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ON THE MINKOWSKI UNITS OF 2-PERIODIC KNOTS

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Abstract. In this paper we give a relationship among the Minkowski units, for all odd prime number including ∞, of 2-periodic knot in S^3, its factor knot, and the 2-component link consisting of the factor knot and the set of fixed points of the periodic action.

1. Introduction

A knot k in S^3 is called an *n-periodic knot* ($n \geq 2$) if there exists a $\mathbb{Z}_n$-action on the pair (S^3, k) such that the fixed point set f of the action is homeomorphic to a 1-sphere in S^3 disjoint from the knot k. It is well known that f is unknotted. Hence the quotient map $p : S^3 \to S^3/\mathbb{Z}_n$ is an n-fold cyclic branched covering branched over $p(f) = f_*$ and $p(k) = k_*$ is also a knot in the orbit space $S^3/\mathbb{Z}_n \cong S^3$, which is called the *factor knot* of k. Several relationships among the invariants of n-periodic knot k, its factor knot k_*, and the 2-component link $\ell = k_* \cup f_*$ have been studied by many authors [2, 6, 7, 9, 10, 12].

The Minkowski unit for a tame knot was first defined by Goeritz for odd prime integers [1]. Such Minkowski units derived from knot diagrams are invariants of the linking pairing on the 2-fold branched covering space. In [11], Murasugi defined the *Minkowski unit* $C_p(\ell)$ for an oriented tame link ℓ by using his symmetric link matrix M [8] of ℓ for any prime integer p, including $p = \infty$, which is a generalization of Goeritz's, although the underlying quadratic form is quite different from the one used by Goeritz.

In section 2, we show that for any prime integer p, including $p = \infty$, the Minkowski unit $C_p(H(L))$ of the modified Goeritz matrix $H(L)$ [13] associated to a regular diagram L of an oriented tame link ℓ is also an

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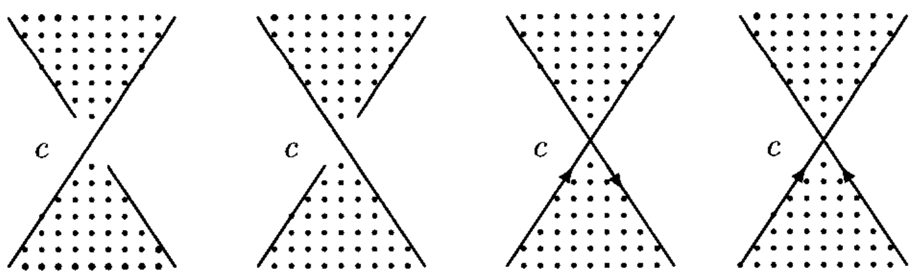
invariant of the link type ℓ and it is equal to the Minkowski unit $C_p(\ell)$ of the link ℓ, as defined by Murasugi.

In section 3, for any odd prime integer p, including ∞, we give a relationship among the Minkowski units $C_p(k)$ of a 2-periodic knot k, its factor knot k_*, and the link $\ell = k_* \cup f_*$ together with $|\Delta_{k_*}(-1)|$ and $|\Delta_\ell(-1,-1)|$, where $\Delta_{k_*}(t)$ and $\Delta_\ell(t_1, t_2)$ are the Alexander polynomials of k_* and the 2-component link $\ell = k_* \cup f_*$, respectively.

2. The Minkowski units of the modified Goeritz matrices

Let ℓ be an oriented link in S^3 and let L be its oriented link diagram in the plane $\mathbb{R}^2 \subset \mathbb{R}^3 = S^3 - \{\infty\}$. Colour the regions of $\mathbb{R}^2 - L$ alternately black and white. Denote the white regions by $X_0, X_1, \cdots, X_w$ (We always take the unbounded region to be white and denote it by X_0). Let $C(L)$ be the set of all crossings of L. Assign an incidence number $\eta(c) = \pm 1$ to each crossing $c \in C(L)$ and define a crossing $c \in C(L)$ to be of *type I* or *type II* as indicated in Figure 1.



$\eta(c) = +1$ $\eta(c) = -1$ Type I Type II

FIGURE 1

Let $S(L)$ denote the compact surface with boundary L, which is built up out of discs and bands. Each disc lies in $S^2 = \mathbb{R}^2 \cup \{\infty\}$ and is a closed black region less a small neighbourhood of each crossing. Each crossing gives a small half-twisted band. Let $\beta_0(L)$ denote the number of the connected components of the surface $S(L)$.

