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ON THE GENERALIZED HYERS-ULAM-RASSIAS STABILITY OF A QUADRATIC FUNCTIONAL EQUATION

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ABSTRACT. In this paper, we investigate the generalized Hyers-Ulam-Rassias stability of a quadratic functional equation $f(x+y+z)+f(x-y)+f(y-z)+f(z-x)=3f(x)+3f(y)+3f(z)$ and prove the Hyers-Ulam-Rassias stability of the equation on bounded domain.

1. Introduction

The stability problem for the functional equations was first raised by S. M. Ulam [16]:

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In the next year (1941), D. H. Hyers affirmatively answered the question of Ulam for the case when G_1 and G_2 are Banach spaces, and the result of Hyers was further generalized by Th. M. Rassias ([5], [13]). Since then, the stability problems of functional equations have been extensively investigated by a number of mathematician ([6], [7], [14]). For more detailed definitions for the terminologies, one can refer to [1], [4], [10], [11].

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The quadratic function $f(x) = cx^2$ ($x \in \mathbb{R}$) satisfies the functional equation

$$(1.1) \qquad f(x+y) + f(x-y) = 2f(x) + 2f(y).$$

Hence, the above equation is called the quadratic functional equation or the Euler-Lagrange functional equation, and every solution of the quadratic equation (1.1) is called a quadratic function. It is well known that a function $f : E_1 \to E_2$ between vector spaces is quadratic if and only if there exists a unique symmetric function $B : E_1 \times E_2 \to E_2$, which is additive in x for each fixed y, such that $f(x) = B(x,x)$ for any $x \in E_1$ ([1]).

A Hyers-Ulam stability theorem for the quadratic functional equation (1.1) was proved by F. Skof for functions $f : E_1 \to E_2$ where E_1 is a normed space and E_2 a Banach space ([15]). P. W. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an abelian group. S. Czerwik [4] proved the Hyers- Ulam-Rassias stability of the quadratic functional equation, and this result was generalized by J. M. Rassias [12], C. Borelli and G. L. Forti [2].

Consider the following functional equations:

$$(1.2) \quad f(x+y+z)+f(x)+f(y)+f(z) = f(x+y)+f(y+z)+f(z+x)$$

and

$$(1.3) \quad f(x+y+z)+f(x-y)+f(y-z)+f(z-x) = 3f(x)+3f(y)+3f(z).$$

The functional equation (1.2) was solved by Pl. Kannappan ([9]). Recently, S.-M. Jung investigated in his paper [7] the Hyers-Ulam stability of the equation (1.2) on a restricted (unbounded) domain and applied the result to the study of an asymptotic behavior of the quadratic functions. For more information on the stability of the quadratic equation, we may refer to [14]. The stability problems in connection with the following functional inequality

$$\begin{aligned} &\|f(x+y+z) + f(x-y) + f(y-z) \\ &\quad + f(z-x) - 3f(x) - 3f(y) - 3f(z)\| \\ &\leq \varphi(x,y,z) \end{aligned}$$

will be discussed in section 2. In section 3, the Hyers-Ulam stability of the equation (1.3) on bounded domain will be proved.

By $\mathbb{N}$ and $\mathbb{R}$ we denote the set of positive integers and of real numbers, respectively.

2. Generalized Hyers-Ulam-Rassias stability of Eq. (1.3)

In this section, let E_1 and E_2 be a normed space and a Banach space, respectively. Given a fixed integer $k \geq 2$, we denote by $\varphi : E_1 \times E_1 \times E_1 \to [0, \infty)$ a function such that either

$$\psi_k(x, y, z) := \sum_{i=0}^{\infty} \frac{1}{k^{2i}} \varphi(k^i x, k^i y, k^i z) < \infty \tag{2.1}$$

for all x, y, $z \in E_1$, or

$$\tilde{\psi}_k(x, y, z) := \sum_{i=0}^{\infty} k^{2(i+1)} \varphi\left(\frac{x}{k^{i+1}}, \frac{y}{k^{i+1}}, \frac{z}{k^{i+1}}\right) < \infty \tag{2.2}$$

for all x, y, $z \in E_1$.

