Bull. Korean Math. Soc. **38** (2001), No. 3, pp. 449–461

A GENERALIZATION OF STRONGLY CLOSE-TO-CONVEX FUNCTIONS

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Abstract. The purpose of this paper is to study several geometric properties for the new class $G_k(\beta)$ including geometric interpretation, coefficient estimates, radius of convexity, distortion property and covering theorem.

1. Introduction

Let $C_\alpha(0 \le \alpha \le 1)$ be the class of all functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

analytic in $E = \{z : |z| < 1\}$ that satisfy $f'(z) \neq 0$ and

$$\int_{\theta_1}^{\theta_2} \frac{\partial}{\partial\theta} \arg[e^{i\theta} f'(re^{i\theta})]d\theta \ge -\alpha\pi$$

for all $\theta_1 < \theta_2$ and $0 \le r < 1$. The class C_1 is the class of close-to-convex functions introduced by Kaplan [4], and the class C_α are subclass of C_1. The class C_0 consists of all convex functions.

Ch. Pommerenke [9] showed that a function $f(z)$ of the form (1.1) belongs to C_α if and only if there exists a function $h(z)$ starlike in E with $h(0) = 0, h'(0) = 1$ such that

$$\left|\arg \frac{zf'(z)}{h(z)}\right| \le \frac{\pi}{2}\alpha, \quad (z \in E). \tag{1.2}$$

Received April 18, 2000.

2000 Mathematics Subject Classification: 30C45.

Key words and phrases: strongly close-to-convex function, bounded boundary rotation.

Let $V_k (2 \le k \le 4)$ be the class of all functions $f(z)$ represented by (1.1) in E and satisfy $f'(z) \ne 0$ in E and

$$\limsup_{r\to 1} \int_0^{2\pi} \left| Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| d\theta \le k\pi \quad (z = re^{i\theta}, r < 1).$$

V_k is the class of functions with boundary rotation at most $k\pi$. Every function $f \in V_k$ can be given by the Stieltjes integral representation

$$(1.3) \qquad 1 + \frac{zf''(z)}{f'(z)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + e^{i\theta} z}{1 - e^{i\theta} z} d\psi(\theta),$$

where $\int_0^{2\pi} d\psi(\theta) = 2\pi$, $\int_0^{2\pi} |d\psi(\theta)| \le k\pi$ and $\psi(\theta)$ is a function of bounded variation on $[0, 2\pi]$.

We shall generalize the definition (1.2) of Pommerenke's class of strongly close-to-convex functions by using a function $g(z)$ in the class V_k of bounded boundary rotation for $2 \le k \le 4$. We shall denote this new class of functions by $G_k(\beta)$.

DEFINITION. Let $f(z)$ be a holomorphic function in E with normalizations $f(0) = 0, f'(0) = 1$. $f(z)$ belongs to the class $G_k(\beta)$ if $f'(z) \ne 0$ in E and satisfies the condition

$$|\arg \frac{f'(z)}{g'(z)}| \le \frac{\beta\pi}{2}, \quad (0 \le \beta \le 1, z \in E)$$

for some g(z) in $V_k \quad (2 \le k \le 4)$.

Note that if $k = 2, G_k(\beta)$ reduces to the class of strongly close-to-convex functions of order β which was studied by Ch. Pommerenke [9]. If $k = 2, \beta = 0$, $G_k(\beta)$ reduces to the class of convex functions. If $k = 2, \beta = 1, G_k(\beta)$ reduces to the class of close-to-convex functions. If $\beta = 0, 2 \le k \le 4$, then $G_k(\beta)$ reduces to the class V_k.

In this note, we reduce several geometric properties for the new class $G_k(\beta)$ including geometric interpretation, radius of convexity, distortion property, covering theorem and coefficient estimates.

