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COMPARISON OF p-ADIC THETA FUNCTIONS

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Abstract. In this paper we investigate how p-adic θ-functions of Néron and Tate are related. As a result, we show that the p-adic theta function defined by Néron and that defined by Tate are differ by an analytic function whose values are units.

1. Introduction

We collect necessary definitions and notations in Section 2. We recall Néron's and Tate's theta function in Section 3 and Section 4 respectively. In Section 5 we compare the two theta functions and conclude that the theta functions defined by Néron [6] and Tate [7] are differ by an analytic function whose values are units in a neighborhood of the origin. We restrict our attention to the abelian varieties of dimension 1. We plan to extend our result to higher dimensional abelian varieties.

2. Notations and definitions

Let p be a prime, and let $\mathbb{Q}_p$ the p-adic rational number field, and let K be a finite extension of $\mathbb{Q}_p$. Let $\mathcal{O}$ be the ring of integers of K with the maximal ideal $\mathfrak{p}$ and let k be the residue field of $\mathcal{O}$. Let A be an elliptic curve (= abelian variety of dimension 1) over K with the identity element 0 and let δ be the identity map of A. Let t be a $\mathfrak{p}$-*admissible* coordinate of A at 0 in the sense of Néron; that is t is a K-rational function on A which forms a local coordinate system of A at 0 and further the reduction modulo $\mathfrak{p}$ of the system also form a local coordinate system at the identity of $A \times_{\mathcal{O}} k$. A K-rational divisor D is

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disjoint from 0 mod $\mathfrak{p}$ if any component of the set obtained by reducing the support of D does not contain the identity element of $A \times_{\mathcal{O}} k$.

For a subgroup G of $A(K)$ let $\Lambda = \mathbb{Z}[G]$ be the group ring of G with the coefficients in $\mathbb{Z}$ which is the group of zero cycles with components in G. Let $I = I_G$ be the subgroup of Λ consisting of all $\mathfrak{a} = \sum_{i=1}^{n} n_i(a_i)$ with degree 0, namely $\sum_{i=1}^{n} n_i = 0$. Let I^2 be the ideal of Λ generated by the cycles of the form $(a+b)-(a)-(b)+(0)$. Define the multiplication $*$ on Λ by

$$\mathfrak{a} * \mathfrak{b} = \sum (n_i m_j)(a_i + b_j)$$

for $\mathfrak{a} = \sum_{i=1}^{n} n_i(a_i)$, $\mathfrak{b} = \sum_{j=1}^{m} m_j(b_j)$. Since $(a+b)-(a)-(b)+(0) = \{(a)-(0)\} * \{(b)-(0)\}$ we see that I^2 is the ideal of Λ generated by the cycles of the form $\{(a)-(0)\} * \{(b)-(0)\}$. A divisor and a zero cycle are said to be disjoint if their supports are disjoint. In our case, the divisors are the same as the zero cycles since we work on abelian varieties of dimension 1. We used the terminology because we want to emphasis the fact that the pairing is defined for a divisor and a cycle.

Let f be a rational function and let $D = (f)$ and let $\mathfrak{a} = \sum_{i=1}^{n} n_i(a_i)$ be a zero cycle of degree 0. When D and $\mathfrak{a}$ are disjoint we define a pairing $[D, \mathfrak{a}]$ by

$$[D, \mathfrak{a}] = \prod_{i=1}^{n} f(a_i)^{n_i}$$

which we often write as $f(\mathfrak{a})$. For a divisor D algebraically equivalent to zero (= cycles of degree zero, since we work on abelian varieties of dimension 1) and a cycle $\mathfrak{a} = \sum_{i=1}^{n} n_i(a_i)$ we let $D * \mathfrak{a} = \sum_{i=1}^{n} n_i D_{a_i}$ and let $\mathfrak{a}^- = \sum_{i=1}^{n} n_i(-a_i)$.

