Bull. Korean Math. Soc. **38** (2001), No. 3, pp. 463–474

RODRIGUES TYPE FORMULA FOR MULTI-VARIATE ORTHOGONAL POLYNOMIALS

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ABSTRACT. We find Rodrigues type formula for multi-variate orthogonal polynomial solutions of second order partial differential equations.

1. Introduction and preliminaries

Rodrigues formula for classical orthogonal polynomials in one variable is well developed in [1, 2, 6].

In this work, we are concerned with Rodrigues type formula for multi-variate orthogonal polynomial solutions of a second order partial differential equation of spectral type in d variables:

$$(1.1)\qquad L[u] = \sum_{i,j=1}^{d} A_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} B_i \frac{\partial u}{\partial x_i} = \lambda_n u, \quad n = 0, 1, 2, \cdots ,$$

where $d \geq 2$ is an integer.

Let $\mathbb{N}_0$ be the set of nonnegative integers and $\mathbb{R}$ the set of real numbers. For $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}_0^d$ and $\mathbf{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d$ we write $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$. For any integer $n \in \mathbb{N}_0$, let Π_n^d be the space of real polynomials in d variables of (total) degree $\leq n$ and Π^d the space of all real polynomials in d variables. Also, let r_n^d be the number of monomials of degree exactly n. Then

$$\dim \Pi_n^d = \binom{n+d}{d} \text{ and } r_n^d = \binom{n+d-1}{n}.$$

Received November 2, 2000.

2000 Mathematics Subject Classification: 33C50, 35P99.

Key words and phrases: multi-variate orthogonal polynomials, partial differential equations, Rodrigues type formula.

This work is partially supported by BK-21 project and KOSEF(99-2-101-001-5).

By a polynomial system(PS), we mean a sequence of polynomials

$$\{\phi_\alpha(\mathbf{x})\}_{\alpha\in\mathbb{N}_0^d}$$

such that $\deg(\phi_\alpha) = |\alpha|$ and $\{\phi_\alpha\}_{|\alpha|=n}$ are linearly independent modulo Π_{n-1}^d for $n \in \mathbb{N}_0$ ($\Pi_{-1}^d = \{0\}$).

For $n \in \mathbb{N}_0$, and $\mathbf{x} \in \mathbb{R}^d$, and any PS $\{\phi_\alpha(\mathbf{x})\}_{\alpha\in\mathbb{N}_0^d}$, we write

$$\mathbf{x}^n := [\mathbf{x}^\alpha : |\alpha| = n]^T \text{ and } \Phi_n := [\phi_\alpha : |\alpha| = n]^T$$

which are vectors in $\mathbb{R}^{r_n^d}$ whose elements are arranged according to the lexicographical order of $\{\alpha \in \mathbb{N}_0^d : |\alpha| = n\}$ and use also $\{\Phi_n\}_{n=0}^\infty$ to denote the PS $\{\phi_\alpha\}_{\alpha\in\mathbb{N}_0^d}$.

For a matrix $\Psi = [\psi_{ij}]_{i=0,j=0}^{m\ \ n}$ of polynomials in Π^d and a moment functional(i.e., a linear functional) σ on Π^d, we let

$$\langle\sigma, \Psi\rangle = [\langle\sigma, \psi_{ij}\rangle]_{i=0,j=0}^{m\ \ n}.$$

A PS $\{\mathbb{P}_n\}_{n=0}^\infty$ is said to be monic if

$$\mathbb{P}_n(\mathbf{x}) = \mathbf{x}^n \text{ modulo } \Pi_{n-1}^d, \quad n \in \mathbb{N}_0.$$

For any PS $\{\Phi_n\}_{n=0}^\infty$, where $\Phi_n = A_n\mathbf{x}^n$ modulo Π_{n-1}^d, A_n, $n \geq 0$, is an $r_n^d \times r_n^d$ constant non-singular matrix. We then call the monic PS $\{\mathbb{P}_n\}_{n=0}^\infty$ the normalization of $\{\Phi_n\}_{n=0}^\infty$, where $\mathbb{P}_n := A_n^{-1}\Phi_n$.

For any moment functional σ on Π^d, we let

$$\langle\frac{\partial\sigma}{\partial x_i}, \phi\rangle = -\langle\sigma, \frac{\partial\phi}{\partial x_i}\rangle, \quad i = 1, 2, \cdots, d,$$

and

$$\langle\psi\sigma, \phi\rangle = \langle\sigma, \psi\phi\rangle$$

for any polynomials $\phi(\mathbf{x})$ and $\psi(\mathbf{x})$.