Let $G(L) = (g_{ij})_{1 \leq i,j \leq w}$, where $g_{ij} = -\sum_{c \in C_L(X_i, X_j)} \eta(c)$ for $i \neq j$ and $g_{ii} = \sum_{c \in C_L(X_i)} \eta(c)$, where $C_L(X_i, X_j) = \{c \in C(L) |\ c$ is incident to both X_i and $X_j\}$ and $C_L(X_i) = \{c \in C(L) | c$ is incident to $X_i\}$. The symmetric integral matrix $G(L)$ is called *Goeritz matrix* of ℓ associated to L[1, 3].

Let $C_{II}(L) = \{c_1, c_2, \cdots, c_d\}$ denote the set of all crossings of type II in L and let $A(L) = \text{diag}(-\eta(c_1), -\eta(c_2), \cdots, -\eta(c_d))$, the $d \times d$ diagonal matrix. Then Traldi [13] defined the *modified Goeritz matrix* $H(L)$ of ℓ associated to L by $H(L) = G(L) \oplus A(L) \oplus B(L)$, where $B(L)$ denotes the $(\beta_0(L) - 1) \times (\beta_0(L) - 1)$ zero matrix.

Two integral matrices H_1 and H_2 are said to be *equivalent* if they can be transformed into each other by a finite number of the following two types of transformations and their inverses:

$T_1 : H \to UHU^t$, where U is a unimodular integral matrix,

$T_2 : H \to H \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

If L_1 and L_2 are link diagrams of ambient isotopic oriented links, then the modified Goeritz matrices $H(L_1)$ and $H(L_2)$ are equivalent. The signature $\sigma(\ell)$ and the nullity $n(\ell)$ of an oriented link ℓ in S^3 are given by the formulas: $\sigma(\ell) = \sigma(H(L))$, $n(\ell) = n(H(L)) + 1$, where $\sigma(H(L))$ and $n(H(L))$ are the signature and the nullity of the matrix $H(L)$, respectively [13]. The absolute value of the determinant, $\det(H(L))$, of the modified Goeritz matrix $H(L)$ associated to a diagram L of a link ℓ is clearly an invariant of the link type ℓ. Let $\Delta_k(t)$ denote the Alexander polynomial of a knot k. Then it is well known that $|\Delta_k(-1)| = |\det(G(K))| = |\det(H(K))|$ for any diagram K of the knot k.

Now two symmetric rational matrices A_1 and A_2 are said to be *R-equivalent* if they can be transformed into each other by a finite number of the following two types of transformations and their inverses:

$Q_1 : A \to RAR^t$, where R is a nonsingular rational matrix,

$Q_2 : A \to A \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Any $n \times n$ nonzero symmetric rational matrix A can be transformed by Q_1 into a matrix of the form:

$$\begin{pmatrix} B & O \\ O & O \end{pmatrix},$$

where B is a nonsingular matrix. In particular, if A is a symmetric integral matrix, then A may be transformed by T_1 into the same form. The matrix B is called a *nonsingular matrix associated to A*.

Let A be an $n \times n$ symmetric integral matrix of rank r and B a nonsingular integral matrix associated to A. Then there is a sequence

$B_1, B_2, \cdots, B_r$, called the *σ-series of A,* of principal minors of B such that

(1) B_i is of order i and is a principal minor of B_{i+1},
(2) For $i = 1, 2, \cdots, r-1$, no consecutive matrices B_i and B_{i+1} are both singular.

Denote $D_i = \det(B_i)$. Then for any prime integer p, we define

$$c_p(B) = (-1, -D_r)_p \prod_{i=1}^{r-1} (D_i, -D_{i+1})_p,$$

where $(a,b)_p$ denotes the *Hilbert symbol.* If $D_{i+1} = 0$, then $(D_i, -D_{i+1})_p$ $(D_{i+1}, -D_{i+2})_p$ is interpreted to be $(D_i, -h)_p$ $(h, -D_{i+2})_p$, where h is an arbitrary nonzero integer. Note that $c_p(B)$ is independent of the choice of σ-series of B [5, 11].