If D is a divisor algebraically equivalent to zero and $\mathfrak{a}$ and $\mathfrak{b}$ are cycles of degree zero, then $D * \mathfrak{a}^-$ is linearly equivalent to zero; hence $[D * \mathfrak{a}^-, \mathfrak{b}]$ make sense. As is well known, this is well defined by a reciprocity law [3, Ch. 11]. And we define

$$[D, \mathfrak{a} * \mathfrak{b}] = [D * \mathfrak{a}^-, \mathfrak{b}].$$

3. Theta function of Néron

Néron [5] defined his theta function in the following way: Let G be the open subgroup of the group of all K-rational points $A(K)$ consisting of those a such that $t(a) \in \mathfrak{p}$, where t is a $\mathfrak{p}$-admissible coordinate system of A at 0. Let D be a K-rational divisor on A algebraically equivalent to 0 and disjoint from 0 mod $\mathfrak{p}$. For $\mathfrak{a} \in I$, Néron showed the limit

$$\theta_D^\nu(\mathfrak{a}) = \lim_{\nu \to \infty} [D, p^\nu \mathfrak{a} - p^\nu \delta \mathfrak{a}]^{1/p^\nu}$$

exists in [5].

THEOREM 1 (NÉRON). *The limit* $\lim_{\nu \to \infty} [D, p^\nu \mathfrak{a} - p^\nu \delta \mathfrak{a}]^{1/p^\nu}$ *converges and* $\theta_D^\nu(\mathfrak{a}) = [D, \mathfrak{a}]$ *for* $\mathfrak{a} \in I^2$. *The function* $\theta_D^\nu(a) = \theta_D^\nu((a) - (0))$ *for* $a \in G$ *is an* $\mathcal{O}$*-analytic function with values in* $1 + \mathfrak{p}$.

In general for a K-rational divisor D not necessarily disjoint from 0 mod $\mathfrak{p}$ choose a rational function f so that $\operatorname{div}(f) + D$ is disjoint form 0 mod $\mathfrak{p}$ and define $\theta_D^\nu(\mathfrak{a}) = \theta_{D'}^\nu(\mathfrak{a}) f(\mathfrak{a})^{-1}$. We will call θ^ν a *theta function of Néron* corresponding to the divisor D.

For a divisor $D = \operatorname{div}(f)$ linearly equivalent to 0, the theta function $\theta_D^\nu(a)$ differs from the function f by a unit. See [5, Section 4(b)].

4. Theta function of Tate

Now we recall the theta functions defined by Tate: Let K be a finite extension of $\mathbb{Q}_p$ as before. The Tate curve A of period $q \in K^*$ is defined by the analytic parameterization

$$0 \to q^{\mathbb{Z}} \to K^* \xrightarrow{\pi} A \to 0.$$

Let D be a divisor on A of degree 0. Then we have the corresponding exact sequence

$$E(D) : 0 \to K^* \to X_D \to A \to 0.$$

The extension has a rational section $s_D : A \to X_D$ which is determined up to a constant.

By Mazur and Tate [4, 2], X_D can be described in the following way: For a finite extension L of K, $X_D(L)$ is the set of all pairs $(\mathfrak{a}, c)$ where $\mathfrak{a}$ is a zero cycle of degree 0 and $c \in L^*$ subjecting to the rules; $c(\mathfrak{a},1) = (\mathfrak{a},c)$, $(\mathfrak{a}_1,c_1)(\mathfrak{a}_2,c_2) = (\mathfrak{a}_1+\mathfrak{a}_2)$ and if $\mathfrak{a} \in I^2$ then $(\mathfrak{a},c) = (0,[D,\mathfrak{a}]c)$.

The section s_D is given by $s_D : a \mapsto ((a)-(0))$ which is a K-rational section of π and s_D is regular on $A(L) \setminus |D|$.

