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SOME LIMIT THEOREMS FOR POSITIVE RECURRENT AGE-DEPENDENT BRANCHING PROCESSES

Hye-Jeong Kang

Abstract. In this paper we consider an age dependent branching process whose particles move according to a Markov process with continuous state space. The Markov process is assumed to be stationary with independent increments and positive recurrent. We find some sufficient conditions for the Markov motion process such that the empirical distribution of the positions converges to the limiting distribution of the motion process.

1. Introduction

We consider an age-dependent branching process evolving from one particle. That is, the process starts at time 0 with one particle of age 0 and it dies at time λ and produces ξ offsprings where λ and ξ are independent random variables with distributions G and $\{p_k\}$ respectively. Then each particle dies and produces independently of each other in the same way as its parent, an so on.

We superimpose on this process the additional structure of movement. A particle whose parent was at x at its time of birth moves until it dies according to a Markov process starting at x. The motions of different particles are assumed independent. When the underlying motion is Markovian and null-recurrent it is known that the empirical distribution of the geographical state appropriately scaled does converge(see Kang(1999)).

On the other hand, if the underlying motion process is positive recurrent, one expects that the geographical distribution should mimic the long run distribution of the Markovian motion. That is, the empirical distribution (with no scaling) of the position at time t should approach to the limiting

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distribution of the Markov motion process. In this paper we find some sufficient conditions for the motion process which justify this conjecture. The corresponding results for Galton-Watson processes are available in Athreya and Kang(1998).

2. Preliminary results

Let $\{Z(t); t \geq 0\}$ be an age-dependent branching process evolving from one particle at time $t = 0$ whose lifetime distribution is G and offspring distribution is $\{p_k\}$. We make the following assumptions throughout. Sometimes they will appear in lemmas and theorems explicitely and sometimes not, but they will always be in force.

(A 1) $p_0 = 0$,
(A 2) $1 < \mu \equiv \sum_{j=0}^{\infty} j p_j < \infty$,
(A 3) $\sum_{j=1}^{\infty} (j \log j) p_j < \infty$.

The assumption (A 1) is primarily for convenience of exposition. Otherwise, the extinction probability is positive and so one has to keep qualifying "on the set of non-extinction". A branching process satisfying (A 2) is said to be supercritical. (A 3) gurantees (see Athreya and Ney(1972)) the existence of the almost sure positive limit W of $W(t) \equiv e^{-\alpha t} Z(t)$ with finite moment, where $\alpha = \alpha(\mu, G)$ is the Malthusian parameter for μ and G defined by the root of the equation $\mu \int_0^\infty e^{-\alpha t} dG(t) = 1$. That is, there exists a random variable W such that

$$(1) \qquad \lim_{t\to\infty} e^{-\alpha t} Z(t) = W \quad \text{a.s.}, \quad P(W > 0) = 1, \quad \text{and} \quad E(W) < \infty.$$

See Athreya and Ney(1972) for details.

Now let $\{X(t); t \geq 0\}$ be the underlying Markov process governing motion such that $X(0) = 0$ a.s. Suppose that $\{X(t); t \geq 0\}$ is stationary with independent increments and has a regeneration set. Further we assume that the process is positive recurrent. See Assmussen(1986) for these definitions.

For a Borel set B, write $P^t(x, B) = P_x(X(t) \in B)$ and $P^t(x, b) = P^t(x, (-\infty, b])$ simply. Then it is known that there exists a unique limiting distribution π for $X(t)$ such that the P_x-distribution of $X(t)$ converges to π in total variation. That is, for any $x \in R$,

$$||P_x(X(t) \in \cdot\,) - \pi(\cdot)|| \equiv \sup_{B \in \mathcal{B}} |P_x(X(t) \in B) - \pi(B)| \to 0 \quad \text{as} \quad t \to \infty,$$

where $\mathcal{B}$ is the Borel σ-algebra.

