Comm. Korean Math. Soc. **16** (2001), No. 3, pp. 437–443

INVERSE PROBLEM FOR INTERIOR SPECTRAL DATA OF THE DIRAC OPERATOR

KIYOSHI MOCHIZUKI AND IGOR TROOSHIN

ABSTRACT. In this paper the inverse problems for the Dirac Operator are studied. A set of values of eigenfunctions in some internal point and spectrum are taken as a data. Uniqueness theorems are obtained. The approach that was used in the investigation of inverse problems for interior spectral data of the Sturm-Liouville operator is employed.

1. Introduction

We consider the canonical form of the Dirac operator L (see [5]), generated by the differential expression

$$l(y) = By' + Q(x)y \quad (0 \le x \le 1)$$

with

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}, \quad y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$

subject to the boundary conditions

$$y_1(0) = y_1(1) = 0, \tag{1}$$

where $p(x),\ q(x) \in C^1[0,1]$ are real-valued functions.

As is well known, the operator L has a discrete spectrum consisting of simple eigenvalues $\lambda_n,\ n \in Z$. We denote by $y_n(x) = (y_1^n(x), y_2^n(x))^T$, $n \in Z$, the corresponding eigenfunctions.

Research of inverse problems for Dirac operator follows investigations of closely related inverse problems for Sturm-Liouville operator. T. N. Arutyunyan [1] obtained an analog of Marchenko theorem [7], [8] ; he proved that the eigenvalues $\lambda_n,\ n = 0,1,2,\dots$ and normalising coefficients $\alpha_n = \|y_n\|_{\{L^2(0.1)\}^2},\ n = 0,1,2,\dots$ (for $y_{n,1}(0) = 0,\ y_{n,2}(0) = -1$)

Received March 4, 2001. Revised April 28, 2001.
2000 Mathematics Subject Classification: 34A55, 34L40.
Key words and phrases: Dirac operator, eigenvalue, inverse problem.

uniquely determined the potential $Q(x)$. M. M. Malamud [6] proved an analog of Borg theorem [2]; he showed that the spectra of two boundary value problems for an operator with different boundary conditions at one end (and identical conditions repeated at the other end) uniquely determined the potential $Q(x)$. He also proved an analog of the theorem of Hochstadt and Lieberman [4]; one spectrum and a potential on the interval $(0, \frac{1}{2})$ uniquely determined the potential $Q(x)$ on the whole interval $[0, 1]$.

The aim of our present work is to investigate a possibility to recover potential from the known eigenvalues and some information on eigenfunctions in the internal point $b \in (0, 1)$. The similar problem for the Sturm-Liouville operator is formulated and studied in our papers [9], [10]. In these articles we have employed technique similar to those used in [4], [11].

Let us introduce a second Dirac operator $\tilde{L}$ generated by differential expression

$$l(y) = By' + \tilde{Q}(x)y \quad (0 \leq x \leq 1),$$

where

$$\tilde{Q}(x) = \begin{pmatrix} \tilde{p}(x) & q(x) \\ q(x) & -\tilde{p}(x) \end{pmatrix}$$

with a real-valued $\tilde{p}(x), q(x) \in C^1[0, 1]$, subject to the same boundary conditions (1).

We denote by $\tilde{\lambda}_n$, $n \in Z$, and $\tilde{y}_n(x) = (\tilde{y}_1^n(x), \tilde{y}_2^n(x))^T$, $n \in Z$, eigenvalues and corresponding eigenfunctions of operator $\tilde{L}$.

THEOREM. *If for any* $n \in Z$

$$\lambda_n = \tilde{\lambda}_n, \ \frac{y_1^n(\frac{1}{2})}{y_2^n(\frac{1}{2})} = \frac{\tilde{y}_1^n(\frac{1}{2})}{\tilde{y}_2^n(\frac{1}{2})},$$

then $p(x) = \tilde{p}(x)$ *on the* $[0, 1]$.

REMARK. The same result can be obtained by the same method in the more general case of boundary conditions

$$y_1(0)\cos\alpha + y_2(0)\sin\alpha = 0, \ y_1(1)\cos\beta + y_2(1)\sin\beta = 0.$$

2. Preliminary remarks

We shall first mention some results which will be needed later.

Let us denote by $w(x,\lambda) = (w_1(x,\lambda),\ w_2(x,\lambda))^T$ and $\tilde{w}(x,\lambda) = (\tilde{w}_1(x,\lambda),\ \tilde{w}_2(x,\lambda))^T$ the solutions of initial-value problems

$$Bw' + Q(x)w = \lambda w, \tag{2}$$

$$w_1(0) = 0,\ w_2(0) = -1; \tag{3}$$

and

$$B\tilde{w}' + \tilde{Q}(x)\tilde{w} = \lambda\tilde{w}, \tag{4}$$

$$\tilde{w}_1(0) = 0,\ \tilde{w}_2(0) = -1.$$

There exist (see [3], [5]) kernels $K(x,t) = (K_{ij}(x,t))_{i,j=1}^2$, $\tilde{K}(x,t) = (\tilde{K}_{ij}(x,t))_{i,j=1}^2$, with entries continuously differentiable on $0 \le t \le v \le 1$ such that

$$w(x,\lambda) = \phi_0(x,\lambda) + \int_0^x K(x,t)\phi_0(t,\lambda)dt, \tag{5}$$

$$\tilde{w}(x,\lambda) = \phi_0(x,\lambda) + \int_0^x \tilde{K}(x,t)\phi_0(t,\lambda)dt.$$

Here $\phi_0(x,\lambda) = (\sin\lambda x, -\cos\lambda x)^T$.