DEFINITION 2.1. Let B be a nonsingular integral matrix of order r. Then the Minkowski units $C_p(B)$ of B is defined as follows:

(1) For $p = 2$, $C_2(B) = c_2(B)(-1)^\beta$, where

$$\beta = [\frac{r}{4}] + \{1 + [\frac{r}{2}]\}\frac{(d+1)}{2} + \frac{(d^2-1)m}{8},$$

and [ ] denotes the Gaussian symbol, m the power of 2 occurring in $\det(B)$, and $d = 2^{-m}\det(B)$.

(2) For any odd prime integer p,

$$C_p(B) = c_p(B)(\det(B), p)_p^\alpha,$$

where α denotes the exponent of p occurring in $\det(B)$.

(3) For $p = \infty, C_\infty(B) = \prod C_p(B)$, where the product extends over all prime integer p's.

Let A be an $n \times n$ symmetric integral matrix of rank r and let B and B' be any two nonsingular integral matrices of order r associated to A. Then $C_p(B) = C_p(B')$ for any prime integer p, including $p = \infty$. The *Minkowski unit* $C_p(A)$ of A is defined to be the Minkowski unit $C_p(B)$ of B.

THEOREM 2.2. *Let ℓ be an oriented link in S^3 and let $H(L)$ be the modified Goeritz matrix associated to a diagram L of ℓ. Then the Minkowski unit $C_p(H(L))$ of $H(L)$ is an invariant of the link type ℓ, denoted by $C_p(\ell)$, for any prime integer p, including $p = \infty$.*

Proof. Let L_1 and L_2 be two diagrams of the link ℓ and let $H(L_1)$ and $H(L_2)$ be the modified Goeritz matrices associated to L_1 and L_2, respectively. By [11, Lemma 2.4], it suffices to show that $H(L_1)$ and $H(L_2)$ are R-equivalent matrices.

T_1 : Suppose that $H(L_2) = UH(L_1)U^t$ with unimodular integral matrix U. Then it is obvious from Q_1 that $H(L_1)$ and $H(L_2)$ are R-equivalent.

T_2 : Suppose that $H(L_2) = \begin{pmatrix} H(L_1) & O & O \\ O & 1 & 0 \\ O & 0 & -1 \end{pmatrix}$. Observe that

$$\begin{pmatrix} I & O & O \\ O & 1 & -1 \\ O & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} H(L_1) & O & O \\ O & 1 & 0 \\ O & 0 & -1 \end{pmatrix} \begin{pmatrix} I & O & O \\ O & 1 & \frac{1}{2} \\ O & -1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} H(L_1) & O & O \\ O & 0 & 1 \\ O & 1 & 0 \end{pmatrix},$$

where I denotes the identity matrix with the same order as $H(L_1)$.

By Q_2, $\begin{pmatrix} H(L_1) & O & O \\ O & 0 & 1 \\ O & 1 & 0 \end{pmatrix}$ is R-equivalent to $H(L_1)$. Since $H(L_1)$ and $H(L_2)$ are transformed into each other by a finite sequence of T_1, T_2, or their inverses, they are R-equivalent matrices from the above observations. This completes the proof. $\square$

Remark 2.3. (1) The set of modified Goeritz matrices H obtained from the various diagrams of a link ℓ contains $M + M^t$ for some Seifert matrix M of ℓ. This implies that $C_p(\ell) = C_p(H)$ is equal to the Minkowski unit $C_p(\ell)$ defined by Murasugi [11].

(2) Let A be a symmetric integral matrix and let B be a nonsingular matrix associated to A. Let ν denote the number of odd primes of the form $4s + 3$ occurring with odd powers in the prime factor decomposition of $\det(B)$. It follows that $C_\infty(A) = (-1)^\gamma$, where $\gamma = [\frac{\sigma(A)-2\nu}{2}] + [\frac{\sigma(A)-2\nu}{4}]$ [4].

3. The Minkowski units of 2-periodic knots

Let $\ell = k_* \cup f_*$ be a 2-component oriented link in S^3 such that the component f_* is unknotted and the linking number λ of k_* and f_*, denoted by $\lambda = \text{link}(k_*, f_*)$, is an odd integer. Then the inverse image $k = p_2^{-1}(k_*)$ of k_* in the 2-fold cyclic branched covering $p_2 : \Sigma^3 \to S^3$ branched over f_* is a 2-periodic knot in $\Sigma^3 \cong S^3$ whose factor knot is

