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THE ITERATION OF ENTIRE FUNCTION

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ABSTRACT. In this paper, we obtain the following results: Let f be a transcendental entire function with $\log M(r,f) = O((\log r)^{\beta} e^{(\log r)^{\alpha}})$ $(0 \le \alpha < 1, \beta > 1)$. Then every component of $N(f)$ is bounded. This result generalizes the result of Baker.

1. Introduction

Let $f : \mathbf{C} \to \mathbf{C}$ denote a nonlinear entire function and f^n, $n \in \mathbf{N}$, the nth iterate of f. The set of normality, $N(f)$, is defined to be the set of points, $z \in \mathbf{C}$, such that the sequence (f^n) forms a normal family in some neighbourhood of z. It is easy to see that $N(f)$ is open and has the property of complete invariance under f, that is $z \in N(f)$ if and only if $f(z) \in N(f)$. The complement $J(f)$, of $N(f)$ is known as the Julia set. This set is clearly closed and completely invariant under f. More details of these and other basic properties of the sets $N(f)$ and $J(f)$ can be found in [4] and [5].

Baker [1] obtained the following result:

THEOREM A. *Let $f(z)$ be a transcendental entire function with* $\log M(r,f) = O\ ((\log r)^p)$, *where $1 < p < 3$. Then every component of $N(f)$ is bounded.*

In the paper, we generalize the condition $\log M(r,f) = O((\log r)^p)$ $(1 < p < 3)$ to the form $\log M(r,f) = O((\log r)^{\beta} e^{(\log r)^{\alpha}})$ $(0 \le \alpha < 1, \beta > 1)$ in Theorem A and obtain the following result:

THEOREM 1. *Let f be a transcendental entire function with*

$$\log M(r,f) = O((\log r)^{\beta} e^{(\log r)^{\alpha}})\ (0 \le \alpha < 1, \beta > 1).$$

Then every component of $N(f)$ is bounded.

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2. Some lemmas

LEMMA 1 [2]. *Let f be an entire function of order zero and $z = re^{i\theta}$. Then*

$$\log|f(re^{i\theta})| - N(2R) - \log|c| < 4Q(2R)$$

for all $z \in \{z : |z| < R\}$.

LEMMA 2 [2]. *Let f be an entire function of order zero and $z = re^{i\theta}$. Then for any $\zeta > 0$, $\eta > 0$, there exist $R_0 = R_0(\zeta, \eta)$, $k = k(\zeta, \eta)$ such that for all $R > R_0$,*

$$\log \mid f(re^{i\theta}) \mid -N(2R) - \log \mid c \mid > -kQ(2R), \zeta R \leq r \leq R,$$

except in a set of circles enclosing the zeros of f, the sum of whose radii is at most ηR. Where $Q(r) = r\int_r^\infty \frac{n(t,1/f)}{t^2}dt$, $N(r) = \int_0^r \frac{n(t,1/f)}{t}dt$.

Combining Lemma 1 and Lemma 2 and taking $\zeta = \eta$ to be small, we obtain:

LEMMA 3. *If f is an entire function of order zero then there exist $R_0 = R_0(\zeta) > 0$ and a constant B such that, for each $R > R_0$, there exists r satisfying*

$$\zeta R \leq r \leq R$$

such that

$$\log m(r, f) > \log M(r, f) - BQ(2R).$$

LEMMA 4 [5]. *Suppose that f is a transcendental entire function and that there exist sequences R_n, $\rho_n \to \infty$ and $c(n) > 1$ such that*

1. $R_{n+1} = M(R_n) = M(R_n, f)$
2. $R_n \leq \rho_n \leq (R_n)^{c(n)}$
3. $m(\rho_n) = m(\rho_n, f) > (R_{n+1})^{c(n+1)}$ *for all sufficiently large n*

Then $N(f)$ contains no unbounded components.