Even though $Z(t)$ denote the total number of particles alive at time t in classical branching processes, we abuse the same notation for the point process describing the positions of particles alive at time t and so $Z(t, B)$

denote the number of particles at time t which are in the set B. Note that $Z(t,R)=Z(t)$. We write $Z(t,b)$ for $Z(t,(-\infty,b])$ unless it does cause any confusion. We add a subscript x and a superscript a to indicate that the process begins with one particle of age a and position x at time 0. Thus $Z_x^a(t)$ is a random counting measure on R and $Z_x^a(t,B)$ denotes the number of particles at time t which are in B when the process begins with one particle of age a at position x at time 0. We write Z^a, Z_x, Z for Z_0^a, Z_x^0, Z_0^0, respectively.

In the proofs to come we make use of the following lemmas. The first one can be found in Kang(1999).

LEMMA 1. *Put* $M=\sup_{s\geq 0}\sup_{a\geq 0}\{e^{-\alpha s}Z^a(s)\}$. *If* $\sum_{j=1}^{\infty}(j\log j)p_j<\infty$, *then* $E(M)<\infty$.

Put

$$G^y(t)=\frac{G(t+y)-G(y)}{1-G(y)},$$
$$V(y)=\mu\int_0^{\infty}e^{-\alpha t}G^y(dt),$$
$$A(a)=\frac{\int_0^a e^{-\alpha t}(1-G(t))dt}{\int_0^{\infty}e^{-\alpha t}(1-G(t))dt},$$
$$n_1=\frac{\int_0^{\infty}e^{-\alpha t}(1-G(t))dt}{\mu\int_0^{\infty}e^{-\alpha t}G(dt)},$$
$$V_t=\sum_{j=1}^{Z(t)}V(a_j),$$

where $\{a_j; j=1,\cdots,Z(t)\}$ *is the age-chart at time* t. *The following two lemmas are in Athreya and Kaplan(1976).*

LEMMA 2. *Define* $m^y(s)=E(Z^y(s))$, *then*

$$\sup_{y\geq 0}|m^y(s)e^{-\alpha s}-n_1V(y)|\to 0 \quad \text{as} \quad s\to\infty.$$

LEMMA 3. *Suppose* $\sum_{j=1}^{\infty}(j\log j)p_j<\infty$. *Then*

$$\lim_{t\to\infty}\frac{V_t}{Z(t)}=n_1^{-1} \quad \text{a.s.}$$

The proof of the following lemma can be found in Athreya and Kang (1998).

LEMMA 4. *Let* $\{\mathcal{F}_n\}_0^\infty$ *be a filtration contained in* $(\Omega, \mathcal{B}, P)$. *Let* $\{X_{ni}; n, i = 1, 2, \cdots\}$ *be a double array of random variables such that for each* n, *conditioned on* $\mathcal{F}_n$ *the sequence* $\{X_{ni}; i = 1, 2, \cdots\}$ *are independent a.s. Let* $\{N_n; n = 1, 2, \cdots\}$ *be a nondecreasing sequence of nonnegative integer valued random variables such that for each* n N_n *is* $\mathcal{F}_n$*-measurable. Assume*

(i) *that there exists a random probability measure* Q *on* $[0, \infty)$ *such that for some constant* $0 < C < \infty$,

$$\sup_{n,i} P(|X_{ni}| > t|\mathcal{F}_n) \leq CQ(t, \infty) \quad \text{for all } t > 0 \text{ a.s.},$$

(ii) *that* $\int_0^\infty Q(t, \infty)dt < \infty$ *a.s.,* and

(iii) *that* $\liminf_n \dfrac{N_{n+1}}{N_n} > 1$ *a.s.*

Then, $\displaystyle \frac{1}{N_n} \sum_{i=1}^{N_n} (X_{ni} - E(X_{ni})) \to 0$ a.s. *as* $n \to \infty$.

3. Results and their proofs

We introduce the following notation: For a Borel set B,

$$H(t, B) \equiv \frac{Z(t, B)}{Z(t)}.$$

We write $H(t, b)$ instead of $H(t, (-\infty, b])$ for simplicity.

