

**A NUMERICAL METHOD FOR THE PROBLEM
OF COEFFICIENT IDENTIFICATION OF
THE WAVE EQUATION BASED ON A
LOCAL OBSERVATION ON THE BOUNDARY**

KENJI SHIROTA

ABSTRACT. The purpose of this paper is to propose a numerical algorithm for the problem of coefficient identification of the scalar wave equation based on a local observation on the boundary: Determine the unknown coefficient function with the knowledge of simultaneous Dirichlet and Neumann boundary values on a part of boundary. To find the unknown coefficient function, the unknown Neumann boundary value is also identified. We recast our inverse problem to a variational problem. The gradient method is applied to find the minimizing functions. We confirm the effectiveness of our algorithm by numerical experiments.

1. Introduction

In this paper, we consider the problem of coefficient identification of the wave equation based on a local observation on the boundary. This problem is to determine the unknown coefficient function with the knowledge of simultaneous Dirichlet and Neumann boundary values on a part of the boundary.

Let $\Omega \subset \mathbf{R}^n$ ($n = 2, 3$) be a bounded domain with two smooth boundaries Γ_i and Γ_d , satisfying $\Gamma_i \cap \Gamma_d = \emptyset$. A conventional problem is to

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find the function u such that

$$(1) \quad \left\{ \begin{array}{ll} \frac{\partial^2 u}{\partial t^2} - \nabla \cdot (K \nabla u) = 0 & \text{in } \Omega \times (0, T], \\ u = \frac{\partial u}{\partial t} = 0 & \text{on } \Omega \times \{0\}, \\ u = \bar{u} & \text{on } \Gamma_i \times (0, T], \\ K \frac{\partial u}{\partial n} = g & \text{on } \Gamma_d \times (0, T]. \end{array} \right.$$

Here we assume that the coefficient function K belongs to $L^\infty(\Omega)$ and satisfies the condition $K(\mathbf{x}) \geq C > 0$ for all $\mathbf{x} \in \Omega$, where C is a given positive constant.

Our inverse problem is to determine the unknown coefficient function K with the knowledge of a set of the Dirichlet boundary value \bar{u} and the Neumann boundary value \bar{g} on the boundary Γ_i . If a set of these boundary values is given on the whole boundary, the uniqueness and stability of the problem of coefficient identification were guaranteed under the appropriate assumption [2, 3]. However, our problem is not uniquely solvable.

The purpose of this paper is to present an algorithm for the numerical resolution of our inverse problem. To find the unknown coefficient function, we also identify the unknown Neumann boundary value on the boundary Γ_d . We introduce an object functional to be minimized, then our problem is recast to a variational problem. We adopt the direct variational method using the gradient method. We confirm the effectiveness of our algorithm by numerical experiments.

2. Variational method and numerical algorithm

2.1. Variational method

Let V be a convex subset of $L^\infty(\Omega)$ characterized by

$$V := \{K \in L^\infty(\Omega) \mid K(\mathbf{x}) \geq C > 0 \text{ for all } \mathbf{x} \in \Omega\}.$$

We denote that the function $u[K, g]$ is the solution of the initial-boundary value problem (1) with the coefficient function K and the Neumann boundary value g on Γ_d . The unknown coefficient function K and the unknown Neumann boundary value g on Γ_d are determined by minimizing the functional $F : V \times L^2(\Gamma_d \times (0, T]) \rightarrow \mathbf{R}_+ := [0, \infty)$, defined

by

$$F(K, g) = \int_0^T \int_{\Gamma_i} |q[K, g] - \bar{q}|^2 ds dt,$$

where $q[K, g] := K \frac{\partial u[K, g]}{\partial n} \Big|_{\Gamma_i \times (0, T)}$. The function K is a solution of our problem if $F(K, g) = 0$. However it is difficult to find one of the minima of the functional F directly, because two unknown functions K and g are involved. We notice that K and g are mutually independent.

To find the minimal K and g , we make use of the gradient method: For $j = 0, 1, 2, \dots$

$$(2) \quad K_{j+1} = K_j - \alpha_j r_j, \quad g_{j+1} = g_j - \beta_j s_j,$$

where r_j and s_j are the search directions. The search direction r_j is determined by using the projected gradient method:

$$r_j(\mathbf{x}) = K_j(\mathbf{x}) - P(K_j - F_K(K_j, g_j))(\mathbf{x}),$$

where

$$(PK)(\mathbf{x}) = \begin{cases} K(\mathbf{x}) & (K(\mathbf{x}) \geq C), \\ C & (K(\mathbf{x}) < C). \end{cases}$$

Here $F_K(K, g)$ means the first variation with respect to the function K , defined by

$$F(K + \delta K, g) - F(K, g) = (F_K(K, g), \delta K)_\Omega + o(\|\delta K\|_\Omega),$$

where

$$(\varphi, \psi)_\Omega := \int_\Omega \varphi \psi dx, \quad \|\varphi\|_\Omega := \left(\int_\Omega |\varphi|^2 dx \right)^{\frac{1}{2}}.$$

We can guarantee that the updated coefficient function K_j generated by (2) belongs to V if the initial coefficient function K_0 belongs to V and the step size $0 < \alpha_j \leq 1$.