For convenience, we use the following abbreviations:

$$\begin{aligned} Df(x, y, z) = f(x + y + z) + f(x - y) + f(y - z) + f(z - x) \\ - 3f(x) - 3f(y) - 3f(z). \end{aligned}$$

In this section, we assume that a function $f : E_1 \to E_2$ satisfies the inequality

$$\|Df(x, y, z)\| \leq \varphi(x, y, z) \tag{2.3}$$

for all $x, y, z \in E_1$. We define a real sequence (b_i) by

$$b_1 = 1, \; b_2 = 2 \quad \text{and} \quad b_i = 2b_{i-1} + b_{i-2} \; (i \geq 3).$$

First, we will introduce the following lemma before we prove the generalized Hyers-Ulam stability of the quadratic equation (1.3).

LEMMA 2.1. *Under the above assumptions,*

$$\left\| \frac{1}{k^{2n}} f(k^n x) \right\| \leq \frac{1}{2k^{2n}} \varphi(0, k^n x, 0) \tag{2.4}$$

is true for any $x \in E_1$ if f is odd function, and

$$\left\| \frac{1}{k^{2n}} f(k^n x) - f(x) \right\| \leq \frac{1}{k^2} \sum_{j=1}^{k-1} b_j \sum_{i=0}^{n-1} \frac{1}{k^{2i}} \varphi((k - j)k^i x, k^i x, 0) \tag{2.5}$$

is true for any $x \in E_1$ if f is even function and $f(0) = 0$.

Proof. (Odd case). If we replace x, y, z in (2.3) by $0, x, 0$, respectively, then we have

$$\|2f(x)\| \le \varphi(0, x, 0) \tag{2.6}$$

for all $x \in E_1$. In (2.6) put $x = kx$ and divide by $2k^2$. Then

$$\left\| \frac{1}{k^2} f(kx) \right\| \le \frac{1}{2k^2} \varphi(0, kx, 0). \tag{2.7}$$

Make the induction hypothesis

$$\left\| \frac{1}{k^{2n}} f(k^n x) \right\| \le \frac{1}{2k^{2n}} \varphi(0, k^n x, 0), \tag{2.8}$$

which is true for $n = 1$ by (2.7). Assuming (2.8) true, replace x by kx in it and divide by k^2. Then (2.8) remains true with n replaced by $n + 1$, which establishes (2.8) for all $n \in \mathbb{N}$ and all $x \in E_1$.

(Even case). If we replace x, y, z in (2.3) by $ix, x, 0$, respectively, then we have

$$\|f((i+1)x) + f((i-1)x) - 2f(ix) - 2f(x)\| \le \varphi(ix, x, 0)$$

for all integers i, $2 \le i \le k$. Hence, we immediately get

$$\begin{aligned}
&\|f(kx) - k^2 f(x)\| \\
&\le \|2f((k-1)x) - 2(k-1)^2 f(x)\| + \|f((k-2)x) - (k-2)^2 f(x)\| \\
&\qquad + \|f(kx) + f((k-2)x) - 2f((k-1)x) - 2f(x)\| \\
&\le \sum_{i=1}^{k-2} 2b_i \varphi((k-1-i)x, x, 0) \\
&\qquad + \sum_{i=1}^{k-3} b_i \varphi((k-2-i)x, x, 0) + \varphi((k-1)x, x, 0) \\
&\le \sum_{i=2}^{k-1} 2b_{i-1} \varphi((k-i)x, x, 0) \\
&\qquad + \sum_{i=3}^{k-1} b_{i-2} \varphi((k-i)x, x, 0) + \varphi((k-1)x, x, 0) \\
&= \sum_{j=1}^{k-1} b_j \varphi((k-j)x, x, 0).
\end{aligned}$$