We can show that

$$\begin{aligned} &\langle Jw(x,\lambda), \tilde{w}(x,\lambda)\rangle \\ &= -\cos(2\lambda x) + \int_0^x R_1(x,t)\cos(2\lambda t)dt + \int_0^x R_2(x,t)\sin(2\lambda t)dt, \end{aligned} \tag{6}$$

where $\langle (a_1,a_2)^T, (b_1,b_2)^T\rangle = a_1b_1 + a_2b_2$,

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $R_l\,(x,t) = (R_{ij}^l\,(x,t))_{i,j=1}^2$, $l = 1,2$, with entries piecewise-continuously differentiable on $0 \le t \le v \le 1$.

3. Proof of Theorem

If we multiply (2) (in the sense of scalar product in R^2) by $\tilde{w}(x,\lambda)$ and (4) by $w(x,\lambda)$ and subtract, after integrating from 0 to $\frac{1}{2}$, we obtain

$$\begin{aligned} &\int_0^{\frac{1}{2}} \langle (Q(x) - \tilde{Q}(x))w(x,\lambda), \tilde{w}(x,\lambda)\rangle dx \\ &= (\langle \tilde{w}_2(x,\lambda), w_1(x,\lambda)\rangle - \langle \tilde{w}_1(x,\lambda), w_2(x,\lambda)\rangle)\big|_0^{\frac{1}{2}}. \end{aligned}$$

Functions $w(x,\lambda)$ and $\tilde{w}(x,\lambda)$ satisfy the same initial conditions (3), i.e.

$$\langle \tilde{w}_2(0,\lambda), w_1(0,\lambda)\rangle - \langle \tilde{w}_1(0,\lambda), w_2(0,\lambda)\rangle = 0.$$

Define

$$P(x) = Q(x) - \tilde{Q}(x),$$

and

$$H(\lambda) = \int_0^{\frac{1}{2}} \langle P(x)w(x,\lambda), \tilde{w}(x,\lambda)\rangle dx.$$

Therefore from the conditions of the theorem it follows that

$$\langle \tilde{w}_2(\frac{1}{2},\lambda_n), w_1(\frac{1}{2},\lambda_n)\rangle - \langle \tilde{w}_1(\frac{1}{2},\lambda_n), w_2(\frac{1}{2},\lambda_n)\rangle = 0$$

and hence

$$H(\lambda_n) = 0,\ n \in Z.$$

In the case of consideration we have

$$H(\lambda) = \int_0^{\frac{1}{2}} p(x)\langle Jw(x,\lambda), \tilde{w}(x,\lambda)\rangle dx,$$

and from (6)

$$\begin{aligned} H(\lambda) = & \int_0^{\frac{1}{2}} p(x)[-\cos(2\lambda x) \\ & + \int_0^x R_1(x,t)\cos(2\lambda t)dt \\ & + \int_0^x R_2(x,t)\sin(2\lambda t)dt]dx. \end{aligned} \tag{7}$$

Therefore it follows that $H(\lambda)$ is an entire function of order no greater than 1.

We next define the function

$$\omega(\lambda) = w_1(1,\lambda).$$

It follows from the formula (5) that

$$\omega(\lambda) = \sin\lambda + \int_0^1 [K(1,t)\phi_0(t,\lambda)]_1 dt.$$

Integrating by parts we obtain the following asymptotic relations :

$$\omega(\lambda) = \sin\lambda + O(\frac{\exp(|\Im\lambda|)}{\lambda}). \tag{8}$$

The zeros of $\omega(\lambda)$ are the eigenvalues of L and hence it has only simple zeros λ_n.

From this and the estimates for $\omega(\lambda)$ and $H(\lambda)$ it follows that

$$\chi(\lambda) = \frac{H(\lambda)}{\omega(\lambda)}$$

is the entire function of order no greater than 1.

It follows from asymptotic (8) and formula (7) that $\chi(\lambda)$ is bounded on any nonreal axis with vertex in origin of λ-plane. Then it follows from the Phragmen-Lindelöf theorem that $\chi(\lambda) = K$ is constant on the whole λ-plane.

