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ON A CHARACTERIZATION OF LINEAR OPERATORS

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ABSTRACT. We obtain a characterization of linear operators on vector spaces and homomorphisms on algebras applying the stability properties of functional equations.

1. Introduction

In 1941 Hyers [3] showed that if $\delta > 0$ and $f : E_1 \to E_2$, is a mapping with E_1 and E_2 Banach spaces, such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta \quad \text{for all} \quad x, y \in E_1,$$

then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E_1$, and if $f(tx)$ is continuous in t for each fixed x, then T is a linear mapping.

Rassias [7] and Gajda [1] gave some generalizations of the Hyers' result in the following ways : Let $f : E_1 \to E_2$ be a mapping such that $f(tx)$ is continuous in t for each fixed x. Assume that there exist $\theta \geq 0$ and $p \neq 1$ such that

$$\frac{\|f(x+y) - f(x) - f(y)\|}{\|x\|^p + \|y\|^p} \leq \theta \quad \text{for all } x, y \in E_1.$$

Then there exists a unique linear mapping $T : E_1 \to E_2$ such that

$$\frac{\|T(x) - f(x)\|}{\|x\|^p} \leq \frac{2\theta}{2 - 2^p} \quad \text{for all} \;\; x \in E_1.$$

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However, it was showed that the similar result for the case $p = 1$ does not hold(see [8]). Recently, Găvruta [2] also obtained a further generalization of the Hyers-Rassias' theorem. Székelyhidi [9] gave the following result: Let X be a real or complex vector space, B a normed space and let $f, g, h : X \to B$ be functions. Let

$$\Phi(x, y, \lambda) = f(\lambda x + y) - \lambda g(x) - h(y)$$

for any $x, y \in X$ and scalar λ. If Φ is bounded for large λ, then g is linear. In this paper, we obtain a characterization of linear operators on vector spaces and homomorphisms on algebras applying the stability properties of functional equations.

2. Main results

Jun, Shin and Kim [6] generalized the result of Găvruta in the following theorem.

THEOREM 2.1. *Let G be an abelian group and X a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a mapping such that*

$$\tilde{\varphi}(x, y) = \sum_{k=0}^{\infty} 2^{-k} \varphi(2^k x, 2^k y) < \infty$$

for all $x, y \in G$. Suppose $f, g, h : G \to X$ are mappings satisfying

$$\|f(x + y) - g(x) - h(y)\| \leq \varphi(x, y) \tag{1}$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to X$ such that

$$\|f(x) - T(x)\| \leq \|g(0)\| + \|h(0)\| + \frac{1}{2}\tilde{\varphi}(x, x) + \frac{1}{2}\tilde{\varphi}(x, 0) + \frac{1}{2}\tilde{\varphi}(0, x)$$

for all $x \in G$.

REMARK. In the proof of [6], they also showed that $T(x) = \lim_{n\to\infty} f(2^n x)/2^n = \lim_{n\to\infty} g(2^n x)/2^n = \lim_{n\to\infty} h(2^n x)/2^n$. Furthermore, we have

$$\|g(x) - T(x)\| \leq \|f(0)\| + \|h(0)\| + \frac{1}{2}\tilde{\varphi}(2x, -x) + \frac{1}{2}\tilde{\varphi}(x, 0) + \frac{1}{2}\tilde{\varphi}(x, -x)$$

for all $x \in G$.

$$\|h(x) - T(x)\| \leq \|f(0)\| + \|g(0)\| + \frac{1}{2}\tilde{\varphi}(-x, 2x) + \frac{1}{2}\tilde{\varphi}(0, x) + \frac{1}{2}\tilde{\varphi}(-x, x)$$

for all $x \in G$.

THEOREM 2.2. *Let G be a real or complex vector space and X a normed space. Let $\varphi(x,y)$ be as in Theorem 2.1. Suppose $f,g,h : G \to X$ are mappings satisfying*

$$\|f(2^n x + y) - 2^n g(x) - h(y)\| \le \varphi(2^n x, y) \tag{2}$$

for any $x, y \in G$ and for any positive integer n. Then g is an additive mapping such that

$$\|f(x) - g(x)\| \le \|h(0)\| + \varphi(x,0) \quad \text{for all } x \in G.$$

In particular, $g(x) = \lim_{n\to\infty} f(2^n x)/2^n$ for all $x \in G$.