2. Properties for the class $G_k(\beta)$

THEOREM 2.1. *For the class of functions $G_k(\beta), 0 \le \beta \le 1,\ 2 \le k \le 4$, the inequality*

$$\int_{\theta_1}^{\theta_2} Re\Big(1 + re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\Big)d\theta > \frac{-k\beta}{2}\pi, \tag{2.1}$$

holds for all $\theta_1 < \theta_2$ and for all $0 \le r < 1$.

Proof. Suppose $f \in G_k(\beta)$, and let g be an associated bounded boundary rotation function. Then for a suitable choice of arguments,

$$|\arg f'(z) - \arg g'(z)| < \frac{\beta\pi}{2} \le \frac{k\beta}{4}\pi, \quad 0 \le \beta \le 1.$$

Let

$$F(r,\theta) = \arg\{\frac{\partial}{\partial\theta}f(re^{i\theta})\} = \arg f'(re^{i\theta}) + \frac{\pi}{2} + \theta$$

and

$$G(r,\theta) = \arg\{\frac{\partial}{\partial\theta}g(re^{i\theta})\} = \arg g'(re^{i\theta}) + \frac{\pi}{2} + \theta.$$

Since g is a bounded boundary rotation function, $G(r,\theta)$ is an increasing function of θ. The definition of $G_k(\beta)$ gives us that

$$|F(r,\theta) - G(r,\theta)| \le \frac{k\beta}{4}\pi, \quad 0 \le \beta \le 1.$$

Thus for $\theta_1 < \theta_2$,

$$\begin{aligned}
&F(r,\theta_2) - F(r,\theta_1)\\
&= [F(r,\theta_2) - G(r,\theta_2)] + [G(r,\theta_2) - G(r,\theta_1)] + [G(r,\theta_1) - F(r,\theta_1)]\\
&> -\frac{k\beta}{4}\pi + 0 - \frac{k\beta}{4}\pi\\
&= -\frac{k\beta}{2}\pi,
\end{aligned}$$

which is equivalent to the condition (2.1). □

REMARK. From Theorem 2.1, we can interpret some geometric meaning for the class $G_k(\beta)$. If we suppose that the image domain is bounded by an analytic curve C, the outward drawn normal at a point on C has an angle $\arg[e^{i\theta}f'(e^{i\theta})]$. Then from (2.1), it follows that the angle of the outward drawn normal turns back at most $\frac{k\beta}{2}\pi$. This is a necessary condition for a function f to belong to $G_k(\beta)$. It will be interesting to see if this condition is also sufficient.

LEMMA 1 ([2]). *Let $g \in V_k$ $(2 \le k \le 4)$. Then there are two starlike functions s_1 and s_2 such that for $z \in E$*

$$g'(z) = \frac{\left(\frac{s_1(z)}{z}\right)^{\frac{k}{4}+\frac{1}{2}}}{\left(\frac{s_2(z)}{z}\right)^{\frac{k}{4}-\frac{1}{2}}}.$$

THEOREM 2.2. *Let $C(\beta)$ denote the class of close-to-convex functions of order β. Then $f \in G_k(\beta)$, $0 \le \beta \le 1$, $2 \le k \le 4$ if and only if*

$$f'(z) = \frac{m_1'(z)^{\frac{k}{4}+\frac{1}{2}}}{m_2'(z)^{\frac{k}{4}-\frac{1}{2}}}, \quad m_1(z), m_2(z) \in C(\beta).$$

Proof. From definition of $G_k(\beta)$, we have

$$f'(z) = g'(z)h^{\beta}(z), \quad g \in V_k \text{ and } |\arg h(z)| < \frac{1}{2}\pi.$$

Using Lemma 1, we know that there are two starlike functions s_1 and s_2 such that for $z \in E$,

$$g'(z) = \frac{\left(\frac{s_1(z)}{z}\right)^{\frac{k}{4}+\frac{1}{2}}}{\left(\frac{s_2(z)}{z}\right)^{\frac{k}{4}-\frac{1}{2}}}.$$