Definition 1.1. ([4]) A PS $\{\Phi_n\}_{n=0}^\infty$ is a weak orthogonal polynomial system (WOPS) if there is a non-zero moment functional σ such that

$$\langle\sigma, \Phi_m\Phi_n^T\rangle = K_n\delta_{mn} \quad \text{if } m \neq n \text{ and } m, n \in \mathbb{N}_0$$

where $K_n := \langle\sigma, \Phi_n\Phi_n^T\rangle, n \in \mathbb{N}_0$, is an $r_n^d \times r_n^d$ constant diagonal matrix. If furthermore $K_n, n \in \mathbb{N}_0$, is nonsingular (respectively, positive-definite) diagonal matrix, we call $\{\Phi_n\}_{n=0}^\infty$ an orthogonal polynomial system (OPS) (respectively, a positive-definite OPS). In this case, we say that $\{\Phi_n\}_{n=0}^\infty$ is a WOPS or an OPS relative to σ.

A PS $\{\Phi_n\}_{n=0}^{\infty}$ is a WOPS relative to σ if and only if $\langle \sigma, \Phi_n R \rangle = 0$ for any polynomial $R(\mathbf{x}) \in \Pi_{n-1}^d$.

For any PS $\{\Phi_n\}_{n=0}^{\infty}$, there is a unique moment functional σ, called the canonical moment functional of $\{\Phi_n\}_{n=0}^{\infty}$, defined by the conditions

$$\langle \sigma, 1 \rangle = 1 \text{ and } \langle \sigma, \Phi_n \rangle = 0, \; n \geq 1.$$

Note that if $\{\Phi_n\}_{n=0}^{\infty}$ is a WOPS relative to σ, then σ must be a non-zero constant multiple of the canonical moment functional of $\{\Phi_n\}_{n=0}^{\infty}$.

DEFINITION 1.2. A moment functional σ is quasi-definite (respectively, positive-definite) if there is an OPS (respectively, a positive-definite OPS) relative to σ.

The following was proved in [4](see also [5]).

PROPOSITION 1.1. *For a moment functional $\sigma \neq 0$, the following statements are all equivalent:*

(i) *σ is quasi-definite (respectively, positive-definite);*
(ii) *There is a unique monic WOPS $\{\mathbb{P}_n\}_{n=0}^{\infty}$ relative to σ;*
(iii) *There is a monic WOPS $\{\mathbb{P}_n\}_{n=0}^{\infty}$ relative to σ such that $H_n := \langle \sigma, \mathbb{P}_n \mathbb{P}_n^T \rangle$, $n \in \mathbb{N}_0$, is a nonsingular (respectively, positive-definite) symmetric matrix;*
(iv) *D_n is nonsingular (respectively, positive-definite), where*

$$D_n := \left[\sigma_{\alpha+\beta}\right]_{|\alpha|=0,|\beta|=0}^{n \quad\; n}, \quad n \in \mathbb{N}_0,$$

and $\sigma_\alpha = \langle \sigma, \mathbf{x}^\alpha \rangle$, $\alpha \in \mathbb{N}_0^d$, are the moments of σ.

Let $\{\mathbb{P}_n\}_{n=0}^{\infty}$ be the monic WOPS relative to a quasi-definite moment functional σ and A_n the $r_n^d \times r_n^d$ nonsingular matrix such that $A_n H_n A_n^T = \langle \sigma, (A_n \mathbb{P}_n)(A_n \mathbb{P}_n)^T \rangle$ is diagonal. Then $\{\Phi_n := A_n \mathbb{P}_n\}_{n=0}^{\infty}$ is an OPS relative to σ. It is also easy(cf. [8]) to see that σ is positive-definite if and only if $\langle \sigma, \phi^2 \rangle > 0$ for any polynomial $\phi(\mathbf{x}) \not\equiv 0$.

LEMMA 1.2. (see Lemma 2.2 in [3]) *Let σ and τ be moment functionals and $R(\mathbf{x})$ a polynomial in Π^d. Then*

(i) *$\sigma = 0$ if and only if $\dfrac{\partial \sigma}{\partial x_i} = 0$ for some $i = 1, \cdots, d$.*

Assume that σ is quasi-definite and let $\{\Phi_n\}_{n=0}^{\infty}$ be an OPS relative to σ. Then

(ii) *$R(\mathbf{x})\sigma = 0$ if and only if $R(\mathbf{x}) = 0$;*

(iii) $\langle \tau, \phi_\alpha \rangle = 0$, $|\alpha| > k$ $(k \in \mathbb{N}_0)$ *if and only if* $\tau = \psi(\mathbf{x})\sigma$ *for some polynomial* $\psi(\mathbf{x}) \in \Pi^d_k$.

