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STRICT STATIONARITY AND FUNCTIONAL CENTRAL LIMIT THEOREM FOR ARCH/GARCH MODELS

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ABSTRACT. In this paper we consider the (generalized) autoregressive models with conditional heteroscedasticity (ARCH/GARCH models). We will give conditions under which strict stationarity, ergodicity and the functional central limit theorem hold for the corresponding models.

1. Introduction

Let $\eta_n, n \in \mathbf{Z}$ be a sequence of independent and identically distributed(i.i.d.) random variables. The generalized autoregressive conditional heteroscedastic model of order p and q (GARCH(p,q)) $\{Y_n : n \in \mathbf{Z}\}$ is then given by $Y_n = \eta_n\sqrt{V_n}$ and $V_n = \delta + \sum_{i=1}^{p}\beta_i V_{n-i} + \sum_{i=1}^{q}\alpha_i Y_{n-i}^2$, $n \in \mathbf{Z}$. For $p = 0$ the process reduces to the autoregressive conditional heteroscedastic model of order q (ARCH(q)). ARCH process was introduced by Engel(1982) and was extended to GARCH process by Bollerslev(1986). While conventional time series and econometric models operate under the assumption of constant variance, ARCH/GARCH process admits a nonconstant conditional variance given the past information. In ARCH(q) process the conditional variance is specified as a linear function of the past sample variances only, whereas the GARCH(p,q) process allows lagged conditional variance to enter as well.

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The ARCH/GARCH processes have been proved useful in modelling various economic phenomena and have received a great amount of attention in the economic literature. Statistical properties of this parametric class of models have been studied by many authors, for example, Weiss(1984), Nelson(1990), Bera and Higgins(1992), Guégan and Diebolt(1994), Borkovec(2000) and references therein. Given such models, interests are the conditions under which such a model has properties such as strict stationarity, ergodicity, existence of moments and central limit theorem. Those properties are of great importance in statistical inference for time series models. These kinds of results for ARCH/GARCH models can be found in Bollerslev(1986), Nelson(1990), Bougerol and Picard(1992), Lu(1996), An, Chen and Huang(1997), etc.

Bollerslev(1986) showed that if $\delta > 0$, the GARCH model defines a second order stationary solution if and only if $\sum_{i=1}^{q} \alpha_i + \sum_{i=1}^{p} \beta_i < 1$. Nelson(1990) gave the necessary and sufficient conditions for the strict stationarity and ergodicity for the GARCH$(1,1)$ model. Guégan and Diebolt(1994) proved the geometrical ergodicity and existence of moments of the $\beta-$ARCH(1) model. Lu(1996) obtained a sharp condition under which ARCH(p) process is geometrically ergodic. An, Chen and Huang(1997) showed the geometric ergodicity and the existence of higher order moments for $\beta-$ARCH(p) model. Bougerol and Picard (1992) examined that the GARCH (p,q) has a unique strictly stationary ergodic solution if and only if the Lyapounov exponent associated with the appropriately given matrices $\{A_n\}$ is strictly negative. However, as they pointed out in their paper, the conditions are very difficult to verify and can only be estimated by Monte Carlo simulations. Therefore we need to find conditions easy to check. There are a lot of literatures considering the functional central limit theorem for various types of nonlinear time series. For example, see Bhattacharya and Lee(1988), Meyn and Tweedie(1993), Glynn and Meyn(1996), Lee(1997), Benda(1998) etc. Rudolph(1998) considered the central limit theorem for GARCH(p,q) models.

In this paper, we consider the ARCH/GARCH models and give sufficient conditions for the existence of a stationary ergodic solution and then find a class of functions for which the functional central limit theorem holds. Toward this end, we rephrase the given process as a properly defined Markov chain and prove its asymptotic properties by using Makov chain techniques and then induce the desired results.

A short overview of this paper is as follows. In Section 2, we represent the GARCH(p,q) model as a Markov chain and obtain conditions under which the chain is stationary and ergodic. Section 3 is devoted to consider the functional central limit theorem. The proofs of Theorem 2.1, Corollary 2.1 Theorem 3.1 and Corollary 3.1 are presented in Section 4.