Proof. Replacing $y = 0$, (2) gives

$$\left\|\frac{f(2^n x)}{2^n} - g(x)\right\| \le \frac{\|h(0)\|}{2^n} + \frac{\varphi(2^n x, 0)}{2^n}.$$

Since $\tilde{\varphi}(x,0) < \infty$, $\lim_{n\to\infty} \varphi(2^n x, 0)/2^n = 0$. From this $\lim_{n\to\infty} f(2^n x)/2^n$ exists for all fixed x in X and is equal to $g(x)$. For the case $n = 0$, (2) can be written by

$$\|f(x+y) - g(x) - h(y)\| \le \varphi(x,y) \quad \text{for all} \quad x, y \in G.$$

We can regard $f,g,h : G \to X$ as $f,g,h : G \to \overline{X}$, where $\overline{X}$ is a completion of X. By Theorem 2.1, $\lim_{n\to\infty} f(2^n x)/2^n = g(x)$ is additive.□

COROLLARY 2.3. *Let G, X and φ be as in Theorem 2.2. Suppose $f, g : G \to X$ are mappings satisfying*

$$\|f(2^n x + y) - 2^n f(x) - g(y)\| \le \varphi(2^n x, y) \tag{2}$$

for any $x, y \in G$ and for any positive integer n. Then f is additive.

THEOREM 2.4. *Let G, X and φ be as in Theorem 2.2. If $f, g, h : G \to X$ are mappings satisfying*

$$\|f(\lambda x + y) - \lambda g(x) - h(y)\| \le \varphi(\lambda x, y) \tag{3}$$

for any $x, y \in G$ and scalar λ, then g is a linear mapping such that

$$\|f(x) - g(x)\| \le \|h(0)\| + \varphi(x,0) \quad \text{for all } x \in G.$$

Proof. By Theorem 2.2, g is additive and $\lim_{n\to\infty} f(2^n x)/2^n = g(x)$ for all $x \in G$. Let μ be a fixed nonzero scalar. Replacing $y = 0$ and $\lambda = 2^n\mu$, (3) gives

$$\|\frac{f(2^n\mu x)}{2^n} - \mu g(x)\| \leq \frac{\|h(0)\|}{2^n} + \frac{\varphi(2^n\mu x, 0)}{2^n}.$$

From this we have

$$\lim_{n\to\infty} \frac{f(2^n\mu x)}{2^n} = \mu g(x). \tag{4}$$

On the other hand,

$$\lim_{n\to\infty} \frac{f(2^n\mu x)}{2^n} = g(\mu x). \tag{5}$$

By (4) and (5),

$$g(\mu x) = \mu g(x).$$

Since g is additive, g is linear. □

COROLLARY 2.5. *Let G, X and φ be as in Theorem 2.2. If $f, g : G \to X$ are mappings satisfying*

$$\|f(\lambda x + y) - \lambda f(x) - g(y)\| \leq \varphi(\lambda x, y) \tag{6}$$

for any $x, y \in G$ and scalar λ, then f is linear.

The above technique can be employed for a stability-type characterization of homomorphism on algebras.

THEOREM 2.6. *Let G be a real or complex algebra with the identity e and X a normed algebra with identity. Let φ be as in Theorem 2.2. Let $f, g, h, k : G \to X$ be mappings where $g(e), h(e)$ are invertible. If*

$$\|f(\lambda xy + z) - \lambda g(x)h(y) - k(z)\| \leq \varphi(\lambda xy, z) \tag{7}$$

for any $x, y, z \in G$ and scalar λ, then $g(e)^{-1}g$ and $h(\cdot)h(e)^{-1}$ are homomorphisms from G into X such that

$$\|f(x) - g(x)h(e)\| \leq \|k(0)\| + \varphi(x, 0) \quad \text{for all } x \in G.$$