Proof. (i) and (ii) are obvious.

(iii) $\Leftarrow$: It is trivial from the orthogonality of $\{\Phi_n\}_{n=0}^\infty$ relative to σ.

(iii) $\Rightarrow$: Consider a moment functional $\widetilde{\tau} = (\sum_{j=0}^{k} \mathbb{C}_j \Phi_j)\sigma$, where $\mathbb{C}_j = [C_\alpha]_{|\alpha|=j}$, $0 \le j \le k$, are arbitrary constant row vectors. Then

$$\langle \widetilde{\tau}, \Phi_n \rangle = \sum_{j=0}^{k} \langle \sigma, \Phi_n \Phi_j^T \rangle \mathbb{C}_j^T = \begin{cases} 0, & \text{if} \quad n > k \\ \langle \sigma, \Phi_n \Phi_n^T \rangle C_n^T, & \text{if} \quad 0 \le n \le k. \end{cases}$$

Hence, if we take $\mathbb{C}_n = \langle \tau, \Phi_n \rangle^T H_n^{-1}$, $0 \le n \le k$, then $\langle \tau, \Phi_n \rangle = \langle \widetilde{\tau}, \Phi_n \rangle$, $n \in \mathbb{N}_0$, so that $\tau = \widetilde{\tau}$. □

2. Differential operator $L[\cdot]$

We now recall quickly results from [3], which we need to develop Rodrigues type formula. If the differential equation (1.1) has a PS $\{\Phi_n\}_{n=0}^\infty$ as solutions, then it must be of the form

$$\begin{aligned} L[u] &= \sum_{i,j=1}^{d} A_{ij}(\mathbf{x}) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} B_i(\mathbf{x}) \frac{\partial u}{\partial x_i} \\ &= \sum_{i,j=1}^{d} \left(a x_i x_j + \sum_{k=1}^{d} b_{ij}^k x_k + c_{ij} \right) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} (g x_i + h_i) \frac{\partial u}{\partial x_i} \\ &= \lambda_n u, \quad n \in \mathbb{N}_0, \end{aligned} \tag{2.1}$$

where $\lambda_n := an(n-1) + gn$. Without the loss of generality, we may assume that the matrix $[A_{ij}(\mathbf{x})]_{i,j=1}^d$ is symmetric, that is, $A_{ij}(\mathbf{x}) = A_{ji}(\mathbf{x})$, $1 \le i, j \le d$. We also assume that $\sum_{i,j=0}^{d} |A_{ij}| \not\equiv 0$ and $|a| + |g| \neq 0$ since otherwise the differential equation (2.1) can not have an OPS as solutions. The differential operator $L[\cdot]$ in (2.1) is called to be admissible if $\lambda_m \neq \lambda_n$ for $m \neq n$ or equivalently $an + g \neq 0$ for $n \ge 0$. It is then easy to see (cf. [3, 4]) that the differential equation (2.1) is admissible if and only if the differential equation (2.1) has a unique monic PS as solutions.

From now on, we always assume that $L[\cdot]$ is admissible.

PROPOSITION 2.1. (see Theorem 3.7 in [3]) *Let σ be the canonical moment functional of a PS $\{\Phi_n\}_{n=0}^{\infty}$ of solutions to the differential equation (2.1). Then the following statements are all equivalent :*

(i) $\{\Phi_n\}_{n=0}^{\infty}$ *is a WOPS relative to* σ;

(ii) $M_i[\sigma] := \sum_{j=1}^{d} \frac{\partial(A_{ij}\sigma)}{\partial x_j} - B_i\sigma = 0, \quad i = 1, \cdots, d;$

(iii) $\langle \sigma, x_i\Phi_n \rangle = 0, \quad n \geq 2$ *and* $i = 1, \cdots, d.$

We say that the differential operator $L[\cdot]$ is symmetric if $L[\cdot] = L^*[\cdot]$ where $L^*[\cdot]$ is the formal Lagrange adjoint of $L[\cdot]$ defined by

$$L^*[u] := \sum_{i,j=1}^{d} \frac{\partial^2(A_{ij}u)}{\partial x_i \partial x_j} - \sum_{i=1}^{d} \frac{\partial(B_i u)}{\partial x_i}.$$