We refer to Meyn and Tweedie(1993) for general terminologies and relevant results in Markov chain theory.

2. Strict stationarity and ergodicity

A sequence of univariate stochastic process $Y_n,\ n \in \mathbf{Z}$ is said to be a GARCH(p, q) process if it satisfies the equation $Y_n = \eta_n \sqrt{V_n}$ with

$$V_n = \delta + \sum_{i=1}^{p} \beta_i V_{n-i} + \sum_{i=1}^{q} \alpha_i Y_{n-i}^2, \quad n \in \mathbf{Z}, \tag{2.1}$$

where, $\eta_n,\ n \in \mathbf{Z}$ are i.i.d. random variables with mean $E(\eta_n) = 0$ and variance 1. Assume that $\delta > 0$, $p \geq 0$ and $q > 0$. If $p = 0$, then the process is said to be an ARCH(q) process.

Following Bougerol and Picard(1992), for GARCH(p, q) process given in (2.1), we define a $(p + q - 1) \times (p + q - 1)$ matrix A_n:

$$A_n = \begin{bmatrix} \tau_n, & \beta_p, & \alpha, & \alpha_q \\ I_{p-1}, & 0, & 0, & 0 \\ \xi_n, & 0, & 0, & 0 \\ 0, & 0, & I_{q-2}, & 0 \end{bmatrix}, \tag{2.2}$$

where

$$\tau_n = (\beta_1 + \alpha_1 \eta_n^2, \beta_2, \cdots, \beta_{p-1}) \in R^{p-1},$$
$$\xi_n = (\eta_n^2, 0, 0, \cdots, 0) \in R^{p-1},$$
$$\alpha = (\alpha_2, \alpha_3, \cdots, \alpha_{q-1}) \in R^{q-2},$$

and I_{p-1} and I_{q-2} are the identity matrices of size $p - 1$ and $q - 2$, respectively. Then $\{A_n\}$ are independent and identically distributed random matrices. We will always assume that $p, q \geq 2$, by adding some α_i or β_i equal to zero if needed. Now let

$$B = (\delta, 0, 0, \cdots, 0)^t \in R^{p+q-1},$$
$$X_n = (V_{n+1}, \cdots, V_{n-p+2}, Y_n^2, \cdots, Y_{n-q+2}^2)^t.$$

Then Y_n is a solution of (2.1) if and only if X_n is a solution of

$$X_{n+1} = A_{n+1} X_n + B, \quad n \in \mathbf{Z}. \tag{2.3}$$

Since $A_k,\ k \geq n + 1$ are independent of X_n, $\{X_n : n \geq 0\}$ with arbitrarily specified random vector X_0 independent of $\{\eta_n : n \geq 1\}$ can be regarded as a Markov chain with its transition probability function, say, $p(x, dy)$.

Let T be the transition operator on the linear space of all real valued bounded measurable functions on R^{p+q-1} defined by

$$Tf(x) = \int f(y)p(x, dy). \tag{2.4}$$

Its adjoint T^* is defined on the space of all finite signed measures on the Borel sigma-field $\mathcal{B}(R^{p+q-1})$ of R^{p+q-1} by

$$(T^*\mu)(C) = \int p(x, C)\mu(dx), \quad C \in \mathcal{B}(R^{p+q-1}). \tag{2.5}$$

Note that if the distribution of X_0 is μ, then the $T^*\mu$ in (2.5) is the distribution of $X_1 = A_1X_0 + B$ and $T^{*n}\mu = (T^n)^*\mu$ is the distribution of X_n, since $\{X_n : n \geq 0\}$ is the Markov chain with transition probability function $p(x, dy)$.

We define for any $x \in R^{p+q-1}$, $\|x\| = (x^tx)^{\frac{1}{2}}$, where x^t denotes the transpose of x and define a matrix norm $|||\cdot|||$ for $(p+q-1) \times (p+q-1)$ matrix A by $|||A||| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$.