Thus

$$f'(z) = \frac{\left(\frac{s_1(z)}{z}\right)^{\frac{k}{4}+\frac{1}{2}}}{\left(\frac{s_2(z)}{z}\right)^{\frac{k}{4}-\frac{1}{2}}}h^{\beta}(z) = \frac{\left(\frac{s_1(z)h^{\beta}(z)}{z}\right)^{\frac{k}{4}+\frac{1}{2}}}{\left(\frac{s_2(z)h^{\beta}(z)}{z}\right)^{\frac{k}{4}-\frac{1}{2}}} = \frac{(m_1'(z))^{\frac{k}{4}+\frac{1}{2}}}{(m_2'(z))^{\frac{k}{4}-\frac{1}{2}}},$$

where m_1 and m_2 are two suitable selected close-to-convex functions of order β. □

THEOREM 2.3. *Let $f \in G_k(\beta)$. Then the radius r_0 of convexity for the function f is given by*

$$r_0 = \frac{1}{2}[(k+2\beta) - \sqrt{k^2 + 4\beta(k+\beta) - 4}], \quad 2 \leq k \leq 4.$$

Proof. By definition of $G_k(\beta)$

$$zf'(z) = zg'(z)h^{\beta}(z), \quad g \in V_k, \ |arg\ h(z)| < \frac{\pi}{2}.$$

Thus

$$\frac{(zf'(z))'}{f'(z)} = \frac{(zg'(z))'}{g'(z)} + \frac{zh'(z)}{h(z)}$$

and so

$$Re\frac{(zf'(z))'}{f'(z)} \geq Re\frac{(zg'(z))'}{g'(z)} - \left|\frac{zh'(z)}{h(z)}\right|.$$

For $g \in V_k$, it is well-known that, for $z = re^{i\theta}$, $0 \leq r < 1$,

$$Re\frac{(zg'(z))'}{g'(z)} \geq \frac{r^2 - kr + 1}{1 - r^2}.$$

Hence

$$Re\frac{(zf'(z))'}{f'(z)} \geq \frac{r^2 - kr + 1}{1 - r^2} - \beta\frac{2r}{1 - r^2} = \frac{r^2 - (k+2\beta)r + 1}{1 - r^2}.$$

Therefore the radius of convexity $r_0 = \frac{1}{2}[(k+2\beta) - \sqrt{k^2 + 4\beta(k+\beta) - 4}]$.□

LEMMA 2 ([11]). *Let $Q(z)$ be analytic for $z \in E$ with $Q(0) = 1$. Then $Re\ Q(z) \geq \gamma$ if and only if*

$$Q(z) = \frac{1 + (1 - 2\gamma)g(z)}{1 - g(z)},$$

where $g(z)$ is analytic, $g(0) = 0$ and $|g(z)| < 1$ for $z \in E$.

THEOREM 2.4. *Let* $f(z) \in G_k(\beta)$, $0 \le \beta \le 1$, *then*

(1) $|f'(z)| \le \frac{(1+r)^{\frac{k}{2}-1}}{(1-r)^{\frac{k}{2}+1}}(\frac{1+r}{1-r})^\beta$.

(2) $|f'(z)| \ge \frac{(1-r)^{\frac{k}{2}-1}}{(1+r)^{\frac{k}{2}+1}}(\frac{1-r}{1+r})^\beta$.

Equality holds in (1) *for the function*

$$f_1(z) = \int_0^z \frac{(1+t)^{\frac{k}{2}-1}}{(1-t)^{\frac{k}{2}+1}}\Big(\frac{1+t}{1-t}\Big)^\beta dt$$

and equality holds in (2) *for the function*

$$f_2(z) = \int_0^z \frac{(1-t)^{\frac{k}{2}-1}}{(1+t)^{\frac{k}{2}+1}}\Big(\frac{1-t}{1+t}\Big)^\beta dt.$$