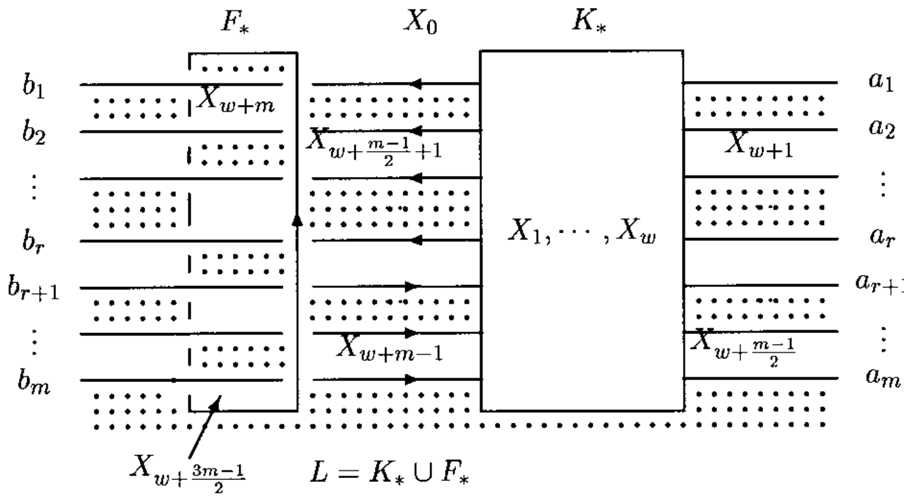


FIGURE 2

the knot k_*. Conversely, every 2-periodic knots in S^3 arises in the this manner.

Now let $L = K_* \cup F_*$ be a regular diagram of $\ell = k_* \cup f_*$ in $\mathbb{R}^2$ which has the form as shown in Figure 2, where the points $a_1, a_2, \cdots, a_m$ are identified with the points $b_1, b_2, \cdots, b_m$. Colour the regions of $\mathbb{R}^2 - L$ alternately black and white. Let w denote the number of white regions in the coloured diagram which does not intersect with the trivial component F_* and let a and b denote the number of the crossings of type II in $K_* = L - F_*$ with incidence number $+1$ and -1, respectively.

In [6, Section 3], the authors discovered a relationship among the modified Goeritz matrices of the 2-periodic knot(or link) k, its factor k_*, and the link $\ell = k_* \cup f_*$ which can be summarized as the following Theorem 3.1.

THEOREM 3.1. *Let $\ell = k_* \cup f_*$ be an oriented 2-component link in S^3 such that f_* is unknotted and $\lambda = \mathrm{link}(k_*, f_*)$ is an odd integer. Let L be a link diagram of ℓ as shown in Figure 2. Then*

(1) *The modified Goeritz matrix $H(L)$ of ℓ associated to L equivalent to the symmetric integral matrix of the form:*

$$H(L) = \begin{pmatrix} M & P & Q & O \\ P^t & N_1 & R & J \\ Q^t & R^t & N_2 & J \\ O & J^t & J^t & S \end{pmatrix} \oplus (-I_a \oplus I_b) \oplus E_r,$$

where M *(*$w\times w$ *matrix),* P, Q, R, N_1, N_2 *are some integral matrices,* $S = \begin{pmatrix} O & O \\ O & 2 \end{pmatrix}$, r *is the positive integer with* $\lambda = 2r - m$, $E_r = -I_r \oplus I_{m-r-1}$ *if* r *is even,* $E_r = -I_{r+1} \oplus I_{m-r}$ *if* r *is odd, and* J *is the* $\frac{m-1}{2} \times (\frac{m-1}{2}+1)$ *matrix of the form: for* $m = 1$, $J = \emptyset$ *and for* $m > 1$,

$$J = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix}.$$

(2) *The modified Goeritz matrix* $H(K_*)$, $K_* = L - F_*$, *of the component* k_* *of* ℓ *is given by*

$$H(K_*) = \begin{pmatrix} M & P+Q \\ P^t + Q^t & N_1 + N_2 + R + R^t \end{pmatrix} \oplus (-I_a \oplus I_b).$$

(3) *Let* $p_2 : \Sigma^3 \to S^3$ *be the 2-fold cyclic branched covering space branched over* f_*. *Then the modified Goeritz matrix* $H(K)$ *of the 2-periodic knot* $k = p_2^{-1}(k_*)$ *in* $\Sigma^3 \cong S^3$ *is given by the symmetric matrix of the form:*

$$H(K) = \begin{pmatrix} M & P & O & Q \\ P^t & N_1+N_2 & Q^t & R+R^t \\ O & Q & M & P \\ Q^t & R+R^t & P^t & N_1+N_2 \end{pmatrix} \oplus (-I_a \oplus I_b) \oplus (-I_a \oplus I_b).$$