We also say that $L[\cdot]$ is symmetrizable if there is a non-zero C^2-function $s(\mathbf{x})$ in some open subset of $\mathbb{R}^d$ such that $sL[\cdot]$ is symmetric. In this case, we call $s(\mathbf{x})$ a symmetry factor of $L[\cdot]$. In fact(see [3]), a C^2-function $s(\mathbf{x})$ $(\not\equiv 0)$ is a symmetry factor of $L[\cdot]$ if and only if $s(\mathbf{x})$ is a non-zero solution of the following so-called symmetry equations :

$$M_i[s] = \sum_{j=1}^{d} \frac{\partial(A_{ij}s)}{\partial x_j} - B_i s = 0, \quad i = 1, \cdots, d. \tag{2.2}$$

PROPOSITION 2.2. (see Lemma 3.9 and Theorem 3.11 in [3]) *If the admissible differential equation* (2.1) *has an OPS* $\{\Phi_n\}_{n=0}^{\infty}$ *as solutions, then* $|[A_{ij}]_{i,j=1}^{d}| \not\equiv 0$ *in any non-empty open subset of* $\mathbb{R}^d$ *and* $L[\cdot]$ *is symmetrizable.*

3. Rodrigues type formula

From now on, we may and shall assume (see Proposition 2.2) that $|[A_{ij}]_{i,j=1}^{d}| \not\equiv 0$ and the differential operator $L[\cdot]$ in (2.1) is symmetrizable. Let $s(\mathbf{x})(\not\equiv 0)$ be a symmetry factor of $L[\cdot]$. That is, $s(\mathbf{x})$ is any non-zero solution of the symmetry equations

$$M_i[s] = \sum_{j=1}^{d} (A_{ij}s)_{x_j} - B_i s = 0, \quad i = 1, \cdots, d. \tag{3.1}$$

Solving the equations (3.1) for s_{x_i}, $i = 1, \cdots, d$, yields

$$\alpha \frac{\partial s}{\partial x_i} = \beta^i s, \quad i = 1, \cdots, d \tag{3.2}$$

where $\alpha := |[A_{ij}]_{i,j=1}^d|$ and β^i is the determinant of the matrix obtained from $[A_{ij}]_{i,j=1}^d$ by replacing i-th column of $[A_{ij}]_{i,j=1}^d$ with

$$\Big[B_1 - \sum_{j=1}^d \frac{\partial A_{1j}}{\partial x_j}, \cdots, B_d - \sum_{j=1}^d \frac{\partial A_{dj}}{\partial x_j}\Big]^T.$$

Note that $\deg(\alpha) \le 2d - 1$ and $\deg(\beta^i) \le 2d - 2$, $i = 1, \cdots, d$.

Decompose A_{ij}, $1 \le i, j \le d$, into

$$\begin{cases} A_{ii} = D_1^i D_2^i, & i = 1, \cdots, d, \\ A_{ij} = D_1^i E^{ij} D_1^j, & 1 \le i, j \le d, \text{ and } i \ne j \end{cases} \tag{3.3}$$

where $D_1^i \not\equiv 0$, $i = 1, \cdots, d$. Then

$$\alpha = D_1^1 \cdots D_1^d \alpha_0 = \alpha_0 \prod_{j=1}^d D_1^j,$$

$$\beta^i = D_1^1 \cdots D_1^{i-1} D_1^{i+1} \cdots D_1^d \beta_0^i = \beta_0^i \prod_{\substack{j=1 \\ j \ne i}}^d D_1^j, \quad i = 1, \cdots, d$$

where

$$\alpha_0 = |\,(l_{ij})\,|, \quad l_{ij} = \begin{cases} D_2^i, & i = j \\ D_1^j E^{ij}, & i \ne j \end{cases}$$

and for $i = 1, \cdots, d$,

$$\beta_0^i = |\,(m_{jk})\,|, \quad m_{jk} = \begin{cases} B_j - \sum_{n=1}^d \frac{\partial(A_{jn})}{\partial x_n}, & k = i \\ D_2^j, & k = j, k \ne i \\ D_1^k E^{jk}, & k \ne i, j. \end{cases}$$

Then the equations (3.2) become

$$p_i \frac{\partial s}{\partial x_i} = \beta_0^i s, \quad i = 1, \cdots, d \tag{3.4}$$

where $p_i = D_1^i \alpha_0$, $i = 1, \cdots, d$.

Note that for each $i = 1, \cdots, d$, $p_i \not\equiv 0$ and $\deg(p_i) \le 2d - 1$, $\deg(\beta_0^i) \le 2d - 2$.