Proof. Let $\frac{f'(z)}{\phi'(z)} = Q^\beta(z)$, where $Re\ Q(z) \ge 0$, $0 \le \beta \le 1$, $\phi(z) \in V_k$. Then from the Lemma 2

$$\frac{f'(z)}{\phi'(z)} = \Big[\frac{1+g(z)}{1-g(z)}\Big]^\beta, \tag{2.2}$$

where $g(0) = 0$ and $|g(z)| < 1$ for $z \in E$. Since $g(z)$ satisfies the conditions of Schwarz's lemma, (2.2) yields

$$\Big[\frac{1-r}{1+r}\Big]^\beta \le \Big|\frac{f'(z)}{\phi'(z)}\Big| \le \Big[\frac{1+r}{1-r}\Big]^\beta. \tag{2.3}$$

In [8] it was shown that

$$\frac{(1-r)^{\frac{k}{2}-1}}{(1+r)^{\frac{k}{2}+1}} \le |\phi'(r)| \le \frac{(1+r)^{\frac{k}{2}-1}}{(1-r)^{\frac{k}{2}+1}}. \tag{2.4}$$

Combining (2.3) and (2.4)

$$\frac{(1-r)^{\frac{k}{2}-1}}{(1+r)^{\frac{k}{2}+1}}\Big(\frac{1-r}{1+r}\Big)^\beta \le |f'(z)| \le \frac{(1+r)^{\frac{k}{2}-1}}{(1-r)^{\frac{k}{2}+1}}\Big(\frac{1+r}{1-r}\Big)^\beta.$$

To prove that $f_1(z) \in G_k(\beta)$ and $f_2(z) \in G_k(\beta)$, let

$$f_1(z) = \int_0^z \frac{(1+t)^{\frac{k}{2}-1}}{(1-t)^{\frac{k}{2}+1}}\Big(\frac{1+t}{1-t}\Big)^\beta dt,$$

$$f_2(z) = \int_0^z \frac{(1-t)^{\frac{k}{2}-1}}{(1+t)^{\frac{k}{2}+1}} (\frac{1-t}{1+t})^\beta dt$$

and

$$\phi_1(z) = \int_0^z \frac{(1+t)^{\frac{k}{2}-1}}{(1-t)^{\frac{k}{2}+1}} dt, \quad \phi_2(z) = \int_0^z \frac{(1-t)^{\frac{k}{2}-1}}{(1+t)^{\frac{k}{2}+1}} dt.$$

Since $\frac{f_1'(z)}{\phi_1'(z)} = (\frac{1+z}{1-z})^\beta, \frac{f_2'(z)}{\phi_2'(z)} = (\frac{1-z}{1+z})^\beta$ have argument which is less than $\frac{\beta\pi}{2}$, it suffices to show that $\phi_1(z), \phi_2(z) \in V_k$. However, it was already shown in [8] that $\phi_1(z), \phi_2(z)$ belongs to V_k. □

THEOREM 2.5. *Let $f(z) \in G_k(\beta), 0 \le \beta \le 1$, then*

$$\int_0^r \frac{(1-t)^{\frac{k}{2}-1}}{(1+t)^{\frac{k}{2}+1}} (\frac{1-t}{1+t})^\beta dt \le |f(z)| \le \int_0^r \frac{(1+t)^{\frac{k}{2}-1}}{(1-t)^{\frac{k}{2}+1}} (\frac{1+t}{1-t})^\beta dt.$$

Equality holds on the right-hand side for

$$f_1(z) = \int_0^z \frac{(1+t)^{\frac{k}{2}-1}}{(1-t)^{\frac{k}{2}+1}} (\frac{1+t}{1-t})^\beta dt$$

and on the left-hand side for

$$f_2(z) = \int_0^z \frac{(1-t)^{\frac{k}{2}-1}}{(1+t)^{\frac{k}{2}+1}} (\frac{1-t}{1+t})^\beta dt.$$