Applying the induction, the assertion (2.5) is true for $n = 1$. Now, we assume that the assertion (2.5) is true. Then

$$\begin{aligned}
&\left\| \frac{1}{k^{2(n+1)}} f(k^{n+1}x) - f(x) \right\| \\
&\leq \left\| \frac{1}{k^{2(n+1)}} f(k^{n+1}x) - \frac{1}{k^{2n}} f(k^n x) \right\| + \left\| \frac{1}{k^{2n}} f(k^n x) - f(x) \right\| \\
&\leq \frac{1}{k^{2n}} \left\| \frac{1}{k^2} f(k \cdot k^n x) - f(k^n x) \right\| + \frac{1}{k^2} \sum_{j=1}^{k-1} b_j \\
&\quad \sum_{i=0}^{n-1} \frac{1}{k^{2i}} \varphi((k-j)k^i x, k^i x, 0) \\
&\leq \frac{1}{k^{2n}} \sum_{j=1}^{k-1} b_j \varphi((k-j)k^n x, k^n x, 0) + \frac{1}{k^2} \sum_{j=1}^{k-1} b_j \\
&\quad \sum_{i=0}^{n-1} \frac{1}{k^{2i}} \varphi((k-j)k^i x, k^i x, 0) \\
&= \frac{1}{k^2} \sum_{j=1}^{k-1} b_j \sum_{i=0}^{n} \frac{1}{k^{2i}} \varphi((k-j)k^i x, k^i x, 0),
\end{aligned}$$

which completes the induction proof. $\square$

In the next theorem, we shall prove the stability of the quadratic equation (1.3).

THEOREM 2.2. *Assume that a function $f : E_1 \to E_2$ satisfies the inequality (2.3) for all $x, y, z \in E_1$ and $f(0) = 0$. Then there exists a unique quadratic function $Q : E_1 \to E_2$ satisfying*

$$\|Q(x) - f(x)\| \leq \frac{1}{k^2} \sum_{j=1}^{k-1} b_j \psi_k((k-j)x, x, 0) + \frac{1}{2} \varphi(0, x, 0) \tag{2.9}$$

for all $x \in E_1$ when $\psi_k(x, y, z) < \infty$ for all $x, y, z \in E_1$, or

$$\|Q(x) - f(x)\| \leq \frac{1}{k^2} \sum_{j=1}^{k-1} b_j \tilde{\psi}_k((k-j)x, x, 0) + \frac{1}{2} \varphi(0, x, 0) \tag{2.10}$$

for all $x \in E_1$ when $\tilde{\psi}_k(x,y,z) < \infty$ for all $x,y,z \in E_1$. If, moreover, f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for every fixed $x \in E_1$, then the function Q satisfies

$$Q(tx) = t^2 Q(x) \tag{2.11}$$

for all $x \in E_1$ and $t \in \mathbb{R}$.

Proof. We first assume that $\psi_k(x,y,z) < \infty$ for all $x,y,z \in E_1$. If f is even function, then it follows from (2.5) and (2.1) that

$$\lim_{n\to\infty}\left\|\frac{1}{k^{2n}}f(k^n x) - f(x)\right\| \le \frac{1}{k^2}\sum_{j=1}^{k-1} b_j \psi_k((k-j)x, x, 0). \tag{2.12}$$

Now, we show that $\{\frac{1}{k^{2n}}f(k^n x)\}$ is a Cauchy sequence: Let m, n be integers with $n > m > 0$. Then, by (2.12) and (2.1), we have

$$\begin{aligned}
&\left\|\frac{1}{k^{2n}}f(k^n x) - \frac{1}{k^{2m}}f(k^m x))\right\| \\
&\le \frac{1}{k^{2m}}\left\|\frac{1}{k^{2(n-m)}}f(k^{n-m}k^m x) - f(k^m x)\right\| \\
&\le \frac{1}{k^{2m}}\frac{1}{k^2}\sum_{j=1}^{k-1} b_j \sum_{i=0}^{n-m-1}\frac{1}{k^{2i}}\varphi((k-j)k^i k^m x, k^i k^m x, 0) \\
&\le \frac{1}{k^2}\sum_{j=1}^{k-1} b_j \sum_{i=m}^{n-1}\frac{1}{k^{2i}}\varphi((k-j)k^i x, k^i x, 0) \to 0 \quad \text{as} \quad m \to \infty.
\end{aligned}$$