Let $\Gamma_2 = \Gamma_2(R^{p+q-1})$ denote the set of probability measures with finite second moments defined on the Borel sigma-field of R^{p+q-1}. On the space Γ_2, define the distance d_2 by

$$d_2(\mu, \nu) = \inf\{(E\|X - Z\|^2)^{\frac{1}{2}}; X \simeq \mu, Z \simeq \nu\}. \tag{2.6}$$

The symbol $\simeq$ denotes the equality in distribution. It is known (see, e.g., Bickel and Freedman(1981)) that the infimum in (2.6) is attained, (Γ_2, d_2) is a complete separable metric space, and the d_2 convergence is equivalent to the weak convergence and the convergence of the second norm moments.

THEOREM 2.1. *Assume* $|||E(A_1^tA_1)||| < 1$. *Then for any* $\mu, \nu \in \Gamma_2$,

(1) $d_2(T^{*n}\mu, T^{*n}\nu) \leq r^n d_2(\mu, \nu)$, *where* $r = |||E(A_1^tA_1)|||^{\frac{1}{2}} < 1$,

(2) *there exists a unique probability measure* $\pi \in \Gamma_2$, *such that*

$$d_2(T^{*n}\mu, \pi) \to 0 \tag{2.7}$$

as $n \to \infty$ *at exponential rate. Here* π *is independent of* μ.

(3) X_n, $n \geq 0$ *in* (2.3) *with* $X_0 \simeq \pi$ *is a strictly stationary and ergodic Markov chain.*

It could occur that $|||E[(A_m \cdots A_1)^t(A_m \cdots A_1)]||| < 1$ for some $m > 0$, but $|||E(A_1^tA_1)||| > 1$.

COROLLARY 2.1. *If for some $m > 0$, $|||E[(A_m \cdots A_1)^t(A_m \cdots A_1)]||| < 1$, then there exists a unique invariant probability measure $\pi \in \Gamma_2$ and X_n, $n \geq 0$ in (2.3) with $X_0 \simeq \pi$ is a strictly stationary and ergodic Markov chain.*

Recall that for i.i.d. sequence of random matrices $\{A_n\}$, top Lyapounov exponent associated to $\{A_n\}$ is defined, provided $E(\log^+ |||A_0|||)$ is finite, by

$$\gamma = \inf \Big\{ E\Big(\frac{1}{n}\log |||A_n A_{n-1} \cdots A_1|||\Big),\ n \in N \Big\}.$$

It is proved by Bougerol and Picard(1992) that if $\gamma < 0$ if and only if X_n in (2.3) has a strictly stationary solution. However, $\gamma < 0$ is difficult to verify and can only be estimated by computer simulation. Note that $\sum_{i=1}^q \alpha_i + \sum_{i=1}^p \beta_i < 1$ implies that $\gamma < 0$.

3. Functional central limit theorem

When we consider the strictly stationary ergodic Markov chain $\{X_n : n \geq 0\}$ with invariant initial distribution π, we are interested in the limiting distribution of the following stochastic process: for each positive integer n and fixed $f \in L^2(R^{p+q-1}, \pi)$,

$$(3.1)\quad F_n(t) = \frac{1}{\sqrt{n}}\left[\sum_{k=0}^{[nt]}(f(X_k) - \bar{f}) + (t - \frac{[nt]}{n})(f(X_{[nt]+1}) - \bar{f})\right], \quad t \geq 0.$$

Here $\bar{f} = \int f d\pi$. We say that the functional central limit theorem (FCLT) holds for $f \in L^2(R^{p+q-1}, \pi)$ if the sequence of stochastic process $F_n(t)$ in (3.1) converges in distribution to a Brownian motion.

We will denote the L^2–norm on $L^2(R^{p+q-1}, \pi)$ by $\|\cdot\|_2$.

THEOREM 3.1. *Suppose $|||E(A_1^t A_1)||| < 1$. Then the following assertions hold.*

(1) *If $X_0 \simeq \pi$, then every Lipschitzian function f holds FCLT.*
(2) *If $X_0 \simeq \mu$ for any $\mu \in \Gamma_2$, every Lipschitzian function f holds FCLT. In particular, if $X_0 \equiv x$ for any x, every Lipschitzian function f holds FCLT.*

Here the variance parameter to the limit Brownian motion is $\|g\|_2^2 - \|Tg\|_2^2$ where $g = -\sum_{n=0}^{\infty} T^n(f - \bar{f})$.