Proof. Integrating along the straight line segment from the origin to $z = re^{i\theta}$ and applying Theorem 2.4, we obtain

$$|f(z)| \le \int_0^r |f'(te^{i\theta})| dt \le \int_0^r \frac{(1+t)^{\frac{k}{2}-1}}{(1-t)^{\frac{k}{2}+1}} \left(\frac{1+t}{1-t}\right)^\beta dt,$$

which proves the right-hand inequality. To prove the left-hand inequality, for every r we choose $z_0, |z_0| = r$, such that

$$|f(z_0)| = \min_{|z|=r} |f(z)|.$$

If $L(z_0)$ is the preimage of the segment $\{0, f(z_0)\}$, then

$$|f(z)| \ge |f(z_0)| \ge \int_{L(z_0)} |f'(z)| d|z| \ge \int_0^r \frac{(1-t)^{\frac{k}{2}-1}}{(1+t)^{\frac{k}{2}+1}} \left(\frac{1-t}{1+t}\right)^\beta dt. \quad □$$

THEOREM 2.6. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in* $G_k(\beta)$ *for* $2 \le k \le 4$, $0 \le \beta \le 1$. *Then the coefficients satisfy the inequalities*

$$(2.5)\qquad \begin{aligned} |a_2| &\le \frac{k}{2} + \beta \\ |a_3| &\le \frac{k^2}{4} + \Big(\frac{2\beta+1}{3}\Big)k + \Big(\frac{2\beta^2-1}{3}\Big) \\ |a_4| &\le \frac{k^3}{24} + \frac{3\beta}{8}k^2 + \Big(\frac{\beta^2}{3} + \frac{\beta}{2} + \frac{1}{2}\Big)k + \Big(\frac{\beta^2}{3} + \frac{\beta}{2} - \frac{1}{2}\Big). \end{aligned}$$

Proof. Since $f(z)$ is in $G_k(\beta)$, there exists a function $\phi(z) \in V_k, 2 \le k \le 4$, satisfying $|\arg \frac{f'(z)}{\phi'(z)}| < \frac{\beta\pi}{2}$. We may assume that $f(z)$ and $\phi(z)$ are $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $\phi(z) = z + \sum_{n=2}^{\infty} c_n z^n$. If we write

$$(2.6)\qquad \frac{f'(z)}{\phi'(z)} = [g(z)]^\beta = \Big[1 + \sum_{n=1}^{\infty} b_n z^n\Big]^\beta, \quad |\arg g(z)| < \frac{\pi}{2},$$

then we obtain the following relations among the coefficients

$$(2.7)\qquad \begin{aligned} 2a_2 &= 2c_2 + \beta b_1 \\ 6a_3 &= 6c_3 + 4\beta b_1 c_2 + \beta(\beta-1)b_1^2 + 2\beta b_2 \\ 24a_4 &= 24c_4 + 18\beta b_1 c_3 + 4\beta(\beta-1)b_1^2 c_2 + 8\beta b_2 c_2 \\ &\quad + 2(\beta-1)b_1^2 b_2 + (\beta-1)(\beta-2)b_1^3 \\ &\quad + 6(\beta-1)b_1 b_2 + 4b_2 c_2 + 6b_3. \end{aligned}$$

For the function (2.6), we have the well-known bounds [1]

$$(2.8)\qquad |b_n| \le 2 \quad (n = 1, 2, 3, \cdots)$$

and for the function $\phi(z) \in V_k, 2 \le k \le 4$, we have the inequalities [5]

$$(2.9)\qquad \begin{aligned} |c_2| &\le \frac{k}{2} \\ |c_3| &\le \frac{k^2}{4} + \frac{k-1}{3} \\ |c_4| &\le \frac{k^3+8k}{24}. \end{aligned}$$

From (2.7), (2.8), and (2.9), we obtain the results. □

THEOREM 2.7. *Let* $f(z) \in G_k(\beta)$ *with* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *for* $0 \le \beta \le 1$, $2 \le k \le 4$. *If* $f(z) \ne c$ *for* $z \in E$, *then*

$$|c| \ge \frac{2}{4 + 2\beta + k}.$$

Proof. If $f(z)$ does not assume the value c, then

$$\frac{cf(z)}{c - f(z)} = z + \Big(a_2 + \frac{1}{c}\Big) z^2 + \sum_{n=3}^{\infty} b_n z^n$$