PROPOSITION 3.1. *Let σ be the canonical moment functional of a PS $\{\Phi_n\}_{n=0}^{\infty}$ satisfying the differential equation (2.1). Assume that*

$$\frac{\partial D_1^i}{\partial x_j} = 0, \quad i \neq j \text{ and } 1 \leq i, j \leq d \tag{3.5}$$

and

$$p_i \frac{\partial \sigma}{\partial x_i} = \beta_0^i \sigma, \quad i = 1, \cdots, d. \tag{3.6}$$

Then for any multi-index $\gamma \in \mathbb{N}_0^d$

$$\partial^\gamma \Big[\Big(\prod_{i=1}^{d} p_i^{\gamma_i} \Big) \sigma \Big] = \psi_\gamma \sigma \tag{3.7}$$

where $\psi_\gamma(\mathbf{x})$ is a polynomial of degree $\leq (2d-2)|\gamma|$ and

$$\langle \sigma, \mathbf{x}^{\gamma'} \psi_\gamma(\mathbf{x}) \rangle = 0 \tag{3.8}$$

for any $\gamma' \in \mathbb{N}_0^d$ with $0 \leq |\gamma'| \leq |\gamma|$ and $\gamma' \neq \gamma$.
If moreover, σ is quasi-definite and

$$\deg(p_i) \leq 2 \text{ and } \deg(\beta_0^i) \leq 1, \quad i = 1, \cdots, d, \tag{3.9}$$

then $\{\Psi_n\}_{n=0}^{\infty}$ with $\Psi_n = [\psi_\gamma : |\gamma| = n]^T$ is a WOPS relative to σ and satisfies the differential equation (2.1).

Proof. Assume that the conditions (3.5) and (3.6) hold. Then for any polynomial $\pi(\mathbf{x})$ and any multi-index $\gamma = (\gamma_1, \cdots, \gamma_d) \in \mathbb{N}_0^d$, we have, for each $i = 1, \cdots, d$ with $\gamma_i \neq 0$

$$\frac{\partial(\pi\sigma \prod_{j=1}^{d} p_j^{\gamma_j})}{\partial x_i} = \sigma p_i^{\gamma_i - 1} \pi_i \prod_{\substack{j=1 \\ j \neq i}}^{d} p_j^{\gamma_j},$$

where

$$\pi_i = \gamma_i \pi \frac{\partial p_i}{\partial x_i} + (|\gamma| - \gamma_i) \pi D_1^i \frac{\partial \alpha_0}{\partial x_i} + p_i \pi_{x_i} + \beta_0^i \pi.$$

Since $\deg(\pi_i) \leq \deg(\pi) + \max\{\deg(p_i) - 1, \deg(\beta_0^i)\} \leq \deg(\pi) + 2d - 2$, the first conclusion follows easily by induction on $\gamma \in \mathbb{N}_0^d$. If moreover, the condition (3.9) holds, we have $\deg(\psi_\gamma) \leq |\gamma|$. Now, for all $\gamma' \in \mathbb{N}_0^d$

with $0 \le |\gamma'| \le |\gamma|$ and $\gamma' \ne \gamma$

$$\langle \sigma, \mathbf{x}^{\gamma'} \psi_\gamma \rangle = \langle \psi_\gamma \sigma, \mathbf{x}^{\gamma'} \rangle = \langle \partial^\gamma [(\prod_{i=1}^d p_i^{\gamma_i})\sigma], \mathbf{x}^{\gamma'} \rangle$$
$$= (-1)^\gamma \langle (\prod_{i=1}^d p_i^{\gamma_i})\sigma, \partial^\gamma \mathbf{x}^{\gamma'} \rangle = 0$$

from which (3.8) follows.

Next we further assume that σ is quasi-definite and the condition (3.9) holds. We first claim that $\deg(\psi_\gamma) = |\gamma|$ and $\langle \sigma, \mathbf{x}^\gamma \psi_\gamma \rangle \ne 0$ for $\gamma \in \mathbb{N}_0^d$. Assume that $\deg(\psi_\gamma) \le |\gamma| - 1$ for some $\gamma \in \mathbb{N}_0^d$ with $|\gamma| \ge 1$. Then $\psi_\gamma \equiv 0$ since $\langle \sigma, \pi\psi_\gamma \rangle = 0$ for any $\pi(\mathbf{x})$ in $\Pi^d_{|\gamma|-1}$ by (3.8) and σ is quasi-definite. Hence $\partial^\gamma[(\prod_{i=1}^d p_i^{\gamma_i})\sigma] = 0$ so that $p_i(\mathbf{x}) \equiv 0$ for some i with $1 \le i \le d$ which is a contradiction. Therefore $\deg(\psi_\gamma) = \gamma$ and $\langle \sigma, \mathbf{x}^\gamma \psi_\gamma \rangle \ne 0$ for all $\gamma \in \mathbb{N}_0^d$ by (3.8).