THEOREM 1. *Assume*

$$\sup_x ||P^t(x, \cdot) - \pi(\cdot)|| \to 0 \quad \text{as} \quad t \to \infty. \tag{2}$$

Then for any $b \in R$, $H(t, b) \xrightarrow{\text{a.s.}} \pi(b)$ *as* $t \to \infty$, *where* $\xrightarrow{\text{a.s.}}$ *denotes a convergence with probability 1.*

Since the set of half lines generates the Borel σ-algebra $\mathcal{B}$ we have the following simple

COROLLARY 1. *Under the hypothesis of Theorem 1, for any Borel set* $B \in \mathcal{B}$,

$$H(t, B) \to \pi(B) \quad \text{a.s.} \quad \text{as} \quad t \to \infty.$$

Proof of Theorem 1. By appealing to the additive property of branching process we have the following representation

$$Z(t+s,b)=\sum_{j=1}^{Z(t)} Z_{x_j}^{a_j}(s,b), \tag{3}$$

where $\{(a_j,\, x_j); j = 1,2,\cdots\}$ is the (age, position)-chart at time t and $Z_{x_j}^{a_j}(s,x)$ is the number of particles at time $t+s$ which are in $(-\infty, b]$ in the line of descent initiated by the particle of age a_j and position x_j at time t. Since the motion process is stationary and the motions of particles starting at the same position identically distributed we get

$$\begin{aligned} E(Z_{x_j}^{a_j}(s,b)|\mathcal{F}_t) &= E(Z^{a_j}(s)|\mathcal{F}_t)P^s(x_j,b) \\ &= m^{a_j}(s)P^s(x_j,b), \end{aligned}$$

where $m^a(s) = E(Z^a(s))$ and $\mathcal{F}_t$ is the σ-algebra containing all the information up to time t. Hence starting from (3) we have the following decomposition

$$\begin{aligned} \frac{e^{-\alpha s}}{Z(t)} Z(t+s,b) &= \frac{1}{Z(t)}\sum_{j=1}^{Z(t)}\{Z_{x_j}^{a_j}(s,b) - m^{a_j}(s)P^s(x_j,b)\}e^{-\alpha s} \\ &\quad + \frac{1}{Z(t)}\sum_{j=1}^{Z(t)}\{e^{-\alpha s}m^{a_j}(s) - n_1V(a_j)\}P^s(x_j,b) \\ &\quad + \frac{1}{Z(t)}\sum_{j=1}^{Z(t)} n_1V(a_j)\{P^s(x_j,b)-\pi(b)\} \\ &\quad + \frac{1}{Z(t)}\sum_{j=1}^{Z(t)} n_1V(a_j)\pi(b) \\ &\equiv a(t,s,b) + b(t,s,b) + c(t,s,b) + n_1d(t)\pi(b), \text{ say.} \end{aligned}$$

Consequently, we arrived at

$$\begin{aligned} H(t+s,b) &= \frac{\frac{e^{-\alpha s}}{Z(t)}Z(t+s,b)}{\frac{e^{-\alpha s}}{Z(t)}Z(t+s)} \\ &= \frac{a(t,s,b)+b(t,s,b)+c(t,s,b)+n_1d(t)\pi(b)}{a(t,s,\infty)+b(t,s,\infty)+n_1d(t)}. \end{aligned} \tag{4}$$

It is immediate to see that for any $b \in R \cup \{\infty\}$, and for any $t > 0$

$$b(t,s,b) \xrightarrow{\text{a.s.}} 0, \quad n_1d(t) \xrightarrow{\text{a.s.}} 1 \quad \text{as} \quad s \to \infty \tag{5}$$

from Lemma 2 and Lemma 3, respectively. Noting that $V(y)$ is uniformly bounded in y, we have for any $t > 0$

$$c(t,s,b) \xrightarrow{\text{a.s.}} 0 \quad \text{as} \quad s \to \infty. \tag{6}$$