COROLLARY 3.1. *If for some $m > 0$, $|||E[(A_m \cdots A_1)^t (A_m \cdots A_1)]||| < 1$, then the conclusions of Theorem 3.1 hold.*

COROLLARY 3.2. *Suppose that one of the relations $|||E(A_1^t A_1)||| < 1$ or for some $m > 0$, $|||E[(A_m \cdots A_1)^t (A_m \cdots A_1)]||| < 1$ is satisfied. Let $W_n = (V_{n+1}, \cdots, V_{n-p+2})^t$. Define for $\mathbf{y} = (y_1, y_2, \cdots, y_{p+q-1})^t$,*

$$\pi_1(C) = \pi(\{\mathbf{y} : (y_1, \cdots, y_p)^t \in C\}), \ (C \in \mathcal{B}(R^p)).$$

If $g : R^p \to R$ is a Lipschitzian function, then $G_n(t) = \frac{1}{\sqrt{n}} \sum_{k=0}^{[nt]} [g(W_k) - \bar{g}]$ converges in distribution to a Brownian motion, where $\bar{g} = \int g d\pi_1$.

Proof. For $\mathbf{y} = (y_1, y_2, \cdots, y_{p+q-1})^t$, let $p_1(\mathbf{y}) = (y_1, \cdots, y_p)^t$ and let $f(\mathbf{y}) = g(p_1(\mathbf{y})) = g((y_1, \cdots, y_p)^t)$. If g is a Lipschitzian function on R^p, then f is a Lipschitzian function on R^{p+q-1}. This together with Theorem 3.1 implies the convergence in distribution of $G_n(t)$ to a Brownian motion.

Note that $g(y_1, \cdots, y_p) = \sum_{i=1}^{p} t_i y_i$ for some $(t_1, \cdots, t_p) \in R^p$ is a Lipschitzian function on R^p. □

REMARK 3.1. Consider the ARCH(q) model. From (2.2) and (2.3), A_n and X_n are defined as follows:

$$A_n = \begin{bmatrix} \alpha_1 \eta_n^2 & \alpha & \alpha_q \\ \eta_n^2 & 0 & 0 \\ 0 & I_{q-2} & 0 \end{bmatrix}$$

and

$$X_n = (V_{n+1}, Y_n^2, \cdots, Y_{n-q+2}^2)^t.$$

The theorems and corollaries in Sections 2 and 3 can be applied directly to the ARCH(q) process.

REMARK 3.2. Rudolph(1998) gives sufficient conditions under which the central limit theorem holds for the process $H_n = (V_n, V_{n-1}, \cdots, V_{n-m+1})^t$, where $m = \max\{p, q\}$. Results are obtained under the assumption that H_n is a Markov chain, but that assumption is not true. And conditions given in Theorem 5 do not imply $|||E(\Gamma_n^t \Gamma_n)||| < 1$ which is necessary to derive the central limit theorem for H_n. Here Γ_n is an $m \times m$ random matrix given as a function of $\eta_n, \cdots, \eta_{n-m+1}$ and $H_{n+1} = \Gamma_n H_n + B$.

REMARK 3.3. Theorem 3.1 extends the results of Benda(1998), where it was shown that the FCLT holds for every Lipschitzian function

f if $E(|||A_1|||^2) < 1$ and $E(\|z - A_1 z\|^2) < \infty$. Notice that $|||E(A^t A)||| \leq E(|||A|||^2)$, by Jensen's inequality.

4. Proofs

In order to prove the Theorem 2.1, we need the following lemma in matrix analysis (see Horn and Johnson(1990)).