LEMMA 3.2. *Let* $H(L)$, $H(K_*)$, *and* $H(K)$ *be the modified Goeritz matrices in Theorem 3.1. Then*

(1) *There exists a nonsingular rational matrix* R *such that*

$$R(H(K) \oplus T(r))R^t = 2\{H(K_*) \oplus H(L)\} \oplus -I_{2a} \oplus I_{2b},$$

where $T(r)$ *denotes the diagonal matrix given by*

$$T(r) = \begin{cases} 4(-I_{\frac{m-1}{2}} \oplus I_{\frac{m-1}{2}+1}) \oplus 2(-I_{2a+r} \oplus I_{2b+m-r-1}) & \text{if } r = \text{even}, \\ 4(-I_{\frac{m-1}{2}} \oplus I_{\frac{m-1}{2}+1}) \oplus 2(-I_{2a+r+1} \oplus I_{2b+m-r}) & \text{if } r = \text{odd}. \end{cases}$$

(2) $\det(H(K)) = \frac{1}{2}\det(H(K_*))\det(H(L))(-1)^{\frac{m-1}{2}}$.

Proof. (1) Let

$$U = \begin{pmatrix} I_w & O & O & -QD & O \\ O & I_s & -I_s & (N_2 - R_1)D & O \\ O & O & I_s & O & O \\ O & O & O & D & O \\ O & O & O & Z & 1 \end{pmatrix} \oplus I_{a+b} \oplus I_{t(r)},$$

where $s = \frac{m-1}{2}$, $Z = (1\ 1 \cdots 1), D = (d_{ij})_{1 \le i,j \le s}$ such that $d_{ij} = 1$ for $i \ge j$, otherwise all zero, and $t(r) = m+1$ or $m-1$ according as r is odd or even.

Now define $V = I_{w+s} \oplus U \oplus I_{a+b} \oplus I_{a+b}$ and

$$W = \begin{pmatrix} I_{w+s} & I_{w+s} & O \\ I_{w+s} & -I_{w+s} & O \\ O & O & I_{2(a+b)} \end{pmatrix} \oplus \begin{pmatrix} I_s & -\frac{1}{2}N_2 + I_s & O \\ I_s & -\frac{1}{2}N_2 - I_s & O \\ O & O & 1 \end{pmatrix}^{-1} \oplus I_{2(a+b)+t(r)}.$$

Then V and W are nonsingular rational matrices and we obtain that

$$\begin{aligned} &W\{H(K) \oplus 4(I_s \oplus -I_s \oplus (1)) \oplus 2(-I_a \oplus I_b \oplus -I_a \oplus I_b \oplus E_r)\}W^t \\ &= (XV)\{2(H(K_*) \oplus H(L)) \oplus (-I_a \oplus I_b \oplus -I_a \oplus I_b)\}(XV)^t \end{aligned}$$

for an appropriate permutation matrix X. This implies the result.

(2) By (1), $\det(H(K))\det(R)^2 = 2^{2w-1}\det(H(K_*))\det(H(L))(-1)^{\frac{m-1}{2}}$. Since $2|\det(H(K))| = |\det(H(K_*))||\det(H(L))|$ [6] and $\det(R)^2 = 2^{2w}$. This implies the result. □

LEMMA 3.3. *Let A be an $n \times n$ nonsingular integral matrix and let m denote the power of 2 occurring in* $\det(A)$.

(1) *Let $d = 2^{-m}\det(A)$. Then*

$$C_2(2A) = \begin{cases} C_2(A)(-1)^{\frac{(d^2-1)n}{8}} & \textit{if } n \textit{ is odd,} \\ C_2(A)(2,d)_2 & \textit{if } n \textit{ is even.} \end{cases}$$

(2) *Let p be any odd prime integer and let α be the power of the odd prime p occurring in* $\det(A)$. *Then*

$$C_p(2A) = C_p(A)(-1)^{\frac{(p^2-1)\alpha}{8}}.$$

(3) $C_\infty(2A) = C_\infty(A)$.