Now let $\delta > 0$ and $t_n = n\delta$, then we have from (1) that

$$\begin{aligned} \lim_{n\to\infty} \frac{Z(t_{n+1})}{Z(t_n)} &= \lim_{n\to\infty} \frac{e^{-\alpha t_{n+1}} Z(t_{n+1})}{e^{-\alpha t_n} Z(t_n)} \cdot e^{\alpha(t_{n+1}-t_n)} \\ &= e^{\alpha\delta} > 1, \quad \text{a.s.} \end{aligned} \tag{7}$$

Furthermore, since

$$e^{-\alpha s} Z_{x_j}^{a_j}(s,b) \le M \equiv \sup_{s\ge 0} \sup_{a\ge 0} \{e^{-\alpha s} Z^a(s)\}$$

and $E(M) < \infty$ by Lemma 1 we conclude from Lemma 4 that for any $\{s_n\}$ such that $s_n \to \infty$, $a(n\delta, s_n, b) \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$. Combining this with (5), (6) we get

$$H(n\delta + s_n, b) \xrightarrow{\text{a.s.}} \pi(b) \quad \text{as} \quad n \to \infty. \tag{8}$$

Now choose $s_n = n\delta$ and replacing δ by $\frac{\delta}{2}$ we get

$$H(n\delta, b) \xrightarrow{\text{a.s.}} \pi(b) \quad \text{as} \quad n \to \infty.$$

To complete the proof fix $\varepsilon > 0$ and $\delta > 0$. Let $n\delta \le t < (n+1)\delta$ and define

$$\delta_j = \begin{cases} 1 & \text{if } j\text{th particle at time } n\delta \text{ doesn't split until } (n+1)\delta \\ & \text{and the particle doesn't cover a distance } > \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{(a_j, x_j); j = 1, \cdots, Z(n\delta)\}$ be the (age,position)-chart at time $n\delta$. Since the lifetime and the movement of a particle are independent,

$$\begin{aligned} E(\delta_j|\mathcal{F}_{n\delta}) = P(\delta_j = 1|\mathcal{F}_{n\delta}) &= P(\lambda^{a_j} > \delta|\mathcal{F}_{n\delta}) P(\overline{X}(\delta) \le \varepsilon) \\ &= (1 - G^{a_j}(\delta)) P(\overline{X}(\delta) \le \varepsilon), \end{aligned}$$

where $\overline{X}(\delta) = \sup_{0\le t\le\delta}\{|X(t)|\}$ with $\{X(t); t \ge 0\}$ the underlying motion process starting at 0.

From the definition of δ_j, we have the following inequality

$$Z(t,b) \ge \sum_{j=1}^{Z(n\delta)} I(x_j \le b - \varepsilon)\delta_j,$$

where $I(x \leq b)$ is defined by 1 if $x \leq b$, 0, otherwise. So

$$\begin{aligned}\frac{Z(t,b)}{Z(t)} &\geq \frac{Z(n\delta)}{Z(t)} \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} I(x_j \leq b-\varepsilon)\delta_j \\ &= \frac{Z(n\delta)}{Z(t)}\{A(n\delta,b) + P(\overline{X}(\delta) \leq \varepsilon)B(n\delta,b)\},\end{aligned}$$

where

$$A(n\delta,b) = \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} I(x_j \leq b-\varepsilon) \cdot \{(\delta_j - (1 - G^{a_j}(\delta))P(\overline{X}(\delta) \leq \varepsilon)\},$$

$$B(n\delta,b) = \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} I(x_j \leq b-\varepsilon)(1 - G^{a_j}(\delta)).$$

It is simple to check the conditions in Lemma 4 for the summands in $A(n\delta,b)$ and hence we have $A(n\delta,b) \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$.