Since E_2 is a Banach space, we may define a function $Q : E_1 \to E_2$ by $Q(x) = \lim_{n\to\infty}\frac{1}{k^{2n}}f(k^n x)$ for any $x \in E_1$. By the definition of Q and (2.12),

$$\|Q(x) - f(x)\| \le \frac{1}{k^2}\sum_{j=1}^{k-1} b_j \psi_k((k-j)x, x, 0).$$

If f is odd function, then it follows from (2.4) that $\lim_{n\to\infty}\frac{1}{k^{2n}}f(k^n x) = 0$ for all $x \in E_1$. By (2.6),

$$\|Q(x) - f(x)\| = \|f(x)\| \le \frac{1}{2}\varphi(0, x, 0).$$

Therefore, by $f = f_e + f_o$ (f_e : even part of f, f_o : odd part of f),

$$\begin{aligned}\|Q(x) - f(x)\| &= \|Q(x) - f_e(x)\| + \|f_o(x)\| \\ &\le \frac{1}{k^2}\sum_{j=1}^{k-1} b_j \psi_k((k-j)x, x, 0) + \frac{1}{2}\varphi(0, x, 0).\end{aligned}$$

By replacing x, y and z in (2.3) by $k^n x$, $k^n y$ and $k^n z$, respectively, and dividing the resulting inequality by k^{2n} and by using (2.1), we get

$$\begin{aligned}&\|\frac{1}{k^{2n}}f(k^n(x+y+z)) + \frac{1}{k^{2n}}f(k^n(x-y)) + \frac{1}{k^{2n}}f(k^n(y-z)) \\ &\qquad + \frac{1}{k^{2n}}f(k^n(z-x)) - \frac{3}{k^{2n}}f(k^n x) - \frac{3}{k^{2n}}f(k^n y) - \frac{3}{k^{2n}}f(k^n z)\| \\ &\le \frac{1}{k^{2n}}\varphi(k^n x, k^n y, k^n z) \to 0 \quad \text{as} \quad n \to \infty,\end{aligned}$$

which implies that Q is a quadratic function.

Now, let $Q' : E_1 \to E_2$ be an another quadratic function which satisfies the inequality (2.9). Since Q and Q' are quadratic functions, we can easily show that

$$Q(k^n x) = k^{2n} Q(x) \quad \text{and} \quad Q'(k^n x) = k^{2n} Q'(x) \tag{2.13}$$

for any $n \in N$. Thus, it follows from (2.13), (2.9) and (2.1) that

$$\begin{aligned}\|Q(x) - Q'(x)\| &= \frac{1}{k^{2n}}\|Q(k^n x) - Q'(k^n x)\| \\ &\le \frac{1}{k^{2n}}\Big\|Q(k^n x) - f(k^n x)\| + \|f(k^n x) - Q'(k^n x)\Big\| \\ &\le \frac{2}{k^{2n}}\left(\sum_{j=1}^{k-1}\frac{1}{k^2} b_j \psi_k((k-j)x, x, 0) + \frac{1}{2}\varphi(0, x, 0)\right) \\ &\to 0 \quad \text{as} \quad n \to \infty,\end{aligned}$$

which implies that $Q(x) = Q'(x)$ for all $x \in E_1$.