The extension $E(D)$ can be completed to an exact sequence of commutative diagram of K-holomorphic groups:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & (q,\alpha)^{\mathbb{Z}} & & q^{\mathbb{Z}} & & \\
 & & & & \downarrow \subseteq & & \downarrow \subseteq & & \\
0 & \longrightarrow & K^* & \longrightarrow & K^* \times K^* & \longrightarrow & K^* & \longrightarrow & 0 \\
 & & \| & & \downarrow \phi & & \downarrow \pi & & \\
0 & \longrightarrow & K^* & \longrightarrow & X_D & \longrightarrow & A & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where $\alpha \in K^*$ is defined by D up to a power of q. The K^* action on $K^* \times K^*$ is given by $c'(a,c) = (a,c'c)$ for $c' \in K^*$ and $(a,c) \in K^* \times K^*$. See [8].

A *theta function* $\theta(x)$ of type (D,α) is defined to be a K-meromorphic function on K^* with $\theta(qx) = \alpha\theta(x)$ and $\text{div}\theta = D$. Now we recall some properties of nonarchimedean theta functions. See [7] for the proofs.

PROPOSITION 1. (i) *There are theta functions of type (D,α) and any two of them are only differ by a constant in K^*.*

(ii) *If $\theta(x)$ is a theta function of type (D,α) then $x^m\theta(x)$ is a theta function of type $(D,\alpha q^m)$.*

(iii) *For $a_1,\ldots,a_r \in K^*$ and $m_1,\ldots,m_r \in \mathbb{Z}$ with $\sum m_i = 0$ put $D = \sum m_i(\pi(a_i))$ and $\alpha = q^m \prod a_i^{m_i}$ then*

$$\theta_D(x) = x^m \prod_{i=1}^{r} \theta_0\left(\frac{x}{a_i}\right)^{m_i}$$

with $\theta_0(x) = \prod_{n\geq 0}(1-q^n x)\prod_{n\geq 1}(1-q^n x^{-1})$ *is a theta function of type* (D,α).

Our section $s_D : A \to X_D$ lifts to a map $x \mapsto (x, \theta_D(x))$ from $K^*\backslash|D|$ to $K^* \times K^*$, with a theta function of type (D,α).

We start with identifying two descriptions of X_D; one given by Mazur and Tate and $K^* \times K^*/(q,\alpha)^{\mathbb{Z}}$.

LEMMA 1. *The identification between the descriptions of Mazur and Tate and* $K^*\times K^*/(q,\alpha)^{\mathbb{Z}}$ *is given by* $((x)-(y),c) \mapsto (\tilde{x}-\tilde{y}, c\theta(\tilde{x})\theta(\tilde{y})^{-1})$ *where* $\tilde{x},\tilde{y}$ *are lifts of* x,y *in* K^* *respectively and we used additive notation for the operation on the left side of* $K^*\times K^*$.

Proof. Since we know both fit in the exact sequence $E(D)$ we need to construct a well defined map $\psi : X_D \to K^* \times K^*/(q,\alpha)^{\mathbb{Z}}$ which is functorial and compatible with K^*–action making the diagram

$$\begin{array}{ccccc} X_D & \longrightarrow & A & \longrightarrow & 0 \\ \downarrow{\scriptstyle\psi} & & \| & & \\ K^*\times K^*/(q,\alpha)^{\mathbb{Z}} & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

To see the map θ is well defined it suffices to check on the elements of the form $((x)-(0),1)$ which maps to $(\tilde{x},\theta(\tilde{x}))$. But altering $\tilde{x}$ by q changes $\theta(x)$ by α which shows that ψ is well defined.

Now it is easy to check that ψ is compatible with K^*-action. Obviously ψ is functorial and makes the above diagram commutative. □

Now we prove a simple fact saying that any lift of s_D is in fact, same as a theta function of Tate. [cf. 9, p. 306, Lemma 4.6]

PROPOSITION 2. *The map* $x \mapsto (x,\theta_D(x))$ *is the unique (up to a constant) analytic lifting of the section* s_D.

