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C^∞ EXTENSIONS OF HOLOMORPHIC FUNCTIONS FROM SUBVARIETIES OF A CONVEX DOMAIN

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ABSTRACT. Let Ω be a bounded convex domain in $\mathbb{C}^n$ with smooth boundary. Let M be a subvariety of Ω which intersects $\partial\Omega$ transversally. Suppose that Ω is totally convex at any point of ∂M in the complex tangential directions. For $f \in \mathcal{O}(M) \cap C^\infty(\overline{M})$, there exists $F \in \mathcal{O}(\Omega) \cap C^\infty(\overline{\Omega})$ such that $F(z) = f(z)$ for $z \in M$.

1. Introduction

In this paper we study the problem of extending holomorphic functions from a subvariety of a convex domain. More precisely, we get the C^∞ boundary regularity of extension functions from a subvariety of a convex domain. For the construction of an extension function we will use the integral formula constructed along the lines in Berndtsson [6].

Henkin [10] introduced methods of integral representations in order to obtain the bounded extensions of holomorphic functions from submanifolds to strongly pseudoconvex domains. Since then, many works on regularity problems of extension functions have been done in various function spaces ([1], [2], [5], [7], [8], [9], [11]). It is a natural question to ask whether similar regularity results hold in general pseudoconvex domains. It turns out the bounded extension is false in general, even in the case of complex ellipsoids [12]. In [4] we obtained the Lipschitz extensions of holomorphic functions from a subvariety of dimension 1 to a convex domain. In [3] we studied the problem of extending holomorphic functions from a subvariety of an analytic polyhedron and obtained

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$L^p(1 \le p < \infty)$ and $H^p(1 < p \le \infty)$ extensions of holomorphic functions.

Let $\Omega = \{\rho < 0\}$ be a bounded convex domain in $\mathbb{C}^n$ with C^∞ boundary. Let $\widetilde{M}$ be a subvariety of dimension m in a neighborhood of $\overline{\Omega}$ given as

$$\widetilde{M} = \{z; h_1(z) = \cdots = h_{n-m}(z) = 0\},$$

where $h_j \in \mathcal{O}(\overline{\Omega})$. Let $M = \widetilde{M} \cap \Omega$ and $\partial M = \widetilde{M} \cap \partial\Omega$. We assume the transverse assumption that

$$(1.1) \qquad \partial h_1 \wedge \cdots \wedge \partial h_{n-m} \wedge \partial \rho \neq 0 \quad \text{on} \quad \partial M.$$

In [13], Range introduced the total convexity to study the boundary behavior of the Carathéodory metric and holomorphic mappings.

DEFINITION 1.1. Let ζ be a boundary point of Ω. We say that Ω is *totally convex at ζ in the complex tangential directions* if

$$\overline{\Omega} \cap (T^{\mathbb{C}}_\zeta(\partial\Omega) + \{\zeta\}) = \{\zeta\},$$

where $T^{\mathbb{C}}_\zeta(\partial\Omega)$ is the complex tangent space of $\partial\Omega$ at ζ.

EXAMPLE 1.2. Let

$$\Omega = \left\{ z \in \mathbb{C}^2; |z_1|^2 + e^\nu \cdot \exp\left(-\frac{1}{|z_2|^2}\right) < 1 \right\}.$$

If $\nu > \frac{3}{2}$, then Ω is a convex domain with C^∞ boundary, and Ω is totally convex at any point of $\partial\Omega$ in the complex tangential directions.

THEOREM 1.3. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded convex domain with C^∞ boundary. Let M be a subvariety of Ω of dimension m satisfying the transverse assumption* (1.1). *Suppose that Ω is totally convex at any point of ∂M in the complex tangential directions. Let $f \in \mathcal{O}(M) \cap C^\infty(\overline{M})$. Then there exists a holomorphic function $F \in \mathcal{O}(\Omega) \cap C^\infty(\overline{\Omega})$ such that $F(z) = f(z)$ for $z \in M$.*

2. Proof of Theorem 1.3

Let $h_{jk}(\zeta, z)$ be holomorphic functions in $\overline{\Omega} \times \overline{\Omega}$ such that

$$h_j(\zeta) - h_j(z) = \sum_{k=1}^{n} h_{jk}(\zeta, z)(\zeta_k - z_k),$$

h_{jk} are so-called Hefer functions to h_j, and define the (1,0)-forms $H_j = \sum_{k=1}^{n} h_{jk} d\zeta_k$. Then

$$\mu = \frac{H_1 \wedge \cdots \wedge H_{n-m} \wedge \overline{\partial h_1} \wedge \cdots \wedge \overline{\partial h_{n-m}}}{|\partial h|^2} dS$$

is a $(n-m, n-m)$-current whose coefficients are measures supported on M, and which depending holomorphically on $z \in \Omega$. Here $|\partial h|$ is the Euclidean norm of the form $\partial h = \partial h_1 \wedge \cdots \wedge \partial h_{n-m}$ and dS is the measure on M induced by the Euclidean metric.