LEMMA 5. *Let $\phi(r)$, $H(r)$ be two positive functions tending to ∞(as $r \to \infty$), and $A = A(r) > 1$, and $\phi(Ar)/\phi(r) \to c(r \to \infty), (c \geq 1)$. If $H(r) = O(\phi(r))$, then there exists $r_0 > 1$, $H(Ar)/H(r)$ is upper bounded in $[r_0, +\infty)$.*

Proof. Suppose that $H(Ar)/H(r)$ is not bounded from above in $[r_0, +\infty)$, then there exists a sequence $\{r_n\}$, $r_n \to \infty (n \to \infty)$, for arbitrary $G > 0$, there exists a natural number n_0 such that for $n > n_0$, we have

$$H(Ar_n)/H(r_n) > G. \tag{1}$$

Since $\overline{\lim_{r\to\infty}} H(r)/\phi(r) = k \quad (0 \le k < +\infty)$, we see that

1. If $k \neq 0$, then by (1) we obtain $H(Ar_n)/\phi(Ar_n) > G\frac{H(r_n)}{\phi(r_n)}\frac{\phi(r_n)}{\phi(Ar_n)}$, take $G = 2c$, thus

$$\overline{\lim_{n\to\infty}} H(Ar_n)/\phi(Ar_n) \ge G\,\overline{\lim_{n\to\infty}}\frac{H(r_n)}{\phi(r_n)}\lim_{n\to\infty}\frac{\phi(r_n)}{\phi(Ar_n)},$$

i.e., $k \ge 2c(1/c)k = 2k$, this is a contradiction.

2. If $k = 0$, then $\lim_{r\to\infty} H(r)/\phi(r) = 0$ take $G = 4c$, when $n > n_0$, by (1) we get $H(Ar_n)/H(r_n) > 4c$, so, when $n > n_0$, for arbitrary natural number m, we have

$$H(A^m r_n) > 4cH(A^{m-1}r_n) > \cdots > (4c)^m H(r_n). \tag{2}$$

Since $\lim_{n\to\infty}\phi(Ar_n)/\phi(r_n) = c$, take $\epsilon_0 = c > 0$, there exists a $n_1 > n_0$, such that for $n > n_1$ we obtain

$$|\,\phi(Ar_n)/\phi(r_n) - c\,| < \epsilon_0 = c, \quad \text{i.e.,} \quad \phi(Ar_n) < 2c\phi(r_n),$$

thus, for arbitrary natural number m, we get

$$\phi(A^m r_n) < 2c\phi(A^{m-1}r_n) < \cdots < (2c)^m\phi(r_n), \tag{3}$$

take $n = n_2 > n_1 > n_0$, by (2) and (3) we have

(4)

$$H(A^m r_{n_2})/\phi(A^m r_{n_2}) > \frac{(4c)^m H(r_{n_2})}{(2c)^m\phi(r_{n_2})} = 2^m\frac{H(r_{n_2})}{\phi(r_{n_2})} \to \infty (m \to \infty).$$

As $m \to \infty$, we obtain $A^m r_{n_2} \to \infty$, so, (4) is a contradiction with $\lim_{r\to\infty} H(r)/\phi(r) = 0$.

This completes the proof of the Lemma 5. □

LEMMA 6. *Let f be a transcendental entire function with* $\log M(r,f) = O((\log r)^\beta\, e^{(\log r)^\alpha})$ $(0 \le \alpha < 1,\ \beta > 1)$. *Then*

$$\log M(\delta r, f) \sim \log M(r,f) \qquad (r \to \infty, \delta \ge 2).$$

Proof. (i) We may assume $f(0) = 1$, otherwise, we only need to make a transformation

$$F(z) = f(z) - f(0) + 1.$$

By Jensen's theorem

$$
\begin{aligned}
N(r,1/f) &= \int_0^r \frac{n(t,1/f)}{t}dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log \mid f(re^{i\theta}) \mid d\theta \le \log M(r,f),
\end{aligned} \tag{5}
$$

for $r > 1$ and $A > 1$, by (5) we have

$$
n(r,1/f)\log A \le \int_r^{Ar} \frac{n(t,1/f)}{t}dt \le N(Ar,1/f) \le \log M(Ar,f).
$$

So

$$
n(r,1/f) \le \frac{\log M(Ar,f)}{\log A}. \tag{6}
$$

Since

$$
\log M(r,f) = O((\log r)^{\beta} e^{(\log r)^{\alpha}}). \tag{7}
$$

Take $A = r^{\sigma(r)}$ and $\sigma(r) = \frac{1}{(\log r)^{\alpha}}$. By (6) we have

$$
n(r,1/f) \le \frac{\log M(r^{1+\sigma(r)},f)}{\sigma(r)\log r}. \tag{8}
$$

Therefore, there exists $r_0 > 1$ and $k > 0$, such that for $r > r_0$, and put $r = e^u$, by(7) we obtain