Proof. Suppose $\tilde{s}_D = (f(x), g(x))$ be a lift of s_D. Then commutativity of the diagram

$$\begin{array}{ccc} K^*\times K^* & \xrightarrow{\tilde{p}} & K^* \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\pi} \\ X_D & \xrightarrow{p} & A. \end{array}$$

implies that $\tilde{p}\tilde{s}_D(x) \equiv x \ (\mathrm{mod} q^{\mathbb{Z}})$. Hence $x - f(x)$ is an analytic function with values in $q^{\mathbb{Z}}$. By Strassman's theorem [1, p. 95 Cor. 4.2.7], we conclude that $x - f(x)$ is a constant. In fact, suppose $a - f(a) = q^n$ for some a and n. Then some open neighborhood U_a of a is mapped to q^n. Since U_a contains infinitely many points we conclude that $x - f(x) = q^n$.

Now suppose s_D has two lifts $(x, \theta_D), (x, g(x))$. We claim that $g(qx) = \alpha g(x)$. In fact, $\tilde{s}_D(x) - \tilde{s}_D(qx) = (x, g(x)) - (qx, g(qx)) \in (\alpha, q)^{\mathbb{Z}}$. Hence $g(qx)/g(x) = \alpha$ as contended.

Therefore $\theta_D(x)/g(x)$ is q-periodic without zeros and poles since $\mathrm{div} f$ is the same as $\mathrm{div}(\theta_D)$. Thus $\theta_D/f(x)$ is a constant. □

5. Comparison of theta functions

In this section we compare theta functions of Néron and Tate and conclude that they are differ by an analytic function whose values are units in a neighborhood of the origin. In this section G is the group defined in Section 3.

PROPOSITION 3. *Let D be a divisor of degree 0 and disjoint from 0 mod $\mathfrak{p}$. Let $a \in K^*$ such that $\pi(a) \in G$ and let $\mathfrak{a} = (\pi(a)) - (0)$. For a positive integer n we have*

$$[D, n\mathfrak{a} - n\delta\mathfrak{a}] = \frac{\theta_D(a)^n}{\theta_D(na)}$$

up to a constant. Here we used additive notation for the operation of K^*.

Proof. Consider the commutative diagram:

$$\begin{array}{ccc} K^* \times K^* & \xrightarrow{\tilde{p}} & K^* \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \pi} \\ X_D & \xrightarrow{p} & A. \end{array}$$

We first have, $(n\mathfrak{a} - n\delta\mathfrak{a}, 1) = (0, [D, n\mathfrak{a} - n\delta\mathfrak{a}]) = [D, n\mathfrak{a} - n\delta\mathfrak{a}](0, 1)$. On the other hand, $(n\mathfrak{a} - n\delta\mathfrak{a}, 1) = (n\mathfrak{a}, 1) - (n\delta\mathfrak{a}, 1)$. Now $(n\mathfrak{a}, 1) = n(\mathfrak{a}, 1)$ lifts to $(na, \theta_D(a)^n)$ and $(n\delta\mathfrak{a}, 1)$ lifts to $(na, \theta_D(na))$. Since $[D, n\mathfrak{a} - n\delta\mathfrak{a}](0, 1)$ lifts to $[D, n\mathfrak{a} - n\delta\mathfrak{a}]\theta_D(0)$ we have,

$$[D, n\mathfrak{a} - n\delta\mathfrak{a}]\theta_D(0) \equiv \frac{\theta_D(a)^n}{\theta_D(na)} \mod \alpha^{\mathbb{Z}}.$$

By Strassman's theorem again they are equal up to a constant. □

We normalize the theta function θ_D so that its value at 0 is 1; we denote such theta functions by $\tilde{\theta}_D$ i.e., $\tilde{\theta}_D(0) = 1$. For the proof of the following lemma we use the trick of Néron [6]. Let $\mathfrak{m}$ be the maximal ideal of 0 of A. Write $\pi_i : A \times A \to A$ $(i = 1, 2)$ be the projection to the i-th factor. Let $\mathfrak{m}_i$ be the ideal of local ring of the origin of $A \times A$ obtained by extending $\mathfrak{m}$ by π_i.