We define

$$\phi(\zeta, z) = \sum_{j=1}^{n} \frac{\partial \rho}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) - \rho(\zeta).$$

The followings are well-known consequences of the convexity of Ω.

LEMMA 2.1. *The function $\phi : \overline{\Omega} \times \overline{\Omega} \to \mathbb{C}$ has the following properties:*

(1) $\phi \in C^\infty(\overline{\Omega} \times \overline{\Omega})$ *and* $\phi(\zeta, \cdot) \in \mathcal{O}(\Omega)$ *for all* $\zeta \in \overline{\Omega}$;
(2) *For* $\zeta \in \partial\Omega, H_\zeta(\partial\Omega) + \{\zeta\} = \{z \in \mathbb{C}^2; \phi(\zeta, z) = 0\}$;
(3) $2\mathrm{Re}\, \phi(\zeta, z) \geq -\rho(\zeta) - \rho(z)$ *for all* $\zeta, z \in \overline{\Omega}$.

For any $f \in \mathcal{O}(M) \cap C^1(\overline{M})$, we have a representation [6]

$$f(z) = \int_{\zeta \in M} f(\zeta) P(\zeta, z) \quad \text{for} \quad z \in M, \tag{2.1}$$

where

$$P(\zeta, z) = c \frac{\rho^{m+1}(\zeta)}{\phi^{m+1}(\zeta, z)} \left(\partial\overline{\partial} \log \frac{1}{-\rho(\zeta)} \right)^m \wedge \mu.$$

We write M in the integral in (2.1) to emphasize that the integration is performed only over M, even though it would be more correct to write Ω, since the kernel is a (n,n)-current in Ω. Note that

$$\partial\bar{\partial}\log\frac{1}{-\rho}=\frac{\partial\rho\wedge\bar{\partial}\rho}{\rho^2}-\frac{\partial\bar{\partial}\rho}{\rho}.$$

Hence we have

$$f(z)=\int_{\zeta\in M}\frac{f(\zeta)\omega(\zeta,z)}{\phi^{m+1}(\zeta,z)},$$

where ω is a bidegree (m,m) in ζ and $(0,0)$ in z which is smooth in a neighborhood of $\overline{M}\times\overline{\Omega}$ and holomorphic in $z\in\Omega$. Therefore,

$$F(z)=\int_{\zeta\in M}\frac{f(\zeta)\omega(\zeta,z)}{\phi^{m+1}(\zeta,z)}\quad\text{for}\quad z\in\Omega$$

is a holomorphic extension of f to Ω.

For the proof of Theorem 1.3, it is enough to prove that the function

$$G(z)=\int_{\zeta\in M}\frac{f(\zeta)A(\zeta,z)}{\phi^{m+1}(\zeta,z)}dS\quad\text{for}\quad z\in\Omega$$

is C^∞ up to the boundary $\partial\Omega$, where $A\in C^\infty(\overline{M}\times\overline{\Omega})$ is holomorphic in $z\in\Omega$ and dS is a surface measure on M.

LEMMA 2.2 ([9]). *Let $(\zeta_0,z_0)\in\partial\Omega\times\partial\Omega$ such that $\phi(\zeta_0,z_0)=0$. Then there exist neighborhoods W of ζ_0 and V of z_0, such that for each $z\in V$, there exists a C^∞ local coordinate system $\zeta\mapsto t^{(z)}(\zeta)=(t_1,\dots,t_{2n})$ on W with the following properties:*

$$t_1(\zeta)=\rho(\zeta),\ t_2(\zeta)=\operatorname{Im}\phi(\zeta,z),\ t_3(z)=\cdots=t_{2n}(z)=0;$$

$$|t^{(z)}(\zeta)-t^{(z)}(\zeta')|\sim|\zeta-\zeta'| \tag{2.2}$$

for all $\zeta,\zeta'\in W$, with the constants in (2.2) independent of $z\in V$.

