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LINEAR DERIVATIONS IN BANACH ALGEBRAS

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ABSTRACT. The main goal of this paper is to show the following: Let d and g be (continuous or discontinuous) linear derivations on a Banach algebra A over a complex field $\mathbb{C}$ such that $\alpha d^3 + dg$ is a linear Jordan derivation for some $\alpha \in \mathbb{C}$. Then the product dg maps A into the Jacobson radical of A.

1. Introduction

Throughout this paper A will represent an associative algebra over a complex field $\mathbb{C}$, and the Jacobson radical of A will be denoted by $rad(A)$. Let I be any closed (2-sided) ideal of a Banach algebra A. Then we let Q_I denote the canonical quotient map from A onto A/I. Recall that a ring R is prime if $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$, and is semiprime if $aRa = \{0\}$ implies that $a = 0$. An additive mapping d from R to R is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. A derivation d is inner if there exists $a \in R$ such that $d(x) = [a, x]$ holds for all $x \in R$, where $[x, y]$ denotes the commutator $xy - yx$. An additive mapping d from R to R is called a Jordan derivation if $d(x^2) = d(x)x + xd(x)$ is fulfilled for all $x \in R$. Obviously, every derivation is a Jordan derivation. The converse is, in general, not true, but Brešar showed that every Jordan derivation on a 2-torsion free semiprime ring is a derivation [1].

The Singer-Wermer theorem, which is a classical theorem of a Banach algebra theory, states that every continuous linear derivation on a commutative Banach algebra maps into its Jacobson radical [6], and Thomas generalized the Singer-Wermer theorem by proving that the continuity assumption can be removed [7]. On the other hand, Posner [4, Theorem

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1] obtained a fundamental theorem (so-called Posner's first theorem) in 1957, which asserts that if d and g are derivations on a 2-torsion free prime ring such that the product dg is also a derivation, then either $d = 0$ or $g = 0$. But we can prove nothing in case d and d^3 (the triple product of d) are derivations. For, let

$$M_2(\mathbb{C}) = \left\{ \begin{pmatrix} x & y \\ w & z \end{pmatrix} : x,\ y,\ w,\ z \in \mathbb{C} \right\},$$

and let d be an inner derivation on a ring $M_2(\mathbb{C})$ defined by

$$d(\begin{pmatrix} x & y \\ w & z \end{pmatrix}) = \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} x & y \\ w & z \end{pmatrix} \right].$$

Since $M_2(\mathbb{C})$ is primitive, it is also prime. A simple calculation shows that $d^3 = 0$ and so d^3 is a derivation, but $d \neq 0$. In connection with the above facts, Vukman proved the following result as a non-commutative version of the Singer-Wermer Theorem [9, Theorem 1]: if d is a continuous linear derivation on a Banach algebra A such that $\alpha d^3 + d^2$ is a linear derivation for some $\alpha \in \mathbb{C}$, then $d(A) \subseteq rad(A)$. In the same paper Vukman raised the question of whether the continuity assumption of d can be dropped. It seems that the answer to this question has not been given until now. The purpose of this note is to give answers to the question.

2. The results

We will use the following lemmas in our main theorem.

Lemma 2.1. *Let R be a 2-torsion free prime ring. Suppose that there exist derivations d_1, d_2, and d_3 on R such that $d_1(d_1^2(x)+d_2(x)) = d_3(x)$ holds for all $x \in R$. Then either $d_1 = 0$ or $d_2 = 0$.*

Proof. See Theorem 2 in [9]. □

Lemma 2.2. *Let d be a linear derivation on a Banach algebra A and J a primitive ideal of A. If there exists a real constant $K > 0$ such that $\|Q_J d^n\| \leq K^n$ for all $n \in \mathbb{N}$, then $d(J) \subseteq J$.*

Proof. See Lemma 1.2 in [8]. □

Let T be a linear operator from a Banach space X into a Banach space Y. Then the separating space of T is defined as

$$\mathcal{S}(T) = \{y \in Y : \text{ there exists } x_n \to 0 \text{ in } X \quad \text{with } T(x_n) \to y\},$$

and T is continuous if and only if $\mathcal{S}(T) = \{0\}$ (see [5, Lemma 1.2]).

Our main theorem is as follows:

THEOREM 2.3. *Let d and g be (continuous or discontinuous) linear derivations on a Banach algebra A such that $\alpha d^3 + dg$ is a linear Jordan derivation for some $\alpha \in \mathbb{C}$. Then $dg(A) \subseteq rad(A)$.*