We now claim that for each $n \ge 0$, $\{\psi_\gamma\}_{|\gamma|=n}$ are linearly independent modulo Π^d_{n-1} so that $\{\Psi_n\}_{n=0}^\infty$ is a PS. For any integer $n \ge 1$, let $C_\gamma,\ \gamma \in \mathbb{N}_0^d$ with $|\gamma| = n$, be constants such that $\phi(\mathbf{x}) = \sum_{|\gamma|=n} C_\gamma \psi_\gamma$ is of degree $\le n-1$. Then $\phi(\mathbf{x}) \equiv 0$ since $\langle \sigma, \phi(\mathbf{x})\pi(\mathbf{x}) \rangle = 0$ for any $\pi \in \Pi^d_{n-1}$ by (3.8). Then, for all $\gamma \in \mathbb{N}_0^d$ with $|\gamma| = n$

$$\langle \sigma, \phi(\mathbf{x})\mathbf{x}^\gamma \rangle = C_\gamma \langle \sigma, \psi_\gamma \mathbf{x}^\gamma \rangle = 0$$

so that $C_\gamma = 0,\ \gamma \in \mathbb{N}_0^d$ with $|\gamma| = n$ by the first claim. Hence $\{\psi_\gamma\}_{|\gamma|=n}$ are linearly independent modulo Π^d_{n-1} and so $\{\Psi_n\}_{n=0}^\infty$ is a WOPS relative to σ by (3.8).

Finally, in order to see that $\{\Psi_n\}_{n=0}^\infty$ satisfy the differential equation (2.1), we let $\{\mathbb{P}_n\}_{n=0}^\infty$ and $\{\mathbb{Q}_n\}_{n=0}^\infty$ be the normalizations of $\{\Phi_n\}_{n=0}^\infty$ and $\{\Psi_n\}_{n=0}^\infty$ respectively. Then $\{\mathbb{P}_n\}_{n=0}^\infty$ and $\{\mathbb{Q}_n\}_{n=0}^\infty$ are monic WOPS's relative to σ so that $\mathbb{P}_n = \mathbb{Q}_n$, $n \ge 0$, by Proposition 1.1. Since $\{\Phi_n\}_{n=0}^\infty$ satisfy the differential equation (2.1), $\{\mathbb{Q}_n\}_{n=0}^\infty$ and so $\{\Psi_n\}_{n=0}^\infty$ also satisfy the differential equation (2.1). □

In passing, we note: The moment functional σ in Proposition 3.1 satisfies (3.1) and so (3.2) (see Proposition 2.1). However we can not drop the condition (3.6) since the condition (3.2) for σ : $p_i \dfrac{\partial \sigma}{\partial x_i} = \beta_0^i \sigma$ $(i = 1, \cdots, d)$ does not necessarily imply the condition (3.6).

From Proposition 3.1, we now have:

THEOREM 3.2. *Assume that the differential equation* (2.1) *has an OPS* $\{\Phi_n\}_{n=0}^{\infty}$ *relative to* σ *as solutions. If the conditions* (3.5), (3.6), *and* (3.9) *hold, then the PS* $\{\Psi_n\}_{n=0}^{\infty}$ *with* $\Psi_n = [\psi_\gamma : |\gamma| = n]^T$ *defined by* (3.7) *is a WOPS relative to* σ *and satisfies the differential equation* (2.1).

However, $\langle\sigma, \Psi_n\Psi_n^T\rangle$, with $\{\Psi_n\}_{n=0}^{\infty}$ as in Theorem 3.2, is nonsingular but need not be diagonal, that is, $\{\Psi_n\}_{n=0}^{\infty}$ is a WOPS but need not be an OPS in general(see Example 3.2 below).

We may call (3.7) a (functional) Rodrigues type formula for orthogonal polynomial solutions of the differential equations (2.1). If $s(\mathbf{x})$ happens to be an orthogonalizing weight for a PS $\{\Phi_n\}_{n=0}^{\infty}$, then σ in (3.7) can be replaced by $s(\mathbf{x})$, which is an ordinary Rodrigues type formula.