The conclusion will follow by Sobolev's embedding theorem if we can show that for all multiindices α

$$\left|\frac{\partial^{|\alpha|}}{\partial z^\alpha}G(z)\right|\le C_\alpha\quad\text{uniformly on}\quad\Omega.$$

For $j = 1, \ldots, n$, it follows that

$$\frac{\partial}{\partial z_j} G(z) = \int_{\zeta \in M} \frac{f(\zeta) A_j(\zeta, z)}{\phi^{m+2}(\zeta, z)} \, dS(\zeta),$$

where

$$A_j(\zeta, z) = \frac{\partial A(\zeta, z)}{\partial z_j} \phi(\zeta, z) + (m+1) A(\zeta, z) \frac{\partial \rho}{\partial \zeta_j}(\zeta).$$

Hence, in general, for any multiindex α,

$$\frac{\partial^{|\alpha|}}{\partial z^\alpha} G(z) = \int_M \frac{f(\zeta) A_\alpha(\zeta, z)}{\phi^{m+1+|\alpha|}(\zeta, z)} \, dS(\zeta), \quad z \in \Omega,$$

where $A_\alpha \in C^\infty(\overline{M} \times \overline{\Omega})$. By the compactness of $\partial\Omega$, and a partition of unity, the estimation of $\frac{\partial^{|\alpha|} G}{\partial z^\alpha}$ is reduced to proving the estimate

$$|I(z)| \leq C_\alpha \quad \text{uniformly for all} \quad z \in \Omega \cap V,$$

where

$$I(z) = \int_{M \cap W} \frac{\chi(\zeta) f(\zeta) A_\alpha(\zeta, z)}{\phi^{m+1+|\alpha|}(\zeta, z)} \, dS(\zeta).$$

W, V are neighborhoods as given in Lemma 2.2, and χ has compact support in W. Let $B_\alpha(\zeta, z) = \chi(\zeta) f(\zeta) A_\alpha(\zeta, z)$. Introduce the coordinate system $t = t^{(z)}(\zeta)$ given by Lemma 2.2. Let

$$\begin{aligned} B_{\alpha,0}(t, z) =& B_\alpha(t, z) \\ B_{\alpha,k}(t, z) =& -(2k-1) \frac{\partial^2 \phi}{\partial t_2^2} B_{\alpha,k-1}(t, z) + \frac{\partial \phi}{\partial t_2} \frac{\partial}{\partial t_2} B_{\alpha,k-1}(t, z), \end{aligned}$$

$k = 1, 2, \ldots$. We first integrate by parts in t_2, using the fact that

$$\frac{1}{\phi^{k+1}} = -\frac{1}{k} \left(\frac{\partial \phi}{\partial t_2} \right)^{-1} \frac{\partial}{\partial t_2} \left(\frac{1}{\phi^k} \right).$$

Since B_α has compact support in W, by integration by parts, one obtains

$$\begin{aligned} I(z) &= \int_{|t| \leq c} \frac{B_\alpha(t, z)}{\phi^{m+|\alpha|+1}} \, dt \\ &= \frac{1}{(m + |\alpha|)} \int_{|t| \leq c} \frac{1}{\phi^{m+|\alpha|}} \frac{\partial}{\partial t_2} \left[\left(\frac{\partial \phi}{\partial t_2} \right)^{-1} B_{\alpha,0}(t, z) \right] dt \\ &= \frac{1}{(m + |\alpha|)} \int_{|t| \leq c} \frac{1}{\phi^{m+|\alpha|}} \left(\frac{\partial \phi}{\partial t_2} \right)^{-2} B_{\alpha,1}(t, z) \, dt. \end{aligned}$$

Continuing this process, we obtain

$$\begin{aligned} I(z) &= \frac{1}{(m+|\alpha|)\cdots m}\int_{|t|\le c}\frac{1}{\phi^m}\left(\frac{\partial\phi}{\partial t_2}\right)^{-2(|\alpha|+1)} B_{\alpha,|\alpha|+1}(t,z)\ dt \\ &= \frac{1}{(m+|\alpha|)!}\int_{|t|\le c}\log\phi\ \left(\frac{\partial\phi}{\partial t_2}\right)^{-2(|\alpha|+m+1)} B_{\alpha,|\alpha|+m+1}(t,z)\ dt. \end{aligned}$$

Note that $\frac{\partial\phi}{\partial t_2} = \frac{\partial}{\partial t_2}\mathrm{Re}\ \phi + i$, and so $|\frac{\partial\phi}{\partial t_2}| \ge 1$. Thus $(\partial\phi/\partial t_2)^{-2k}B_{\alpha,k}$ has compact support. By the inequality (3) in Lemma 2.1, we have

$$\begin{aligned} |I(z)| &\le C_\alpha\int_{|t|\le c}|\log|\phi||\ dt \\ &\le C_\alpha\int_{|(t_1,t_2)|\le c}|\log(|t_1|+|t_2|)|dt_1dt_2 \\ &\le C_\alpha\int_0^1|\log r|\ r\ dr \\ &\le C_\alpha \quad \text{uniformly for all} \quad z\in\Omega\cap V. \end{aligned}$$

Thus we get Theorem 1.3.

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