We make use of the steepest descent method to determine the search direction s_j :

$$s_j(\mathbf{x}, t) := F_g(K_{j+1}, g_j)(\mathbf{x}, t),$$

where $F_g(K, g)$ is the first variation with respect to the function g , defined by

$$F(K, g + \delta g) - F(K, g) = (F_g(K, g), \delta g)_{\Gamma_d \times (0, T)} + o(\|\delta g\|_{\Gamma_d \times (0, T)}),$$

where

$$(f, g)_{\Gamma_d \times (0, T)} := \int_0^T \int_{\Gamma_d} f g \, ds \, dt,$$

$$\|f\|_{\Gamma_d \times (0, T)} := \left(\int_0^T \int_{\Gamma_d} |f|^2 \, ds \, dt \right)^{\frac{1}{2}}.$$

To implement this method, we must know concrete expressions of the first variations F_K and F_g .

2.2. First variations

First, we will try to obtain the first variation F_K . We take any admissible g , and fix it.

Let v be the solution of the initial-boundary value problem

$$(3) \quad \begin{cases} \frac{\partial^2 v}{\partial t^2} - \nabla \cdot (K \nabla v) = 0 & \text{in } \Omega \times [0, T), \\ v = w, \frac{\partial v}{\partial t} = 0 & \text{on } \Omega \times \{T\}, \\ v = 2 \left(K \frac{\partial u}{\partial n} - \bar{q} \right) & \text{on } \Gamma_i \times [0, T), \\ K \frac{\partial v}{\partial n} = 0 & \text{on } \Gamma_d \times [0, T), \end{cases}$$

where the function w is the solution of the boundary value problem

$$(4) \quad \begin{cases} \nabla \cdot (K \nabla w) = 0 & \text{in } \Omega, \\ w = 2 \left(K \frac{\partial u}{\partial n}(\cdot, T) - \bar{q}(\cdot, T) \right) & \text{on } \Gamma_i, \\ K \frac{\partial w}{\partial n} = 0 & \text{on } \Gamma_d. \end{cases}$$

For $K \in V$ and $K + \delta K \in V$, we notice that

$$(5) \quad F(K + \delta K, g) - F(K, g) = \int_0^T \int_{\Gamma_i} v \delta q_K \, ds dt + \int_0^T \int_{\Gamma_i} |\delta q_K|^2 \, ds dt$$

where $\delta q_K := q[K + \delta K, g] - q[K, g]$.

Since $(K + \delta K) \frac{\partial u[K + \delta K, g]}{\partial n} |_{\Gamma_d \times (0, T)} - K \frac{\partial u[K, g]}{\partial n} |_{\Gamma_d \times (0, T)} = 0$, we notice that

$$\int_0^T \int_{\Gamma_i} v \delta q_K \, ds dt = \int_0^T \int_{\partial \Omega} v \left\{ K \frac{\partial \delta u_K}{\partial n} + \delta K \frac{\partial u[K + \delta K, g]}{\partial n} \right\} \, ds dt,$$

where $\delta u_K := u[K + \delta K, g] - u[K, g]$. Since $K \frac{\partial v}{\partial n} \Big|_{\Gamma_d \times (0, T)} = 0$ and $\delta u_K|_{\Gamma_i \times (0, T)} = 0$,

$$(6) \quad \int_0^T \int_{\partial\Omega} v \left(K \frac{\partial \delta u_K}{\partial n} \right) ds dt = - \int_0^T \int_{\Omega} \nabla \cdot (K \nabla v) \delta u_K dx dt + \int_0^T \int_{\Omega} v \nabla \cdot (K \nabla \delta u_K) dx dt.$$