Let us show that $K = 0$. We can rewrite the equation $H(\lambda) = Kw(1, \lambda)$ in the form

$$\begin{aligned} &\int_0^{\frac{1}{2}} p(x)\Big[- \cos(2\lambda x) + \int_0^x R_1(x,t)\cos(2\lambda t)dt \\ &+ \int_0^x R_2(x,t)\sin(2\lambda t)dt\Big]\, dx \\ &= K\Big(\sin\lambda + O\Big(\frac{\exp(|\Im\lambda|)}{\lambda}\Big)\Big). \end{aligned} \tag{9}$$

By use of the Rimann-Lebesque lemma, the left side of (9) tends to 0 as $\lambda \to \infty, \lambda \in R$. Thus we obtained that $K = 0$. We are now going to show that $Q(x) = 0$ a.e. on $[0, \frac{1}{2}]$.

We have

$$\begin{aligned} &\int_0^{\frac{1}{2}} p(x)[-\frac{1}{2}\cos(2\lambda x) + \int_0^x R_1(x,t)\cos(2\lambda t)dt \\ &+ \int_0^x R_2(x,t)\sin(2\lambda t)dt]dx = 0. \end{aligned}$$

This can be rewritten as

$$\begin{aligned} &\int_0^{\frac{1}{2}} \cos(2\lambda\tau)[p(\tau) + \int_\tau^{\frac{1}{2}} p(x)R_1(x,t)dx]d\tau \\ &+ \int_0^{\frac{1}{2}} \sin(2\lambda\tau)\int_\tau^{\frac{1}{2}} p(x)R_2(x,t)dx = 0. \end{aligned}$$

Thus from the completeness of the functions $(\cos(2\lambda\tau),\ \sin(2\lambda\tau))^T$ in $L_2(0,1/2)^2$, it follows that

$$p(\tau) + \int_\tau^{\frac{1}{2}} p(x)R_1(x,t)dx = 0 \ \text{ for } \ 0 < \tau < \frac{1}{2}.$$

But this equation is a homogeneous Volterra integral equation and has only the zero solution. Thus we have obtained $p(x) = 0$ on $[0, \frac{1}{2}]$.

To prove that $p(x) = 0$ on $[1/2, 1]$, we should repeat arguments for the supplementary problem

$$l(y) = By' + Q_1(x)y, \; Q_1(x) = Q(1-x) \;\; (0 \le x \le 1)$$

subject to the boundary conditions $y_1(0) = y_1(1) = 0$.

References

[1] T. N. Arutyunyan, *Isospectral Dirac operators*, Izv. Nats. Akad. Nauk Armenii mat. **29** (1994), no. 2, 3–14; English transl. in J. Contemp. Math. Anal., Armen. Acad. Sci. **29** (1994), no. 2, 1–10.

[2] G. Borg, *Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe*, Bestimmung der Differential gleichung durch die Eigenvert, Acta Math. **78** (1946), 1–96.

[3] M. G. Gasymov and B. M. Levitan, *The Inverse Problem for a Dirac System*, Dokl. Akad. Nauk SSSR **167** (1966), 967–970.

[4] H. Hochstadt and B. Lieberman, *An Inverse Sturm-Liouville Problem with Mixed Given Data*, SIAM J. Appl. Math. **34** (1978), 676–680.

[5] B. M. Levitan and I. S. Sargsjan, *Sturm-Liouville and Dirac Operators*, Kluwer Academic Publishers, Dodrecht/Boston/London, 1991.

[6] M. M. Malamud, *Uniqueness Questions in Inverse Problems for Systems of Differential Equations on a Finite Interval*, Trans. Moscow Math. Soc. **60** (1999), 173–224.

[7] V. A. Marchenko, *On certain questions in the theory of differential operators of the second order*, Dokl. Akad. Nauk SSSR **72** (1950), 457–460 (in Russian).

[8] ———, *Some questions in the theory of one-dimensional linear differential operators of the second order. I*, Trudy Moscov. Mat. Obsc. **1** (1952), 327–420 (in Russian); Amer. Math. Soc. Transl. **2** (1973), no. 101, 1–104.

[9] K. Mochizuki and I. Trooshin, *Inverse Problem for Interior Spectral Data of Sturm-Liouville Operator*, Seminar Notes of Mathematical Sciences, Ibaraki University **3** (2000), 44–51.

[10] ———, *Inverse Problem for Interior Spectral Data of Sturm-Liouville Operator*, to appear in "Journal of Inverse and Ill-posed Problems".

[11] A. G. Ramm, *Property C for ODE and applications to inverse problems*, in "Operator theory and applications (Winnipeg, MB, 1998)", Fields Inst. Commun., AMS, Providence, RI **25** (2000), 15–75.

Kiyoshi Mochizuki
Department of Mathematics
Tokyo Metropolitan University
Hachioji Tokyo 192-0397, Japan
E-mail: mochizuk@comp.metro-u.ac.jp

Igor Trooshin
Precision Mechanics and Control Institute
of Russian Academy of Sciences
24, Rabochaya, Saratov 410028, Russia