Also, it can be proved that (2.11) is true if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$ (cf. [4]). We can quite similarly prove the theorem for the case $\tilde{\Psi}_k(x, y, z) < \infty$ for all $x, y, z \in E_1$. $\square$

COROLLARY 2.3. *If a function $f : E_1 \to E_2$ satisfies the functional inequality*

$$\|Df(x,y,z)\| \leq \varepsilon \tag{2.14}$$

for all $x, y, z \in E_1$, then there exists exactly one quadratic function $Q : E_1 \to E_2$ such that

$$\|Q(x) - f(x)\| \leq \frac{5}{6}\varepsilon$$

for all $x \in E_1$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$, then the function Q satisfies (2.11) for all $x \in E_1$ and $t \in \mathbb{R}$.

Proof. If we put $\varphi(x,y,z) = \varepsilon$, then φ satisfies the condition (2.1) for $k = 2$. Hence, it follows from Theorem 2.2 that there exists a unique quadratic function $Q : E_1 \to E_2$ such that

$$\begin{aligned}
\|Q(x) - f(x)\| &\leq \frac{1}{2^2}\psi_2(x,x,0) + \frac{1}{2}\varphi(0,x,0) \\
&\leq \frac{1}{2^2}\sum_{i=0}^{\infty}\frac{1}{2^{2i}}\varphi(2^i x, 2^i x, 0) + \frac{1}{2}\varphi(0,x,0) \\
&= \frac{1}{2^2}\sum_{i=0}^{\infty}\frac{\varepsilon}{2^{2i}} + \frac{\varepsilon}{2} \\
&= \frac{\varepsilon}{3} + \frac{\varepsilon}{2} = \frac{5}{6}\varepsilon
\end{aligned}$$

for all $x \in E_1$. □

COROLLARY 2.4. *If a function $f : E_1 \to E_2$ satisfies the functional inequality*

$$\|Df(x,y,z)\| \leq \varepsilon\,(\|x\|^p + \|y\|^p + \|z\|^p) \tag{2.15}$$

for some $0 < p < 2$ and for all $x, y, z \in E_1$, then there exists a unique quadratic function $Q : E_1 \to E_2$ such that

$$\|Q(x) - f(x)\| \leq \left(\frac{2}{4 - 2^p} + \frac{1}{2}\right)\varepsilon\|x\|^p$$

for all $x \in E_1$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$, then the function Q satisfies (2.11) for all $x \in E_1$ and $t \in \mathbb{R}$.

Proof. Since $\varphi(x,y,z) = \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p)$ satisfies the condition (2.1) with $k = 2$, Theorem 2.2 says that there exists a unique quadratic function $Q : E_1 \to E_2$ such that

$$\begin{aligned}\|Q(x) - f(x)\| &\le \frac{1}{2^2}\psi_2(x,x,0) + \frac{1}{2}\varphi(0,x,0)\\ &= \frac{1}{2^2}\sum_{i=0}^{\infty}\frac{1}{2^{2i}}\varepsilon(\|2^i x\|^p + \|2^i x\|^p) + \frac{1}{2}\varepsilon\|x\|^p\\ &= \left(\frac{2}{4-2^p} + \frac{1}{2}\right)\varepsilon\|x\|^p\end{aligned}$$

for all $x \in E_1$. □

COROLLARY 2.5. *If a function $f : E_1 \to E_2$ satisfies the inequalities (2.15) for some $p > 2$ and for all $x, y, z \in E_1$, then there exists a unique quadratic function $Q : E_1 \to E_2$ such that*

$$\|Q(x) - f(x)\| \;\le\; \left(\frac{2}{2^p-4} + \frac{1}{2}\right)\varepsilon\|x\|^p$$

for all $x \in E_1$. If, in addition, f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$, then the quadratic function Q satisfies (2.11) for all $x \in E_1$ and $t \in \mathbb{R}$.