$$
\begin{aligned}
\frac{\log M(r^{1+\sigma(r)},f)}{r^{1/2}\sigma(r)\log r} &\le \frac{k(\log r^{1+\sigma(r)})^{\beta} e^{(\log r^{1+\sigma(r)})^{\alpha}}}{r^{1/2}\sigma(r)\log r} \\
&= \frac{k(1+\frac{1}{(\log r)^{\alpha}})^{\beta}(\log r)^{\beta} e^{(1+\frac{1}{(\log r)^{\alpha}})^{\alpha}(\log r)^{\alpha}}}{r^{1/2}(\log r)^{1-\alpha}} \\
&= \frac{k(1+1/u^{\alpha})^{\beta} u^{\beta} e^{(1+1/u^{\alpha})^{\alpha} u^{\alpha}}}{(e^u)^{1/2} u^{1-\alpha}} \le \frac{k2^{\beta} u^{\beta+\alpha-1} e^{2^{\alpha} u^{\alpha}}}{e^{u/2}}.
\end{aligned}
$$

Since $\alpha < 1$, $\frac{\log M(r^{1+\sigma(r)},f)}{r^{1/2}\sigma(r)\log r}$ decreases when $r > r_1 > r_0$, and by (8) we have

$$
\begin{aligned}
Q(r) &= r\int_r^{+\infty}\frac{n(t,1/f)}{t^2}dt \\
&\le r\int_r^{+\infty}\frac{\log M(t^{1+\sigma(t)},f)}{t^2\sigma(t)\log t}dt \\
&= \lim_{b\to+\infty} r\int_r^{b}\frac{\log M(t^{1+\sigma(t)},f)}{t^2\sigma(t)\log t}dt \qquad (9)\\
&\le \frac{r^{1/2}\log M(r^{1+\sigma(r)},f)}{\sigma(r)\log r}\int_r^{+\infty} t^{-3/2}dt \\
&= \frac{2\log M(r^{1+\sigma(r)},f)}{\sigma(r)\log r}.
\end{aligned}
$$

Let $\phi(r) = (\log r)^{\beta}e^{(\log r)^{\alpha}}(0 \le \alpha < 1, \beta > 1)$, $A = r^{\sigma(r)}$ and $\sigma(r) = \frac{1}{(\log r)^{\alpha}}$. Then

(10)

$$
\begin{aligned}
\phi(Ar)/\phi(r) &= \frac{(\log r^{1+\sigma(r)})^{\beta}e^{(\log r^{1+\sigma(r)})^{\alpha}}}{(\log r)^{\beta}e^{(\log r)^{\alpha}}} \\
&= (1+\sigma(r))^{\beta}e^{(\log r)^{\alpha}[(1+\sigma(r))^{\alpha}-1]} \\
&= (1+\sigma(r))^{\beta}e^{(\log r)^{\alpha}\alpha\sigma(r)(1+o(1))} \\
&= (1+\frac{1}{(\log r)^{\alpha}})^{\beta}e^{(\log r)^{\alpha}\alpha\frac{1}{(\log r)^{\alpha}}(1+o(1))} \longrightarrow e^{\alpha}(\ge 1) \quad (r\to\infty).
\end{aligned}
$$

So, from (7) $\log M(r,f) = O(\phi(r))$.

On the other hand, from (9), (10), and Lemma 5, there exists $L > 0$, for $r > r_1 > r_0$ we have

$$
\begin{aligned}
\frac{Q(r)}{\log M(r,f)} &\le \frac{2\log M(r^{1+\sigma(r)},f)}{\sigma(r)\log r\log M(r,f)} \\
&= \frac{\log M(Ar,f)}{\log M(r,f)}\frac{2}{\sigma(r)\log r} \\
&\le \frac{2L}{\sigma(r)\log r} = \frac{2L}{(\log r)^{1-\alpha}} \to 0(r\to\infty).
\end{aligned}
$$