EXAMPLE 3.1. Assume that

$$\begin{cases} A_{ij} \equiv 0, & i \neq j, 1 \leq i, j \leq d \\ A_{ii}(\mathbf{x}) = A_{ii}(x_i), & 1 \leq i \leq d \end{cases}$$

so that the differential equation (2.1) is of the form

$$L[u] = \sum_{i=1}^{d} A_{ii}(x_i)\frac{\partial^2 u}{\partial {x_i}^2} + \sum_{i=1}^{d} B_i(x_i)\frac{\partial u}{\partial x_i} = \lambda_n u \tag{3.10}$$

where $A_{ii}(x_i) = d_i x_i + f_i$, $B_i(x_i) = g x_i + h_i$, $i = 1, \cdots, d$, and $g \neq 0$. Then it is shown in [3] (cf. [7]) that

- the differential equation (3.10) has a unique monic PS $\{\mathbb{P}_n\}_{n=0}^{\infty}$ as solutions;
- $P_{n\mathbf{e}_i}(\mathbf{x}) = P_{n\mathbf{e}_i}(x_i)$, $i = 1, \cdots, d$, and $P_\gamma(\mathbf{x}) = P_{\gamma_1\mathbf{e}_1}(x_1) \cdots P_{\gamma_d\mathbf{e}_d}(x_d)$, $\gamma \in \mathbb{N}_0^d$, where $\mathbf{e}_i, i = 1, \cdots, d$, is the i-th fundamental vector in $\mathbb{R}^d$;
- $\{\mathbb{P}_n\}_{n=0}^{\infty}$ is a WOPS;
- For each $i = 1, \cdots, d$, $P_{n\mathbf{e}_i}(x_i)$, $n \geq 0$, satisfy

$$A_{ii}(x_i)P''_{n\mathbf{e}_i}(x_i) + B_i(x_i)P'_{n\mathbf{e}_i}(x_i) = \lambda_n P_{n\mathbf{e}_i}(x_i). \tag{3.11}$$

In decomposition (3.3), we take $D_2^i = -1$, $i = 1, \cdots, d$ and $E^{ij} = 0$, $i, j = 1, \cdots, d$. Then

$$D_1^i = -A_{ii}, \quad \alpha_0 = -1, \quad \beta_0^i = B_i - A'_{ii}, \quad p_i = A_{ii}, \quad i = 1, \cdots, d$$

so that the conditions (3.5) and (3.9) hold. On the other hand, the canonical moment functional σ of $\{\mathbb{P}_n\}_{n=0}^{\infty}$ is equal to $\sigma = \sigma^{(x_1)} \otimes \cdots \otimes$

$\sigma^{(x_d)}$, where for each $i = 1, \cdots, d$, $\sigma^{(x_i)}$ is the canonical moment functional of $\{P_{n\mathbf{e}_i}\}_{n=0}^{\infty}$. Since $A_{ii}(x_i)\frac{\partial}{\partial x_i}\sigma^{(x_i)} = (B_i(x_i) - A'_{ii}(x_i))\sigma^{(x_i)}$ for each $i = 1, \cdots, d$, $\sigma = \sigma^{(x_1)} \otimes \cdots \otimes \sigma^{(x_d)}$ satisfies the condition (3.6). Hence, by Proposition 3.1,

$$\partial^{\gamma}\Big[\Big(\prod_{i=1}^{d} A_{ii}^{\gamma_i}\Big)\sigma\Big] = \psi_{\gamma}\sigma, \quad \gamma \in \mathbb{N}_0^d \tag{3.12}$$

where $\{\psi_{\gamma}\}$ is a WOPS relative to σ. In fact, we have

$$\psi_{\gamma}(\mathbf{x}) = g^{|\gamma|}P_{\gamma_1\mathbf{e}_1}(x_1)\cdots P_{\gamma_d\mathbf{e}_d}(x_d), \quad \gamma \in \mathbb{N}_0^d,$$

so that the Rodrigues type formula (3.12) is nothing but the tensor product of one dimensional Rodrigues formulas for $\{P_{n\mathbf{e}_i}\}_{n=0}^{\infty}$, $i = 1, \cdots, d$(see [1, 2, 6]).

We may, of course, replace σ by a symmetry factor $s(\mathbf{x}) = s_1(x_i)\cdots s_d(x_i)$ of the differential equation (3.10), where for each $i = 1, \cdots, d$, $s_i(x_i)$ is symmetry factor of the differential equations (3.11).