is in the class S of normalized univalent functions in E. Hence,

$$\left| a_2 + \frac{1}{c} \right| \le 2. \tag{2.10}$$

Applying the triangle inequality and the coefficient estimates for $G_k(\beta)$ to (2.10), we obtain

$$\begin{aligned} &\left|\frac{1}{c}\right| - |a_2| \le 2, \\ &\left|\frac{1}{c}\right| \le 2 + |a_2| \le 2 + \beta + \frac{k}{2} = \frac{4 + 2\beta + k}{2}, \\ &|c| \ge \frac{2}{4 + 2\beta + k}. \end{aligned}$$

□

LEMMA 3. *Let* H *be analytic and be defined as*

$$H(z) g'(z) = (z g'(z))', \quad g \in V_k$$

and

$$\begin{aligned} &H(z) = \Big(\frac{k}{4} + \frac{1}{2}\Big) h_1(z) - \Big(\frac{k}{4} - \frac{1}{2}\Big) h_2(z), \\ &|\arg h_1(z)| < \frac{\pi}{2}, \quad |\arg h_2(z)| < \frac{\pi}{2}, \quad h_1(0) = h_2(0) = 1. \end{aligned}$$

Then

$$\frac{1}{2\pi} \int_0^{2\pi} |H(z)|^2 d\theta \le \frac{1 + (k^2 - 1) r^2}{1 - r^2}, \quad z = re^{i\theta}.$$

Proof. From the representation formula by Paatero [7], we can write

$$H(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1+ze^{it}}{1-ze^{it}} d\mu(t),$$

where $\int_0^{2\pi} d\mu(t) = 2\pi$ and $\int_0^{2\pi} |d\mu(t)| \le k\pi$. Let $H(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. Then

$$c_n = \frac{1}{\pi}\int_0^{2\pi} e^{-int} d\mu(t) \quad \text{and} \quad |c_n| \le \frac{1}{\pi}\int_0^{2\pi} |d\mu(t)| \le k.$$

Therefore

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} |H(z)|^2 d\theta &= 1 + \sum_{n=1}^{\infty} |c_n|^2 r^{2n} \le 1 + k^2 \sum_{n=1}^{\infty} r^{2n} \\ &= \frac{1+(k^2-1)r^2}{1-r^2}.\end{aligned}$$

□

THEOREM 2.8. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in G_k(\beta)$. *Then for* $n \ge 1$, $||a_{n+1}| - |a_n|| \le c(k,\beta) n^{\frac{k}{2}-1}$, *where* $c(k,\beta)$ *is a constant and depends on* k *and* β.

Proof. Since $f \in G_k(\beta)$, we can write for $z \in E$,

$$f'(z) = g'(z) h^{\beta}(z), \quad g \in V_k \quad \text{and} \quad |\arg h(z)| < \frac{\pi}{2}.$$

Let

$$F(z) = z(zf'(z))' = zg'(z)[H(z)h(z) + \beta z h'(z)] h^{\beta-1}(z), \tag{2.11}$$

where $|\arg h(z)| < \frac{\pi}{2}$ and $H(z) g'(z) = (zg'(z))'$, with $H(z) = (\frac{k}{4} + \frac{1}{2}) h_1(z) - (\frac{k}{4} - \frac{1}{2}) h_2(z)$,

$$|\arg\ h_1(z)| < \frac{\pi}{2},\ |\arg\ h_2(z)| < \frac{\pi}{2},\ h_1(0) = h_2(0) = 1.$$