Moreover we can get the following relation:

$$(7) \quad \int_0^T \int_{\partial\Omega} v \left(\delta K \frac{\partial u[K + \delta K, g]}{\partial n} \right) ds dt = \int_0^T \int_{\Omega} \delta K \nabla v \cdot \nabla u[K, g] dx dt + \int_0^T \int_{\Omega} \delta K \nabla v \cdot \nabla \delta u_K dx dt + \int_0^T \int_{\Omega} v \nabla \cdot (\delta K \nabla u[K + \delta K, g]) dx dt.$$

Accordingly from (6) and (7), we have

$$(8) \quad \int_0^T \int_{\Gamma_i} v \delta q_K ds dt = \int_0^T \int_{\Omega} \delta K \nabla v \cdot \nabla u[K, g] dx dt + \int_0^T \int_{\Omega} \delta K \nabla v \cdot \nabla \delta u_K dx dt - \int_0^T \int_{\Omega} \nabla \cdot (K \nabla v) \delta u_K dx dt + \int_0^T \int_{\Omega} v \frac{\partial^2 \delta u_K}{\partial t^2} dx dt.$$

Since $\delta u_K(\cdot, 0) = \partial_t \delta u_K(\cdot, 0) = 0$, it holds that

$$(9) \quad \int_0^T \int_{\Omega} v \frac{\partial^2 \delta u_K}{\partial t^2} dx dt = \int_{\Omega} w \frac{\partial \delta u_K}{\partial t}(\cdot, T) dx + \int_0^T \int_{\Omega} \frac{\partial^2 v}{\partial t^2} \delta u_K dx dt.$$

Hence, from (8) and (9), we have

$$(10) \quad F(K + \delta K, g) - F(K, g) = \int_0^T \int_{\Omega} \delta K \nabla v \cdot \nabla u[K, g] dx dt + \int_{\Omega} w \frac{\partial \delta u_K}{\partial t}(\cdot, T) dx + \int_0^T \int_{\Omega} \delta K \nabla v \cdot \nabla \delta u_K dx dt + \int_0^T \int_{\Gamma_i} |\delta q_K|^2 ds dt.$$

Under the appropriate assumption, we can guarantee that

$$\begin{aligned} \int_{\Omega} w \frac{\partial \delta u_K}{\partial t}(\cdot, T) dx &= O(\|\delta K\|_{\Omega}), \\ \int_0^T \int_{\Gamma_i} |\delta q_K|^2 ds dt &= o(\|\delta K\|_{\Omega}), \\ \int_0^T \int_{\Omega} \delta K \nabla v \cdot \nabla \delta u_K dx dt &= o(\|\delta K\|_{\Omega}). \end{aligned}$$

To get the concrete expression of the first variation, we need to calculate the third term of the right hand side of (10). However it is difficult to analyze this term further. Consequently we introduce instead the approximate first variation \widehat{F}_K , defined by

$$\widehat{F}_K(K, g) := \int_0^T \nabla v(\cdot, t) \cdot \nabla u[K, g](\cdot, t) dt.$$

The search direction r_j is then replaced as follows:

$$r_j(\mathbf{x}) = K_j(\mathbf{x}) - P(K_j - \widehat{F}_K(K_j, g_j))(\mathbf{x}).$$

We will try to get the concrete expression of the first variation F_g . We take any admissible $K \in V$, and fix it. In the same way as in (10), we can get the relation

$$\begin{aligned} &F(K, g + \delta g) - F(K, g) \\ &= - \int_0^T \int_{\Gamma_d} v \delta g ds dt + \int_{\Omega} w \frac{\partial \delta u_g}{\partial t}(\cdot, T) dx + \int_0^T \int_{\Gamma_i} |\delta q_g|^2 ds dt, \end{aligned}$$

where $\delta u_g := u[K, g + \delta g] - u[K, g]$ and $\delta q_g := q[K, g + \delta g] - q[K, g]$.

Under the appropriate assumption, we can guarantee that

$$\begin{aligned} \int_{\Omega} w \frac{\partial \delta u_g}{\partial t}(\cdot, T) dx &= O(\|\delta g\|_{\Gamma_d \times (0, T)}), \\ \int_0^T \int_{\Gamma_i} |\delta q_g|^2 ds dt &= o(\|\delta g\|_{\Gamma_d \times (0, T)}). \end{aligned}$$

Therefore we notice that

$$F(K, g + \delta g) - F(K, g) = - \int_0^T \int_{\Gamma_d} v \delta g ds dt + O(\|\delta g\|_{\Gamma_d \times (0, T)}).$$

Similarly to the case of F_K , it is difficult to get the concrete expression of the first variation F_g . Therefore we introduce the approximate first variation instead, defined by

$$\widehat{F}_g(K, g) := -v|_{\Gamma_d \times (0, T)}.$$

Using \widehat{F}_g the search direction s_j is now determined by

$$s_j(\mathbf{x}, t) := \widehat{F}_g(K_{j+1}, g_j)(\mathbf{x}, t).$$

2.3. Numerical algorithm

To implement the gradient method, we must choose the step size α_j and β_j . We employ the Armijo criterion[1] for this purpose:

Armijo criterion

Given the parameters ξ ($0 < \xi < 0.5$), τ ($0 < \tau < 1$) and $\varepsilon > 0$.