On the other hand

(9)

$$\begin{aligned}B(n\delta,b) &= \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} I(x_j \leq b-\varepsilon) - \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} I(x_j \leq b-\varepsilon)G^{a_j}(\delta) \\ &\geq \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} I(x_j \leq b-\varepsilon) - \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} G^{a_j}(\delta).\end{aligned}$$

Since $G_\delta(a) \equiv G^a(\delta)$ is bounded and continuous except on a countable set we have

$$\begin{aligned}\frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} G^{a_j}(\delta) &= \int_0^\infty G_\delta(u)A(du,n\delta) \\ &\xrightarrow{\text{a.s.}} \int_0^\infty G_\delta(u)A(du) \quad \text{as} \quad n \to \infty,\end{aligned}$$

where $A(du,n\delta)$ is the empirical age point measure at time $n\delta$ and $A(du)$ is the limiting stable age distribution. From (8) we conclude that the first term in (9) converges to $\pi(b-\varepsilon)$ a.s. and we obtain

$$\liminf_{n\to\infty} B(n\delta,b) \geq \pi(b-\varepsilon) - \int_0^\infty G_\delta(u)A(du) \quad \text{a.s.}$$

Further hypothesis (A 1) implies that $Z(t)$ is non-decreasing in t and so we have by (7) that

$$\liminf_{n\to\infty}\frac{Z(n\delta)}{Z(t)}\geq\liminf_{n\to\infty}\frac{Z(n\delta)}{Z((n+1)\delta)}=e^{-\alpha\delta}\quad\text{a.s.}$$

Hence

$$\liminf_{t\to\infty}\frac{Z(t,b)}{Z(t)}\geq e^{-\alpha\delta}(\pi(b-\varepsilon)-\int_0^\infty G_\delta(u)A(du))P(\overline{X}(\delta)\leq\varepsilon),\quad\text{a.s.}$$

Since $G_\delta(u)\to 0$ as $\delta\to 0$, we see that $\int_0^\infty G_\delta(u)A(du)\to 0$ as $\delta\to 0$ by Lebesgue convergence theorem. Noting that $P(\overline{X}(\delta)\leq\varepsilon)\to 1$ as $\delta\to 0$ we arrive at the following inequality by letting $\delta\to 0$ and then by letting $\varepsilon\to 0$,

$$\liminf_{t\to\infty}\frac{Z(t,b)}{Z(t)}\geq\pi(b).$$

For the other direction we consider the following inequality

$$Z(t)-Z(t,b)\geq\sum_{j=1}^{Z(n\delta)}I(x_j\geq b+\varepsilon)\delta_j.$$

The same arguments as above establish

$$\liminf_{t\to\infty}(1-H(t,b))\geq e^{-\alpha\delta}\big(1-\pi(b+\varepsilon)-\int_0^\infty G_\delta(u)A(du)\big)P(\overline{X}(\delta)\leq\varepsilon).$$

Letting $\delta\to 0$ and then letting $\varepsilon\to 0$, we get the following inequality

$$\liminf_{t\to\infty}(1-H(t,b))\geq 1-\pi(b)$$

which completes the proof. □

If we weaken the hypothesis we have the following

THEOREM 2. *Assume for each compact set K,*

$$\sup_{x\in K}||P^t(x,\cdot)-\pi(\cdot)||\to 0\quad\text{as}\quad t\to\infty.\tag{11}$$

Then for any $b\in R$,

$$H(t,b)\xrightarrow{\text{pr}}\pi(b)\quad\text{as}\quad t\to\infty,$$

where $\xrightarrow{\text{pr}}$ denotes a convergence in probability.

Proof. We begin with the representation (4). We have shown in the proof of Theorem 1 that for any $\delta>0$, $b\in R\cup\{\infty\}$, and for any $\{s_n\}$ with $s_n\to\infty$,

(12)

$$a(n\delta,s_n,b)\xrightarrow{\text{a.s.}}0,\quad b(n\delta,s_n,b)\xrightarrow{\text{a.s.}}0,\quad n_1d(n\delta,b)\xrightarrow{\text{a.s.}}1\quad\text{as}\quad n\to\infty.$$

Choose $s_n = n\delta$ and we show that

$$\lim_{n\to\infty} P\Big(\frac{n_1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} V(a_j)|P^{n\delta}(x_j,b) - \pi(b)| > \varepsilon\Big) = 0. \tag{13}$$