LEMMA 4.1. *Let A be a symmetric square matrix and $\{\lambda_i\}$ be the eigenvalues of A. Then*

$$\min\{\lambda_i\} \leq \frac{x^t A x}{x^t x} \leq \max\{\lambda_i\}.$$

Proof of the Theorem 2.1. The basic idea of the proof follows that of Theorem 1 in Burton and Rösler(1995). Let μ and ν be probability measures which have finite second norm moments. Choose X and Z independent of $A_n, n \geq 1$ with $X \simeq \mu, Z \simeq \nu$ and $d_2(\mu, \nu) = (E\|X - Z\|^2)^{\frac{1}{2}}$. Then we have

(1)

$$\begin{aligned} d_2(T^*\mu, T^*\nu) &\leq (E\|A_1 X - A_1 Z\|^2)^{\frac{1}{2}} \\ &\leq (E|(X-Z)^t E(A_1^t A_1)(X-Z)|)^{\frac{1}{2}} \\ &\leq |||E(A_1^t A_1)|||^{\frac{1}{2}} (E\|X - Z\|^2)^{\frac{1}{2}} \\ &= r d_2(\mu, \nu), \end{aligned} \tag{4.1}$$

and

$$d_2(T^{*n}\mu, T^{*n}\nu) \leq r^n d_2(\mu, \nu). \tag{4.2}$$

Equality in (4.2) is obtained by repeating the argument used in (4.1) and using the independenc of $\{A_n\}$.

Since $E(A_1^t A_1)$ is a symmetric square matrix, the third inequality in (4.1) follows from Lemma 4.1 and the fact that $|\lambda| < |||A|||$ for any eigenvalue λ of A and any matrix norm $|||\cdot|||$.

(2) For $n > m$,

$$\begin{aligned} d_2(T^{*m}\mu, T^{*n}\mu) &\leq (E\|A_1 A_2 \cdots A_m X - A_1 A_2 \cdots A_m Y\|^2)^{\frac{1}{2}} \\ &\leq r^m (E\|X - Y\|^2)^{\frac{1}{2}}, \end{aligned} \tag{4.3}$$

where $Y = A_{m+1} \cdots A_n X + A_{m+1} \cdots A_{n-1} B + \cdots + A_{m+1} B + B$. The first inequality in (4.3) follows from the fact that the distribution of

$A_1 \cdots A_m X$ is the same as that of $A_m \cdots A_1 X$. Since $(E\|X - Y\|^2)^{\frac{1}{2}} < \infty$, $\forall n, m$, $T^{*n}\mu$ is a Cauchy sequence in the metric d_2. Completeness of (Γ_2, d_2)tells us the existence of a unique probability measure, say π, in Γ_2, such that $T^{*n}\mu$ converges to π at exponential rate. Moreover from (4.2), π does not depend on the initial distribution μ.

(3) implies that if the distribution of X_0 is δ_x(point mass at x), then the distribution of X_n converges weakly to π as n goes to infinity. In this case π is the unique invariant probability measure for a Markov chain X_n, $n \geq 0$ with $\int \|x\|^2 d\pi < \infty$, since X_n, $n \geq 0$ is weak Feller. Therefore X_n with $X_0 \simeq \pi$ is strictly stationary and ergodic (Breiman(1968)). □

Proof of Corollary 2.1. First, note that by assumption, $|||E(A_1^t A_1)||| < \infty$. Choose X and Z as in the proof of Theorem 2.1. By the same process used to prove the inequality (4.2), we obtain that

$$d_2(T^{*n}\mu, T^{*n}\nu) \leq r^{[\frac{n}{m}]} \cdot K, \quad \forall \mu, \nu \in \Gamma_2, \tag{4.4}$$

where $[\frac{n}{m}]$ is the integer part of $\frac{n}{m}$ and

$$\begin{aligned} K &= d_2(T^{*(n-[\frac{n}{m}]m)}\mu, T^{*(n-[\frac{n}{m}]m)}\nu) \\ &= (E(\|A_{n-[\frac{n}{m}]m} \cdots A_1 X - A_{n-[\frac{n}{m}]m} \cdots A_1 Z\|^2))^{\frac{1}{2}} \\ &\leq |||E(A_1^t A_1)|||^{\frac{1}{2}(n-[\frac{n}{m}]m)} (E(\|X - Z\|^2)^{\frac{1}{2}} \\ &< \infty. \end{aligned} \tag{4.5}$$