- (i) If $\|r_j\|_\Omega < \varepsilon$, then stop; else go to next step.
- (ii) $\eta_m := 1$.
- (iii) For $m = 0, 1, 2, \dots$: If

$$F(K_j - \eta_m r_j, g_j) \leq F(K_j, g_j) - \xi \eta_m \int_\Omega \widehat{F}_K(K_j, g_j) r_j \, dx,$$

then $\alpha_j := \eta_m$; else $\eta_{m+1} := \tau \eta_m$.

For choosing the step size β_j , we also use the Armijo criterion.

Thus in order to solve our inverse problem, we can summarize the algorithm as follows:

Numerical algorithm

1. Pick an initial coefficient function $K_0 \in V$ and an initial Neumann boundary value g .
2. For $j = 0, 1, 2, \dots$,
 - (a) Solve the initial-boundary value problem (1) with K_j and g_j .
 - (b) Solve the boundary value problem (4) with K_j .
 - (c) Solve the initial-boundary value problem (3) with K_j .
 - (d) Calculate the approximate first variation

$$\widehat{F}_K(K_j, g_j) = \int_0^T \nabla u[K_j, g_j] \cdot \nabla v \, dt.$$

- (e) Set the search direction: $r_j := K_j - P(K_j - \widehat{F}_K(K_j, g_j))$.
- (f) Choose the step size α_j by using the Armijo criterion.
- (g) Update the coefficient function: $K_{j+1} = K_j - \alpha_j r_j$.
- (h) Solve the initial-boundary value problem (1) with K_{j+1} and g_j .
 - (i) Solve the boundary value problem (4) with K_{j+1} .
 - (j) Solve the initial-boundary value problem (3) with K_{j+1} .
- (k) Set the search direction: $s_j := -v|_{\Gamma_d \times (0, T)}$.
- (l) Choose the step size β_j by using the Armijo criterion.

(m) Update the coefficient function: $g_{j+1} = g_j - \beta_j s_j$.

3. Numerical experiments

In this section, we show a numerical example about our algorithm. We take the example as a bench mark test within the following framework:

1. Let Ω be a unit disk and $T = 4.0$.
2. The corresponding Dirichlet and Neumann data are generated by solving numerically the initial-boundary value problem with the following data: The domain is a circle with radius 3.0. The initial and Neumann boundary values are set to

$$\begin{cases} \sin 1.25\pi(\sqrt{x_1^2 + x_2^2} - 2.6) & (1 \leq \sqrt{x_1^2 + x_2^2} \leq 2.6, x_2 \geq 0) \\ 0.0 & (\text{otherwise}) \end{cases}$$

and 0.0, respectively.

3. The initial-boundary value problems in our algorithm are solved numerically by using the Newmark method for time integration with linear triangular finite elements in space.
4. The domain is divided into triangular mesh as shown in Figure 1.
5. The exact coefficient function K is set as in Figure 2:

$$K(x_1, x_2) = \begin{cases} 2.0 & (0 \leq \sqrt{x_1^2 + x_2^2} < 0.25) \\ 1.0 & (0.25 < \sqrt{x_1^2 + x_2^2} < 1.0) \end{cases}.$$

6. Let $C = 0.9$, and the initial estimates are set as $K_0 = 1.5$ and $g = 0.0$.

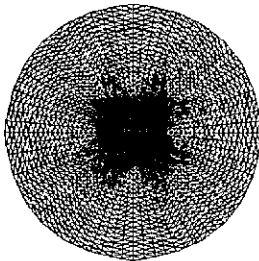


Fig. 1: Finite elements

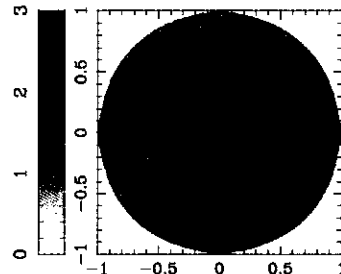


Fig. 2: Exact coefficient K

First, we suppose that

$$\Gamma_i = \left\{ (\cos \theta, \sin \theta) \mid -\frac{3}{8}\pi \leq \theta \leq \frac{11}{8}\pi \right\}.$$

After 30 times of iterations, we have $F = 5.38 \times 10^{-3}$. The calculated coefficient function is shown in Figure 3. Figure 4 shows the distribution of the relative error. This calculated coefficient function is in good agreement with the exact one.