Proof. Let $B_1, B_2, \cdots, B_n$ be a σ-series of A, where $B_n = A$, and let $D_i = \det(B_i)(i = 1, 2, \cdots, n)$. Then $2B_1, 2B_2, \cdots, 2B_n$ is a σ-series of $2A$. Let $\bar{D}_i = \det(2B_i)$. Then $\bar{D}_i = 2^i D_i$ and so, for any prime integer p,

$$\begin{aligned} c_p(2A) &= (-1, -\bar{D}_n)_p \prod_{i=1}^{n-1} (\bar{D}_i, -\bar{D}_{i+1})_p \\ &= \{(-1, -D_n)_p \prod_{i=1}^{n-1} (D_i, -D_{i+1})_p\}\epsilon(p) \\ &= c_p(A)\epsilon(p), \end{aligned}$$

where

$$\begin{aligned} \epsilon(p) &= (-1, 2^n)_p \prod_{i=1}^{n-1} (2^i, -2^{i+1}D_{i+1})_p (2^{i+1}, D_i)_p \\ &= \begin{cases} 1 & \text{if } n \text{ is odd,} \\ (2, \det(A))_p & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

In order to show (1), let m denote the power of 2 occurring in $\det(A)$ and let $d = 2^{-m}\det(A)$. Let $\bar{m}$ be the power of 2 occurring in $\det(2A)$ and let $\bar{d} = 2^{-\bar{m}}\det(2A)$. Then $\bar{m} = m+n$ and $d = \bar{d}$. By Definition 2.1,

$$\begin{aligned} C_2(2A) &= c_2(2A)(-1)^{\bar{\beta}} \\ &= \begin{cases} c_2(A)(-1)^{\bar{\beta}} & \text{if } n \text{ is odd,} \\ c_2(A)(2, \det(A))_2(-1)^{\bar{\beta}} & \text{if } n \text{ is even,} \end{cases} \end{aligned}$$

where $\bar{\beta} = [\frac{n}{4}] + \{1 + [\frac{n}{2}]\}\frac{(\bar{d}+1)}{2} + \frac{(\bar{d}^2-1)\bar{m}}{8} = ([\frac{n}{4}] + \{1 + [\frac{n}{2}]\}\frac{(d+1)}{2} + \frac{(d^2-1)m}{8}) + \frac{(d^2-1)n}{8}$. Since $\det(A) = 2^m d$ and $(2,2)_2 = 1$, we obtain that

$$C_2(2A) = \begin{cases} C_2(A)(-1)^{\frac{(d^2-1)n}{8}} & \text{if } n \text{ is odd,} \\ C_2(A)(2, d)_2 & \text{if } n \text{ is even.} \end{cases}$$

(2) Let α denote the power of p occurring in $\det(A)$. By Definition 2.1, for any odd prime integer p,

$$\begin{aligned} C_p(2A) &= c_p(2A)(\det(2A), p)_p^\alpha = c_p(2A)(\det(A), p)_p^\alpha (2^n, p)_p^\alpha \\ &= \begin{cases} C_p(A)(2, p)_p^\alpha & \text{if } n \text{ is odd,} \\ C_p(A)(2, \det(A))_p & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Note that $(2, \det(A))_p = (2, p)_p^\alpha$ and $(2, p)_p = (-1)^{\frac{p^2-1}{8}}$. Hence $C_p(2A) = C_p(A)(-1)^{\frac{(p^2-1)\alpha}{8}}$.

(3) Since $\sigma(2A) = \sigma(A)$ and the number ν of odd primes of the form $4s + 3$ occurring with odd powers in the prime factor decomposition of $\det(2A)$ is equal to that of $\det(A)$, it follows Remark 2.3(2) that $C_\infty(2A) = C_\infty(A)$. □

From Lemma 3.2, Lemma 3.3, [11, (2.5)], and the properties of Hilbert symbol [5], we obtain the following

LEMMA 3.4. *For any odd prime integer p,*

(1) $C_p(T(r)) = 1$.
(2) $C_p(H(K) \oplus T(r)) = C_p(H(K))$.
(3) $C_p(2\{H(K_*) \oplus H(L)\}) = C_p(H(K_*) \oplus H(L))(-1)^{\frac{(p^2-1)\alpha}{8}}$, *where α denotes the power of p occurring in* $\det(H(K_*) \oplus H(L))$.

