
&+ \frac{(\Delta_i x)^2}{\Delta_{n-i+1}y}(2x_i + x_{i-1})(\eta_{i-1,n-i+1} - \eta_{i-1,n-i})\Big]
\end{aligned}
$$

for $\vec{\eta}$ in $R^{\frac{n(n-1)}{2}}$.

Acknowledgement. We would like to give our thanks to Professor Park for his advise to write this paper.

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