Proof. Let J be any primitive ideal of A. Using Zorn's lemma, we find a minimal prime ideal P contained in J, and hence $d(P) \subseteq P$ and $g(P) \subseteq P$ [3, Lemma]. Suppose first that P is closed. Then linear derivations d and g on A induce linear derivations $\bar{d}$ and $\bar{g}$ on a Banach algebra A/P defined by $\bar{d}(x+P) = d(x) + P$ and $\bar{g}(x+P) = g(x) + P$ $(x \in A)$, respectively. In case A/P is commutative, $\bar{d}(A/P)$ and $\bar{g}(A/P)$ are contained in the Jacobson radical of A/P by [7]. Let us consider the case A/P is noncommutative. The assumption that $\alpha d^3 + dg$ is a linear Jordan derivation gives that $\alpha\bar{d}^3 + \bar{d}\bar{g}$ is a linear Jordan derivation. Note that $\alpha\bar{d}^3 + \bar{d}\bar{g}$ is a linear derivation on A/P by [1, Theorem 1] since A/P is prime and so semiprime. Assume that $\alpha = 0$. Then $\bar{d}\bar{g}$ is a linear derivation. Hence it follows from Posner's first theorem that either $\bar{d} = 0$ or $\bar{g} = 0$ on A/P. If $\alpha \neq 0$, then all the hypotheses of Lemma 2.1 are fulfilled (observe that $\bar{d}$ stands for d_1 and $\bar{g}/\alpha$ for d_2). Thus either $\bar{d} = 0$ or $\bar{g}/\alpha = 0$ on A/P, that is, either $\bar{d} = 0$ or $\bar{g} = 0$ on A/P. In both cases A/P is commutative or not, either $\bar{d}(A/P) \subseteq J/P$ or $\bar{g}(A/P) \subseteq J/P$. Consequently we see that either $d(A) \subseteq J$ or $g(A) \subseteq J$. If P is not closed, then we see that $\mathcal{S}(d) \subseteq P$ (and $\mathcal{S}(g) \subseteq P$) by [2, Lemma 2.3], where $\mathcal{S}(T)$ is the separating space of a linear operator T. Then we have, by [5, Lemma 1.3], $\mathcal{S}(Q_{\bar{P}}d) = \overline{Q_{\bar{P}}(\mathcal{S}(d))} = \{0\}$ on A/P whence $Q_{\bar{P}}d$ is continuous on A. This means that $Q_{\bar{P}}d(\bar{P}) = \{0\}$ on $A/\bar{P}$, that is, $d(\bar{P}) \subseteq \bar{P}$. Hence, from a derivation d on A, we can also induce a derivation $\tilde{d}$ on a Banach algebra $A/\bar{P}$ defined by $\tilde{d}(x+\bar{P}) = d(x) + \bar{P}$ $(x \in A)$. This shows that we can define a map

$$\Psi\tilde{d}^n Q_{\bar{P}} : A \to A/\bar{P} \to A/\bar{P} \to A/J$$

by $\Psi\tilde{d}^n Q_{\bar{P}}(x) = Q_J d^n(x)$ $(x \in A,\ n \in \mathbb{N})$, where Ψ is the canonical inclusion map from $A/\bar{P}$ onto A/J (the relation $\bar{P} \subseteq J$ guarantees its existence). The continuity of $\tilde{d}$ is clear from [5, Lemma 1.4], and yields that $\|Q_J d^n\| \leq \|\tilde{d}\|^n$ for all $n \in \mathbb{N}$. Now, according to Lemma 2.2, we obtain that $d(J) \subseteq J$. Following the same argument as the case $\mathcal{S}(d) \subseteq P$, we see that $g(J) \subseteq J$ in the case $\mathcal{S}(g) \subseteq P$ as well. Then linear derivations d and g on A induce linear derivations $\hat{d}$ and $\hat{g}$ on a Banach algebra A/J defined by $\hat{d}(x+J) = d(x) + J$ and $\hat{g}(x+J) = g(x) + J$ $(x \in A)$, respectively. The remainder follows the similar method to the case P is closed since the primitive algebra A/J is prime. So we also obtain that either $d(A) \subseteq J$ or $g(A) \subseteq J$. Therefore, $dg(A) \subseteq J$. Since J was arbitrary, we arrive at the conclusion that $dg(A) \subseteq rad(A)$. The proof of the theorem is complete. □

COROLLARY 2.4. *Let d be a (continuous or discontinuous) linear derivation on a Banach algebra A such that $\alpha d^3 + d^2$ is a linear Jordan derivation for some $\alpha \in \mathbb{C}$. Then $d(A) \subseteq rad(A)$.*

Proof. If $\alpha d^3 + d^2$ is a linear Jordan derivation for some $\alpha \in \mathbb{C}$, then we see that $d^2(A) \subseteq rad(A)$ by Theorem 2.3. Since the relation

$$2d(x)yd(x) = d^2(xyx) - xd^2(yx) - d^2(xy)x + xd^2(y)x \in rad(A)$$

holds for all $x, y \in A$ and $rad(A)$ is a semiprime ideal of A, we obtain that $d(A) \subseteq rad(A)$. □

COROLLARY 2.5. *Let d be a (continuous or discontinuous) linear derivation on a Banach algebra A such that $\alpha d^3 + d^2$ is a linear derivation for some $\alpha \in \mathbb{C}$. Then $d(A) \subseteq rad(A)$.*

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