Let $\Delta_{k_*}(t)$ and $\Delta_{k_* \cup f_*}(t_1, t_2)$ denote the Alexander polynomials of k_* and $\ell = k_* \cup f_*$, respectively. Then

THEOREM 3.5. *Let k be a 2-periodic knot in S^3 with the fixed point set f and let k_* be its factor knot and f_* be the orbit of f. Then*

(1) *For any odd prime integer p,*

$$C_p(k)(-1)^{\frac{(p^2-1)\alpha}{8}} = C_p(k_*)C_p(k_* \cup f_*)(p,p)_p^{\alpha_1\alpha_2},$$

where α, α_1, and α_2 denote the powers of p occurring in $|\Delta_k(-1)|$, $|\Delta_{k_* \cup f_*}(-1,-1)|$, and $|\Delta_{k_*}(-1)|$, respectively.

(2)
$$C_\infty(k)(-1)^{[\frac{\sigma(k)-2\nu+2\lambda+2}{4}]}$$
$$= C_\infty(k_*)C_\infty(k_* \cup f_*)(-1)^{[\frac{\sigma(k_*)-2\nu_1}{4}]+[\frac{\sigma(k_* \cup f_*)-2\nu_2}{4}]},$$

where ν, ν_1, and ν_2 be the number of odd primes of the form $4s+3$ occurring with odd powers in the prime factor decomposition of $|\Delta_k(-1)|, |\Delta_{k_*}(-1)|$, and $|\Delta_{k_* \cup f_*}(-1,-1)|$, respectively, and [ ] denotes the Gaussian symbol.

Proof. From [11, Lemma 2.4] and Lemma 3.2(1), for any prime integer p, it follows that

$$C_p(H(K) \oplus T(r)) = C_p(2\{H(K_*) \oplus H(L)\} \oplus -I_{2a} \oplus I_{2b}).$$

(1) By [11, (2.5)], Lemma 3.4, and the fact that $C_p(-I_{2a} \oplus I_{2b}) = 1$, we obtain that for any odd prime p,

$$C_p(H(K)) = C_p(H(K_*) \oplus H(L))(-1)^{\frac{(p^2-1)\alpha}{8}},$$

where α denotes the powers of p occurring in $\det(H(K_*) \oplus H(L))$ and

$$C_p(H(K)) = C_p(H(K_*))C_p(H(L))(\det(H(K_*),p)_p^{\alpha_1}(\det(H(L),p)_p^{\alpha_2}$$
$$(\det(H(K_*)), \det(H(L)))_p(-1)^{\frac{(p^2-1)(\alpha)}{8}},$$

where α_1 and α_2 denote the powers of p occurring in $\det(H(L))$ and $\det(H(K_*))$, respectively. Let $d(k_*) = p^{-\alpha_2}\det(H(K_*)), d(\ell) = p^{-\alpha_1}\det(H(L))$. Then $\alpha_1 + \alpha_2 = \alpha$ and

$$\begin{aligned}&(\det(H(K_*),p)_p^{\alpha_1}(\det(H(L),p)_p^{\alpha_2}(\det(H(K_*)), \det(H(L)))_p \\ &= (d(k_*), d(\ell))_p(p,p)_p^{\alpha_1\alpha_2} = (p,p)_p^{\alpha_1\alpha_2}.\end{aligned}$$

It follows from Lemma 3.2(2) that α is equal to the power of p occurring in $\det(H(K))$. This implies the result.

(2) Let ν, ν_1, ν_2 be the number of odd primes of the form $4s+3$ occurring with odd powers in the prime factor decomposition of $\det(H(K))$, $\det(H(K_*))$, $\det(H(L))$, respectively, and let $\gamma = [\frac{\sigma(k)-2\nu}{2}] + [\frac{\sigma(k)-2\nu}{4}]$. Then $C_\infty(k) = C_\infty(H(K)) = (-1)^\gamma$. Since $2\det(H(K)) = \det(H(K_*))$ $\det(H(L))$ and $\sigma(k) = \sigma(k_*) + \sigma(\ell) + \lambda$ [6], $\nu = \nu_1 + \nu_2$ and $\gamma = [\frac{\sigma(k_*)-2\nu_1}{2}] + [\frac{\sigma(k_* \cup f_*)-2\nu_2}{2}] + [\frac{\sigma(k)-2\nu+2\lambda+2}{4}]$. This implies the result and we complete the proof of Theorem 3.5. □

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