EXAMPLE 3.2. Consider the differential equation:

$$L[u] = \sum_{i,j=1}^{d}(x_ix_j - \delta_{ij})\frac{\partial^2 u}{\partial x_i \partial_{x_j}} + \sum_{i=1}^{d} g x_i \frac{\partial u}{\partial x_i} = \lambda_n u. \tag{3.13}$$

In decomposition (3.3), we take $D_1^i = 1$, $i = 1, \cdots, d$ so that $E_{ij} = A_{ij} = x_ix_j$ and

$$\alpha = p = q = \sum_{i=1}^{d} x_i^2 - 1, \quad \beta^i = \beta_0^i = (g-3)x_i, \quad i = 1, \cdots, d.$$

Then if $g \neq 1, 0, -1, \cdots$, then the differential equation (3.13) has an OPS $\{\Phi_n\}_{n=0}^{\infty}$ as solutions and the canonical moment functional σ of $\{\Phi_n\}_{n=0}^{\infty}$ satisfies the condition (3.6) so that by Theorem 3.2, $\{\Psi_n\}_{n=0}^{\infty}$ with $\Psi_n = [\psi_{\gamma} : |\gamma| = n]^T$ defined by

$$\partial^{\gamma}\Big[\Big(\sum_{i=1}^{d} x_i^2 - 1\Big)^{|\gamma|}\sigma\Big] = \psi_{\gamma}(\mathbf{x})\sigma, \quad \gamma \in \mathbb{N}_0^d$$

is a WOPS. But, even in this case, $\{\psi_{\gamma}\}$ is not an OPS. For if we let σ be the canonical moment functional of $\{\psi_{\gamma}\}$, then σ satisfies $L^*[\sigma] = 0$, that is,

$$|\gamma|(|\gamma|-1+g)\sigma_\gamma - \sum_{i=0}^{d} \gamma_i(\gamma_i-1)\sigma_{\gamma-2\mathbf{e}_i} = 0, \quad \gamma \in \mathbb{N}_0^d$$

so that

$$\begin{cases} \sigma_{0,\cdots,0} = 1, & \\ \sigma_{\mathbf{e}_i} = 0, & i=1,\cdots,d \\ \sigma_{\mathbf{e}_i+\mathbf{e}_j} = 0, & i \neq j \text{ and } 1 \leq i,j \leq d \\ \sigma_{2\mathbf{e}_i} = \frac{1}{g+1}, & i=1,\cdots,d \\ \sigma_\gamma = 0, & \gamma \in \mathbb{N}_0^d \text{ with } |\gamma| = 3, \\ \sigma_\gamma = 0, & \gamma \in \mathbb{N}_0^d \text{ with } |\gamma| = 4 \text{ and } \gamma_i = 1 \text{ for some } i \\ \sigma_{4\mathbf{e}_i} = \frac{3}{(g+1)(g+3)}, & i=1,\cdots,d \\ \sigma_{2\mathbf{e}_i+2\mathbf{e}_j} = \frac{1}{(g+1)(g+3)}, & i \neq j \text{ and } 1 \leq i,j \leq d. \end{cases}$$

Hence, it is easy to show that

$$H_2 = \langle \sigma, \Psi_2 \Psi_2^T \rangle$$

is nonsingular but not diagonal.

References

[1] W. C. Brenke, *On polynomial solutions of a class of linear differential equations of the second order*, Bull. Amer. Math. Soc. **36** (1930), 77–84.

[2] C. W. Cryer, *Rodrigues formulas and the classical orthogonal polynomials*, Boll. Unione. Mat. Ital. **25** (1970), 1–11.

[3] Y. J. Kim, K. H. Kwon, and J. K. Lee, *Multi-variate orthogonal polynomials and second order partial differential equation*, Comm. Appl. Anal., to appear.

[4] H. L. Krall and I. M. Sheffer, *Orthogonal polynomials in two variables*, Ann. Mat. Pura Appl. **seri 4, 76** (1967), 325–376.

[5] L. L. Littlejohn, *Orthogonal polynomial solutions to ordinary and partial differential equations*, Proc. 2nd Intern. Symp. Orthog. Polyn. and their Appl., M. Alfaro and al. Ed's, Segovia(Spain), (1986), Lect. Notes Math. **1329**, Springer-Verlag (1988), 98–124.

[6] F. Marcellán, A. Branquinho, and J. Petronilho, *Classical orthogonal polynomials: A functional approach*, Acta Appl. Math. **34** (1994), 283–303.

[7] P. K. Suetin, *Orthogonal polynomials in two variables*, Nauka, Moscow, 1988 (in Russian).

[8] Y. Xu, *On multivariate orthogonal polynomials*, SIAM J. Math. Anal. **4** (1993), 783–794.

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