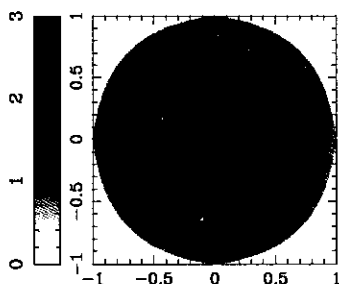


Fig. 3: Calculated K

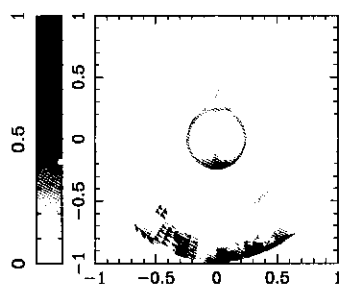


Fig. 4: Relative error

In the previous experiment, the boundary values were given in a wide range of the boundary. The boundary Γ_i is set anew as

$$\Gamma_i = \left\{ (\cos \theta, \sin \theta) \mid 0 \leq \theta \leq \frac{\pi}{2} \right\}.$$

After 30 times of iterations, we have $F = 2.96 \times 10^{-2}$. Figure 5 shows the calculated coefficient function. The distribution of the relative error is shown in Figure 6. This calculated coefficient function becomes worse, but it is still in good agreement with the exact one for the region $x_2 > 0$.

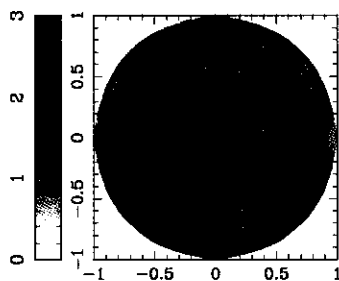


Fig. 5: Calculated K

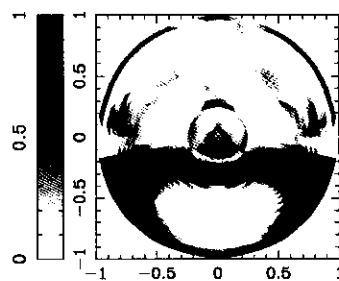


Fig. 6: Relative error

Third, we consider the case of

$$\Gamma_i = \left\{ (\cos \theta, \sin \theta) \mid \frac{\pi}{4} \leq \theta \leq \frac{3}{4}\pi \right\}.$$

After 30 times of iterations, we have $F = 7.66 \times 10^{-3}$. The calculated coefficient function is shown in Figure 7. Figure 8 shows the distribution of the relative error. This calculated coefficient function becomes deteriorated.

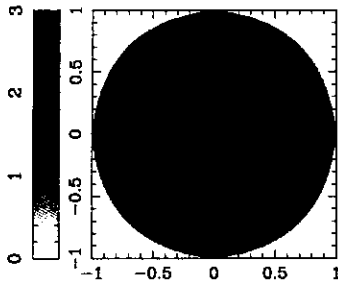
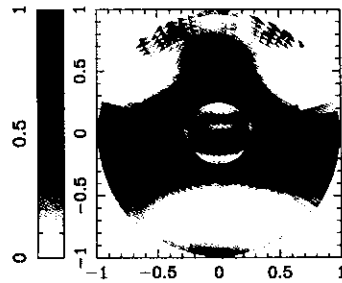
Fig. 7: Calculated K 

Fig. 8: Relative error

4. Concluding remarks

We presented a numerical algorithm for the problem of coefficient identification of the scalar wave equation based on a local observation. To find the unknown coefficient function, the unknown Neumann boundary value on a part of the boundary is also identified. To determine the unknown functions, we make use of the direct variational method. Our numerical algorithm is based on minimizing the functional by the gradient method. We introduce the approximate first variations because it is difficult to get concrete expressions of the exact first variations. By numerical experiment, we confirm that our algorithm is effective when the boundary values are given in a wider range of the boundary.

In our algorithm, we do not use the gradient method exactly, because the search directions are constructed by using approximate first variations. We will investigate theoretically in the future that our method has the convergence property in similar as the conventional gradient method.

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Department of Mathematical Sciences
 Ibaraki University
 2-1-1 Bunkyo, Mito, Ibaraki 310-8512, Japan
E-mail: shiota@mito.ipc.ibaraki.ac.jp