Proof. Putting $y = e$ in (7), we obtain

$$\|f(\lambda x + z) - \lambda g(x)h(e) - k(z)\| \le \varphi(\lambda x, z)$$

for any $x, z \in G$. By Theorem 2.4, we obtain $g(\cdot)h(e)$ is linear,

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = g(x)h(e) \quad \text{for all} \quad x \in G \tag{8}$$

and

$$\|f(x) - g(x)h(e)\| \le \|k(0)\| + \varphi(x, 0) \quad \text{for all} \ \ x \in G.$$

By symmetry, $g(e)h$ is linear and

$$\lim_{n\to\infty} \frac{f(2^n y)}{2^n} = g(e)h(y) \ \ \text{for all} \quad y \in G. \tag{9}$$

Let y be any fixed element of G. Replacing z by zy in (7), we obtain

$$\|f(\lambda xy + zy) - \lambda g(x)h(y) - k(zy)\| \le \varphi(\lambda xy, zy)$$

for any $x, z \in G$. Define $f_1, g_1, h_1 : G \to X$ by $f_1(x) = f(xy)$, $g_1(x) = g(x)h(y)$ and $k_1(z) = k(zy)$. Define $\varphi_1 : G \times G \to [0, \infty)$ by $\varphi_1(x, z) = \varphi(xy, zy)$. We obtain

$$\|f_1(\lambda x + z) - \lambda g_1(x) - k_1(z)\| \le \varphi_1(\lambda x, z)$$

for any $x, z \in G$ and $\tilde{\varphi}_1(x, z) = \tilde{\varphi}(xy, zy) < \infty$ for all $x, z \in G$. By Theorem 2.4,

$$\lim_{n\to\infty} \frac{f_1(2^n x)}{2^n} = g_1(x) \quad \text{for all} \quad x \in G. \tag{10}$$

From (8), (9) and (10),

$$g(xy)h(e) = g(e)h(xy) = g(x)h(y) \quad \text{for any} \quad x, y \in G. \tag{11}$$

Let $\psi : G \to X$ be defined by $\psi(x) = g(e)^{-1}g(x) = h(x)h(e)^{-1}$. Then ψ is a well-defined linear map and

$$\begin{aligned}\psi(xy) &= g(e)^{-1}g(xy) = g(e)^{-1}g(x)h(y)h(e)^{-1} \\ &= g(e)^{-1}g(x)g(e)^{-1}g(y) = \psi(x)\psi(y)\end{aligned}$$

since $g(e)$ and $h(e)$ are invertible. This completes the proof. □

COROLLARY 2.7. *Let G, X and φ be as in Theorem 2.6. Let $f, g, h : G \to X$ be mappings where $f(e), g(e)$ are invertible. If*

$$\|f(\lambda xy + z) - \lambda f(x)g(y) - h(z)\| \leq \varphi(\lambda xy, z)$$

for any $x, y, z \in G$ and scalar λ, then g is a homomorphism.

Proof. By Theorem 2.6, $f(e)^{-1}f$ and $g(\cdot)g(e)^{-1}$ are homomorphisms. By (9),

$$f(e) = \lim_{n\to\infty} \frac{f(2^n e)}{2^n} = f(e)g(e).$$

From this, $g(e)$ is an identity and g is a homomorphism. □

COROLLARY 2.8. *Let G, X and φ be as in Theorem 2.6. Let $f, g : G \to X$ be mappings where $f(e)$ is invertible. If*

$$\|f(\lambda xy + z) - \lambda f(x)f(y) - g(z)\| \leq \varphi(\lambda xy, z)$$

for any $x, y, z \in G$ and scalar λ, then f is a homomorphism.

REMARKS. (I) Let X be a commutative normed algebra with identity. In Theorem 2.6, we can replace the condition that $g(e), h(e)$ are invertible by the condition that there exist $x_0, y_0 \in G$ such that $g(x_0), h(y_0)$ are invertible. (II) Let X be a set of complex numbers. In Theorem 2.6, we can replace the condition that $g(e)$ and $h(e)$ are invertible by the condition that g, h are nonidentically zero function.

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