Proof. Since $\varphi(x,y,z) = \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p)$ satisfies the condition (2.1) with $k = 2$, Theorem 2.2 says that there exists a unique quadratic function $Q : E_1 \to E_2$ such that

$$\begin{aligned}\|Q(x) - f(x)\| &\le \frac{1}{2^2}\tilde{\psi}_2(x,x,0) + \frac{1}{2}\varphi(0,x,0)\\ &= \frac{1}{2^2}\sum_{i=0}^{\infty}2^{2(i+1)}\varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, 0\right) + \frac{1}{2}\varepsilon\|x\|^p\\ &= \frac{1}{2^2}\sum_{i=0}^{\infty}2^{2(i+1)}\varepsilon\left(\left\|\frac{x}{2^{i+1}}\right\|^p + \left\|\frac{x}{2^{i+1}}\right\|^p\right) + \frac{1}{2}\varepsilon\|x\|^p\\ &= \left(\frac{2}{2^p-4} + \frac{1}{2}\right)\varepsilon\|x\|^p\end{aligned}$$

for all $x \in E_1$. □

3. Hyers-Ulam stability of Eq. (1.3) on a bounded domain

Throughout this section, let n be a given positive integer, $r > 0$ a constant, and E a Banach space. For convenience, we use the notation $I^n := [-r, r]^n$.

The stability of the quadratic equation on a bounded real interval was presented by S.-M. Jung as the following theorem [8]:

THEOREM 3.1. *If a function $f : I^n \to E$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \varepsilon$$

for some $\varepsilon > 0$ and for all $x, y \in I^n$ with $x+y$, $x-y \in I^n$, then there exists a quadratic function $q : \mathbb{R}^n \to E$ such that

$$\|f(x) - q(x)\| < (2912n^2 + 1872n + 334)\varepsilon$$

for all $x \in I^n$.

Similarly, the Hyers-Ulam stability of the quadratic equation (1.3) on a bounded domain is obtained as the following:

THEOREM 3.2. *If a function $f : I^n \to E$ satisfies the inequality* (2.14) *for all $x, y, z \in I^n$ with $x+y+z$, $x+y$, $x-y$, $y-z$, $z-x \in I^n$, then there exists a quadratic function $q : \mathbb{R}^n \to E$ such that*

$$\|f(x) - q(x)\| < \frac{13}{5}(2912n^2 + 1872n + 334)\varepsilon \tag{3.1}$$

for every $x \in I^n$.

Proof. Putting $x = y = z = 0$ in (2.14) we get

$$\|f(0)\| \leq \frac{\varepsilon}{5}. \tag{3.2}$$

Putting $y = -x$ in (2.14) and replacing z in (2.14) by x and $-x$, respectively, we have

$$\| - 5f(x) + f(2x) + f(-2x) + f(0) - 3f(-x)\| \ \leq \ \varepsilon$$

and

$$\| - 5f(-x) + f(2x) + f(0) + f(-2x) - 3f(x)\| \ \leq \ \varepsilon$$

for all $x \in I^n$. By the last two inequalities, we obtain

$$\|f(x) - f(-x)\| \leq \varepsilon. \tag{3.3}$$

If we put $z = 0$ in (2.14), we have

$$\|f(x+y) + f(x-y) + f(-x) - 3f(x) - 2f(y) - 3f(0)\| \leq \varepsilon \tag{3.4}$$

for all $x, y \in I^n$ with $x+y, x-y \in I^n$. It then follows from (3.4), (3.3) and (3.2) that the inequality

$$\begin{aligned}
&\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\
&\leq \|f(x+y) + f(x-y) + f(-x) - 3f(x) - 2f(y) - 3f(0)\| \\
&\qquad + \|f(x) - f(-x)\| + \|3f(0)\| \\
&\leq \varepsilon + \varepsilon + \frac{3}{5}\varepsilon \;=\; \frac{13}{5}\varepsilon
\end{aligned}$$

holds for all $x, y \in I^n$ with $x+y, x-y \in I^n$. According to Theorem 3.1, there exists a quadratic function $q : \mathbb{R}^n \to E$ such that the inequality (3.1) holds for any $x \in I^n$. □

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