LEMMA 2. *$\tilde{\theta}_D(x^m)/\tilde{\theta}_D(x)^m \in 1 + \mathfrak{m}^2$ where $\mathfrak{m}$ is the maximal ideal.*

Proof. Let $u_i(x, y) = \tilde{\theta}_D(xy^i)/\tilde{\theta}_D(x)\tilde{\theta}_D(y^i)$ for $i = 1, 2, \ldots, m - 1$. Then we have,

$$\tilde{\theta}_D(x^m)/\tilde{\theta}_D(x)^m = u_1(x, x)u_2(x, x) \cdots u_{m-1}(x, x).$$

In $u_i(x, y)$ if we set $x = 1$ then $u_i = 1$ for any y and similarly if we set $y = 1$ then $u_i = 1$ for any values of x. This means that $u_i(x, y) \in 1 + \mathfrak{m}_1\mathfrak{m}_2$ for $i = 1, 2, \ldots, m - 1$. Now set $x = y$ to get our result. □

THEOREM 2. *Let D be a divisor of degree 0 and disjoint from 0* mod $\mathfrak{p}$. *Then we have:*

(i) *The limit $\lim_{\nu \to \infty} \tilde{\theta}_D(p^\nu a)^{1/p^\nu}$ exists for $|a| \leq 1$.*

(ii) *It takes values in $1 + \mathfrak{p}$ for $|a| < 1$.*

Here we used additive notation for the operation in K^*.

Proof. Let $\xi_\nu = \log \tilde{\theta}(p^\nu a)^{1/p^\nu} = \frac{1}{p^\nu} \log \tilde{\theta}(p^\nu a)$. Then we have

$$\begin{aligned} \xi_\nu - \xi_{\nu-1} &= \frac{1}{p^\nu} \log \tilde{\theta}(p^\nu a) - \frac{1}{p^\nu} \log \tilde{\theta}(p^{\nu-1}a)^p \\ &= \frac{1}{p^\nu} \log \frac{\tilde{\theta}(p \cdot p^{\nu-1}a)}{\tilde{\theta}(p^{\nu-1}a)^p}. \end{aligned}$$

Since $p^{\nu-1}a \in \mathfrak{p}^{\nu-1}$ we have $\log \tilde{\theta}(p \cdot p^{\nu-1}a)/\tilde{\theta}(p^{\nu-1}a)^p \in \mathfrak{p}^{2\nu-2}$ by the lemma. Hence we see $\xi_\nu - \xi_{\nu-1} \in \mathfrak{p}^{\nu-2}$; sequence $\{\xi_\nu\}$ converges.

To see the value of the limit lies in $1 + \mathfrak{p}$ let $f(x) = \tilde{\theta}_D(p^\nu x)$. Then $f(0) = 1$. Hence $f(x) = 1 + x +$ higher degree terms. Since $|a| < 1$ we have $|p^\nu a| \leq \frac{1}{p^{\nu+1}}$ where $|\cdot|$ is the absolute value on K. Now we use the p-adic binomial theorem (See [1] for example.)

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots$$

to conclude $\tilde{\theta}_D(p^\nu a)^{1/p^\nu} \in 1 + \mathfrak{p}$. Since $1 + \mathfrak{p}$ is closed, its limit lies in $1 + \mathfrak{p}$ too. □

Write $\gamma_D(a) = \lim_{\nu \to \infty} \tilde{\theta}_D(p^\nu a)^{1/p^\nu}$.

COROLLARY. *Néron's theta function $\theta_D^\nu(x)$ and Tate's $\theta_D(x)$ is related by*

$$\theta_D^\nu(\bar{x})\gamma_D(x) = \theta_D(x)$$

up to a constant where $\alpha(x)$ is an analytic function on G which takes the value in $1 + \mathfrak{p}$ for $|x| < 1$, $x \in K^*$ with $\bar{x} = \pi(x)$.

Proof. By Proposition 3 we have

$$[D, p^\nu \mathfrak{a} - p^\nu \delta \mathfrak{a}]\tilde{\theta}_D(p^\nu a) = \theta_D(a)^{p^\nu}.$$

Take $1/p^\nu$-th power both sides and take the limit. □

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