Thus $d_2(T^{*n}\mu, T^{*n}\nu) \to 0$ as $n \to \infty$. By the same arguments used in the proof of Theorem 2.1 (2) and (3), we have that $T^{*n}\mu$ is a Cauchy sequence and hence the result follows. □

To prove the Theorem 3.1, we need the following Lemma(see Bhattacharya and Lee(1988), Gordin and Lipsic (1978)). Let T be the transition operator given in (2.4) and I be the identity operator.

LEMMA 4.2. *Let $f \in L^2(R^{p+q-1}, \pi)$. If $\sum_{n=0}^{\infty} \|T^n(f - \bar{f})\|_2 < \infty$, then $f - \bar{f}$ belongs to the range of $T - I$, and hence FCLT holds for f and the variance parameter to the limit Brownian motion is $\|g\|_2^2 - \|Tg\|_2^2$ where $g = -\sum_{n=0}^{\infty} T^n(f - \bar{f})$.*

Proof of theorem 3.1. (1) Write X_n^{μ} for X_n in the case of $X_0 \simeq \mu$ and $X_n(x) = X_n^{\delta_x}$. Let f be any Lipschitzian function on R^{p+q-1}, that is, $|f(x) - f(y)| \leq M\|x - y\|$ for some $M > 0$ and for all x, y. It is easy to check that every Lipschitzian function f is in $L^2(R^{p+q-1}, \pi)$. Then, for $r^2 = |||E(A_1^t A_1)||| < 1$,

$$\begin{aligned} \|T^n(f-\bar{f})\|_2^2 &\le \int \Big[\int E|f(X_n(x)) - f(X_n(y))|\pi(dy) \Big]^2 \pi(dx) \\ &\le M^2 \int\int E\|X_n(x) - X_n(y)\|^2 \pi(dy)\pi(dx) \qquad (4.6) \\ &\le M^2 r^{2n} \int\int \|x-y\|^2 \pi(dy)\pi(dx). \end{aligned}$$

Hence $\|T^n(f-\bar{f})\|_2 \le M r^n (\int\int \|x-y\|^2 \pi(dy)\pi(dx))^{\frac{1}{2}}$. This together with $\pi \in \Gamma_2$ implies that $\sum_{n=0}^{\infty} \|T^n(f-\bar{f})\|_2 < \infty$. Thus the conclusion follows immediately from Lemma 4.2.

(2) Let $F_n(\cdot)$ be the process defined by (3.1) with $X_0 \simeq \pi$, and F_n^μ the corresponding process with $X_0 \simeq \mu$. If f is as in (1),

$$\begin{aligned} E(\max_{0\le t\le 1} |F_n^\mu(t) - F_n(t)|) &\le M\ n^{-\frac{1}{2}} \sum_{k=0}^{n} E\|X_k^\mu - X_k\| \\ &\le M\ n^{-\frac{1}{2}} \sum_{k=0}^{n} r^k \int\int \|x-y\|\pi(dy)\mu(dx), \end{aligned}$$

which converges to 0 as n goes to infinity. Therefore $F_n^\mu(t)$ and $F_n(t)$ have the same limit.

Finally, take $\mu = \delta_x$. □

Proof of Corollary 3.1. Due to Theorem 2.1 and Corollary 2.1, X_n with $X_0 \simeq \pi$ is a strictly stationary ergodic Markov chain. Combining (4.4) and (4.6), we obtain that

$$\begin{aligned} \|T^n(f-\bar{f})\|_2^2 &\le M^2 \int\int E\|X_n(x) - X_n(y)\|^2 \pi(dy)\pi(dx) \\ &\le M^2 r^{2[\frac{n}{m}]} \cdot K, \end{aligned}$$

where K is given in (4.5). Thus $\sum_{n=0}^{\infty} \|T^n(f-\bar{f})\|_2 < \infty$, and the FCLT holds for f. □

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