So,

$$
Q(r) = o(\log M(r,f)). \qquad (11)
$$

(ii) Since $\log M(r,f) = O((\log r)^{\beta} e^{(\log r)^{\alpha}})$, the order ρ of f is equal to zero, and $n(r,1/f) = \circ(r)$ and

$$
\begin{aligned}
\log M(r,f) &\le \log \prod_{n=1}^{+\infty}(1+r/r_n) \\
&= \int_0^{+\infty} \log(1+r/t)dn(t,1/f) \\
&\le \int_0^{+\infty} \frac{r}{t} dn(t,1/f) \\
&= r\int_0^{+\infty} \frac{n(t,1/f)}{t(t+r)}dt \qquad (12)\\
&= r\Big(\int_0^{r} + \int_r^{+\infty}\Big)\frac{n(t,1/f)}{t(t+r)}dt \\
&\le r\frac{1}{r}\int_0^{r} \frac{n(t,1/f)}{t}dt + r\int_r^{+\infty}\frac{n(t,1/f)}{t^2}dt \\
&= N(r) + Q(r) \\
&= N(r,1/f) + Q(r,1/f).
\end{aligned}
$$

So, from Lemma 2, (11), and (12), we obtain

$$
\begin{aligned}
*\log|f(re^{i\theta})| &> N(2R) - kQ(2R) \qquad (\zeta R \le r \le R) \\
&= N(2R) + Q(2R) - (k+1)Q(2R) \\
&\ge \log M(2R,f) - (k+1)\circ(\log M(2R,f)) \\
&= \log M(2R,f)(1-\circ(1)) \qquad (13)\\
&\ge \log M(r,f)(1-\circ(1)). \qquad (14)
\end{aligned}
$$

On the other hand,

$$\log|f(z)| \le \log M(r,f) \le \log M(\delta r,f) \qquad (|z| = r, \delta \ge 2). \tag{15}$$

In (13), let $2R = \delta r, \delta \ge 2$. Then from (13), (14), and (15), we get

$$\log|f(z)| \sim \log M(\delta r,f) \qquad (r \to \infty) \tag{16}$$

$$\log|f(z)| \sim \log M(r,f) \qquad (r \to \infty). \tag{17}$$

By (16) and (17), we get

$$\log M(\delta r,f) \sim \log M(r,f) \qquad (r \to \infty, \delta \ge 2). \tag{18}$$

So, from (18), Lemma 6 is proved. □

3. Proof of Theorem 1

By Lemma 6, we have

$$\log M(\delta r, f) \sim \log M(r, f) = O((\log r)^{\beta} e^{(\log r)^{\alpha}}) \quad (r \to \infty, \delta \geq 2), \tag{19}$$

thus, for $\epsilon \in (0,1)$ and $\delta \geq 2$ we get

$$\begin{aligned} (1-\epsilon)\log M(r,f) &< \log M(\delta r, f) \\ &< (1+\epsilon)\log M(r,f) \\ &< 2\log M(r,f). \end{aligned} \tag{20}$$

By (6) and (20), for $r > 1$ and $A \geq 2$, we obtain

$$\begin{aligned} Q(r) &= r\int_r^{+\infty} \frac{n(t,1/f)}{t^2}dt \\ &\leq r\int_r^{+\infty} \frac{\log M(At,f)}{t^2 \log A}dt \\ &< \frac{2r}{\log A}\int_r^{+\infty} \frac{\log M(t,f)}{t^2}dt. \end{aligned} \tag{21}$$

Therefore, by (19) there exists $r_0 > 1$ and $K > 0$, such that for $r > r_0$, and put $r = e^{\mu}$ we have

$$\begin{aligned} \frac{\log M(r,f)}{r^{1/2}} &\leq \frac{K(\log r)^{\beta} e^{(\log r)^{\alpha}}}{r^{1/2}} \\ &= \frac{K\mu^{\beta} e^{\mu^{\alpha}}}{e^{\mu/2}}. \end{aligned}$$

Since $\alpha < 1$, $\frac{\log M(r,f)}{r^{1/2}}$ decreases when $r > r_1 > r_0$, by (21) we get

$$\begin{aligned} Q(r) &< \frac{2r}{\log A}\int_r^{+\infty} \frac{\log M(t,f)}{t^2}dt \\ &< \frac{2r^{1/2}}{\log A}\log M(r,f)\int_r^{+\infty} t^{-3/2}dt \\ &= 4\frac{\log M(r,f)}{\log A}. \end{aligned} \tag{22}$$