Let $\eta > 0$ be given. Then we can find $h > 0$ by (1) such that

$$P(W < h) < \frac{\eta}{2}. \tag{14}$$

Now choose $N = N(\varepsilon, \eta, h)$ with

$$\pi(I_N^c) \leq \frac{\varepsilon\eta h}{4E(W)}, \tag{15}$$

where $I_N = [-N, N]$ and I_N^c its complement. Then

$$\begin{aligned}
&P\Big(\frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} n_1 V(a_j)|P^{n\delta}(x_j,b) - \pi(b)| > \varepsilon\Big) \\
&\leq P\Big(\frac{1}{Z(n\delta)} \sum_{x_j\in I_N} n_1 V(a_j)|P^{n\delta}(x_j,b) - \pi(b)| > \frac{\varepsilon}{2}\Big) \\
&\quad + P\Big(\frac{1}{Z(n\delta)} \sum_{x_j\in I_N^c} n_1 V(a_j)|P^{n\delta}(x_j,b) - \pi(b)| > \frac{\varepsilon}{2}\Big) \\
&\equiv \beta_n + \gamma_n, \quad \text{say.}
\end{aligned}$$

By the hypothesis (11) there is n_0 such that for $n \geq n_0$

$$\sup_{x\in I_N} |P^{n\delta}(x,b) - \pi(b)| < \frac{\varepsilon}{4}.$$

Hence for $n \geq n_0$ Lemma 3 says that

$$\begin{aligned}
&\frac{1}{Z(n\delta)} \sum_{x_j\in I_N} n_1 V(a_j)|P^{n\delta}(x_j,b) - \pi(b)| \\
&\leq \frac{\varepsilon}{4}\cdot n_1 \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} V(a_j) \xrightarrow{\text{a.s.}} \frac{\varepsilon}{4} \quad \text{as} \quad n\to\infty.
\end{aligned}$$

This implies $\beta_n \to 0$ as $n \to \infty$. On the other hand

$$\begin{aligned}
\gamma_n &= P\Big(\frac{1}{Z(n\delta)} \sum_{x_j\in I_N^c} n_1 V(a_j)|P^{n\delta}(x_j,b) - \pi(b)| > \frac{\varepsilon}{2}\Big) \\
&\leq P\Big(\frac{Z(n\delta, I_N^c)}{Z(n\delta)} > \frac{\varepsilon}{2},\ W(n\delta) > h\Big) + P(W(n\delta) < h) \\
&\leq \frac{2}{\varepsilon h} e^{-\alpha n\delta} E(Z(n\delta, I_N^c)) + P(W(n\delta) < h),
\end{aligned}$$

where the last inequality comes from Chebyshev's inequality. By Wald's equation we have

$$e^{-\alpha n\delta}E(Z(n\delta, I_N^c)) = e^{-\alpha n\delta}E(Z(n\delta))P(X(n\delta) \in I_N^c)$$
$$\rightarrow E(W)\pi(I_N^c) \leq \frac{\varepsilon\eta h}{4}.$$

Combining this with (14) we get

$$\limsup_{n\to\infty} \gamma_n \leq \frac{2}{\varepsilon h} \cdot \frac{\varepsilon\eta h}{4} + \frac{\eta}{2} = \eta.$$

Being $\eta > 0$ arbitrary, we have proved (13), that is, we get $c(n\delta, b) \xrightarrow{\text{pr}} 0$. Substitute $\frac{\delta}{2}$ for δ we conclude from this and (12) that

$$H(n\delta, b) \xrightarrow{\text{pr}} \pi(b) \quad \text{as} \quad n \to \infty.$$