Thus, we have for $\xi \in E$ and $n \geq 1$,

$$|(n+1)^2 \xi a_{n+1} - n^2 a_n| \leq \frac{1}{2\pi r^{n+1}} \int_0^{2\pi} |z-\xi||F(z)|d\theta,$$

and by using Lemma 1 and (2.11),

$$\begin{aligned} &|(n+1)^2 \xi a_{n+1} - n^2 a_n| \\ (2.12) \qquad &\leq \frac{1}{2\pi r^{n+1}} \int_0^{2\pi} |z-\xi| \frac{|s_1(z)|^{\frac{1}{4}k+\frac{1}{2}}}{|s_2(z)|^{\frac{1}{4}k-\frac{1}{2}}} |[H(z)h(z) \\ &+ \beta z h'(z)] h^{\beta-1}(z)| d\theta, \end{aligned}$$

where s_1, s_2 are starlike functions. We know that for starlike function $s \in S^*$,

$$(2.13) \qquad \frac{r}{(1+r)^2} \leq |s(z)| \leq \frac{r}{(1-r)^2}.$$

Let $0 < r < 1$. Then by a result of Goluzin [3], there exists a z_1 with $|z_1| = r$ such that for all $z, |z| = r$,

$$(2.14) \qquad |z - z_1||s(z)| \leq \frac{2r^2}{1-r^2}.$$

From (2.12), (2.13), and (2.14), we have

$$\begin{aligned} &|(n+1)^2 \xi a_{n+1} - n^2 a_n| \\ (2.15) \qquad &\leq \frac{1}{2\pi r^{n+1}} \left(\frac{4}{r}\right)^{\frac{1}{4}k-\frac{1}{2}} \left(\frac{2r^2}{1-r^2}\right) \left(\frac{r}{(1-r)^2}\right)^{\frac{1}{4}k-\frac{1}{2}} \\ &\times \int_0^{2\pi} |[H(z)h(z) + \beta z h'(z)] h^{\beta-1}(z)| d\theta. \end{aligned}$$

Now as in [10], we have with $z = re^{i\theta}$,

$$(2.16) \qquad \frac{1}{2\pi} \int_0^{2\pi} |h(z)|^2 d\theta \leq \frac{1+3r^2}{1-r^2} \qquad \text{and}$$

$$\frac{1}{2\pi} \int_0^{2\pi} |zh'(z)| d\theta \leq \frac{2r}{1-r^2}, \quad \text{where} \quad |\arg h(z)| < \frac{\pi}{2}.$$

Also, by using Schwarz's inequality, Lemma 3 and (2.16),

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}|[H(z)h(z)+\beta zh'(z)]h^{\beta-1}(z)|d\theta \\
(2.17)\qquad &\leq \frac{1}{2\pi}\int_0^{2\pi}|H(z)h^{\beta}(z)|d\theta+\frac{1}{2\pi}\int_0^{2\pi}\beta|zh'(z)h^{\beta-1}(z)|d\theta \\
&\leq \left(\frac{1+(k^2-1)r^2}{1-r^2}\right)^{\frac{1}{2}}\left(\frac{1+3r^2}{1-r^2}\right)^{\frac{\beta}{2}}+\beta\frac{2r}{1-r^2}\left(\frac{1+3r^2}{1-r^2}\right)^{\frac{\beta-1}{2}}.
\end{aligned}
$$

Hence from (2.15) and (2.17),

$$
\begin{aligned}
&|(n+1)^2\xi a_{n+1}-n^2a_n| \\
&\leq \frac{1}{r^{n+1}}2^{\frac{1}{2}k}[(1+(k^2-1)r^2)^{\frac{1}{2}}+r\beta]\frac{1}{(1-r)^{\frac{1}{2}k+1}}\left(\frac{1+3r^2}{1-r^2}\right)^{\frac{\beta-1}{2}},
\end{aligned}
$$

and choosing $|\xi|=r=(\frac{n}{n+1})^2$, we obtain for $n\geq 1$,

$$
\begin{aligned}
&n^2||a_{n+1}|-|a_n|| \\
&\leq (k+\beta)e^2 2^{\frac{1}{2}k+2}\left(\frac{4}{3}\right)^{\frac{1}{2}k+1}\left(\frac{19}{15}\right)^{\frac{\beta-1}{2}}n^{\frac{1}{2}k+1}.
\end{aligned}
$$

Thus $||a_{n+1}|-|a_n||\leq c(k,\beta)n^{\frac{1}{2}k-1}$. □

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