Thus, by Lemma 3 and (22), there exists R_0 and a constant $c > 1$, such that, for each $R > R_0$ and each $A \geq 2$, there exists r satisfying

$$R/\delta \leq r \leq R$$

with

$$\log m(r,f) > \log M(r,f) - c\frac{\log M(2R,f)}{\log A}. \tag{23}$$

On the other hand, since f is transcendental there exists r_2 such that for all $r > r_2 > r_1$, $M(r,f) \geq r^2$, take

$$R_1 > \max(e^2, r_2) \qquad \text{and} \qquad R_{n+1} = M(R_n, f). \tag{24}$$

We now separate two cases:

Case 1. Suppose that

$$\log M((R_n)^{2+1/n}/8, f) \geq 4\log M(R_n, f). \tag{25}$$

Put $A = e^{8c}$, by (23) for sufficiently large n there exists ρ_n satisfying

$$R_n < \frac{1}{8}(R_n)^{2+1/n} \leq \rho_n \leq (R_n)^{2+1/n} \tag{26}$$

and

$$\log m(\rho_n, f) > \log M(\rho_n, f) - c\frac{\log M(2(R_n)^{2+1/n}, f)}{\log e^{8c}}.$$

By (20), for sufficiently large n we have

$$\begin{aligned} &\log M(2(R_n)^{2+1/n}, f) \\ = &\log M(16(R_n)^{2+1/n}/8, f) \\ < &2\log M((R_n)^{2+1/n}/8, f). \end{aligned}$$

So

$$\begin{aligned} \log m(\rho_n, f) &> \log M(\rho_n, f) - \frac{1}{4}\log M((R_n)^{2+1/n}/8, f) \\ &> \frac{3}{4}\log M((R_n)^{2+1/n}/8, f). \end{aligned}$$

By (25), we get

$$\log m(\rho_n, f) > 3\log M(R_n, f).$$

Thus

$$m(\rho_n, f) > (M(R_n, f))^3 = (R_{n+1})^3 > (R_{n+1})^{2+\frac{1}{n+1}}. \tag{27}$$

Case 2. Suppose that

$$\log M((R_n)^{2+1/n}/8, f) < 4\log M(R_n, f). \tag{28}$$

Put $A = e^{cn^3}$, by (23) for sufficiently large n, there exists ρ_n satisfying

$$R_n < (R_n)^{2+1/n-1/n^3} < \frac{1}{64}(R_n)^{2+1/n} \leq \rho_n < \frac{1}{8}(R_n)^{2+1/n} \tag{29}$$

and

$$\log m(\rho_n, f) > \log M(\rho_n, f) - c\frac{\log M(2(R_n)^{2+1/n}/8, f)}{\log e^{cn^3}}.$$

By (20) and (28), for sufficiently large n, we obtain

$$\log m(\rho_n, f) > \log M(\rho_n, f) - \frac{2\log M(\frac{1}{8}(R_n)^{2+1/n}, f)}{n^3}$$

$$> \log M(\rho_n, f) - \frac{8\log M(R_n, f)}{n^3}. \tag{30}$$

By (29), $\rho_n > (R_n)^{2+1/n-1/n^3}$ and $\log M(r, f)$ is a convex function of $\log r$, for sufficiently large n, we have

$$\log M(\rho_n, f) > (2+1/n-1/n^3)[\log M(R_n, f) - \log M(1, f)] + \log M(1, f)$$

$$> (2 + 1/n - 1/n^3)\log M(R_n, f). \tag{31}$$

Thus, by (30) and (31), for sufficiently large n, we get

$$\log m(\rho_n, f) > (2 + 1/n - 2/n^3 - 8/n^3)\log M(R_n, f)$$

$$> (2 + \frac{1}{n+1})\log M(R_n, f)$$

so

$$m(\rho_n, f) > (R_{n+1})^{2+\frac{1}{n+1}}. \tag{32}$$

Hence, by (24), (26), and (27) or (24), (29), and (32), we obtain the following result:

If f is a transcendental entire function with $\log M(r, f) = O((\log r)^\beta e^{(\log r)^\alpha})$ $(0 \le \alpha < 1, \beta > 1)$, then there exists sequences $R_n, \rho_n \to \infty$ such that

1. $R_{n+1} = M(R_n, f)$,
2. for sufficiently large n,

$$R_n \le \rho_n \le (R_n)^{2+1/n} \qquad \text{and} \qquad m(\rho_n, f) > (R_{n+1})^{2+\frac{1}{n+1}}.$$

Put $c(n) = 2 + 1/n > 1$, by Lemma 4, we get every component of $N(f)$ is bounded.

Theorem 1 is proved. □

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