Adapting the same technique used to complete the proof of Theorem 1 we can show that

$$H(t, b) \xrightarrow{\text{pr}} \pi(b) \quad \text{as} \quad t \to \infty.$$ □

COROLLARY 2. *Suppose that $\{P^t(x, b); t \geq 0\}$ is equicontinuous in x on any compact set K. Then for any $b \in R$,*

$$H(t, b) \xrightarrow{\text{pr}} \pi(b) \quad \text{as} \quad t \to \infty.$$

Proof. It is enough to show that $H(n\delta, b) \xrightarrow{\text{pr}} \pi(b)$ as $n \to \infty$ for any $\delta > 0$. Given $\varepsilon > 0$, $\eta > 0$ let $h > 0$ and $N = N(\varepsilon, \eta, h)$ be as in (14) and (15), respectively. Then we have seen in the proof of Theorem 2 that

$$\limsup_{n\to\infty} \gamma_n \leq \eta.$$

Now we show that $\beta_n = 0$ for sufficiently large n. Since $P^{n\delta}(x, b)$ is uniformly equicontinuous on the compact set I_N, there exists $\varepsilon' > 0$ such that for any $n \geq 1$, and for $x, y \in I_N$,

$$|P^{n\delta}(x, b) - P^{n\delta}(y, b)| < \frac{\varepsilon}{4} \quad \text{if} \quad |x - y| < \varepsilon'. \tag{16}$$

By the compactness of I_N, we can find finite points $y_1, \cdots, y_k$ in I_N such that

$$I_N \subset \bigcup_{i=1}^{k} B(y_i, \varepsilon'),$$

where $B(y, \varepsilon) = (y - \varepsilon, y + \varepsilon)$. So for each $x \in I_N$, there exists $x' \in \{y_1, \cdots, y_k\}$ such that $|x - x'| < \varepsilon'$. Now choose n_0 such that for $n \geq n_0$

$$\sup_{1\leq i\leq k} |P^{n\delta}(y_i, b) - \pi(b)| \leq \frac{\varepsilon}{4}. \tag{17}$$

So if $n \geq n_0$ we have

$$\begin{aligned}
&\frac{1}{Z(n\delta)} \sum_{x_j \in I_N} |P^{n\delta}(x_j, b) - \pi(b)| \\
&\leq \frac{1}{Z(n\delta)} \sum_{x_j \in I_N} \{|P^{n\delta}(x_j, b) - P^{n\delta}(x'_j, b)| + |P^{n\delta}(x'_j, b) - \pi(b)|\} \\
&\leq \frac{1}{Z(n\delta)} \sum_{x_j \in I_N} (\frac{\varepsilon}{4} + \frac{\varepsilon}{4}) \quad \text{by (16) and (17)} \\
&\leq \frac{\varepsilon}{2} \cdot \frac{Z(n\delta, I_N)}{Z(n\delta)} \leq \frac{\varepsilon}{2}.
\end{aligned}$$

That is, for $n \geq n_0$ we have $\beta_n = 0$. Hence

$$\limsup_{n\to\infty} P(\frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} n_1 V(a_j)|P^{n\delta}(x_j, b) - \pi(b)| > \varepsilon) \leq \eta.$$

Being $\eta > 0$ arbitrary, we conclude that

$$\lim_{n\to\infty} P(\frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} n_1 V(a_j)|P^{n\delta}(x_j, b) - \pi(b)| > \varepsilon) = 0. \qquad \square$$

References

[1] S. Asmussen, *Applied Probability and Queues*, John Wiley & Sons, Chichester, 1987.

[2] K. B. Athreya and H-J. Kang, *Some limit theorems for positive recurrent branching Markov chains* : I Advances in Applied Probability **30/3** (1998), 693–710.

[3] K. B. Athreya and N. Kaplan, *Convergence of the age-distribution in the one-dimensional age-dependent branching processes*, Annals of Probability **4** (1976), 38–50.

[4] K. B. Athreya and P. Ney, *Branching Processes*, Springer-Verlag, New York, 1972.

[5] H-J. Kang, *Law of large numbers for branching Brownian motion*, J. Korean Math. Soc. **36** (1999), no. 1, 139–157.

UNIBOSS Inc.
25F Hanhwa Securities Bldg
Yeoido-dong Youngdeungpo-gu
Seoul 150-010, Korea

Mailing address :
Statistical Research Center for Complex Systems
Seoul National University
Seoul 151-742, Korea