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DIRECT DETERMINATION OF THE DERIVATIVES OF CONDUCTIVITY AT THE BOUNDARY FROM THE LOCALIZED DIRICHLET TO NEUMANN MAP

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Abstract. We consider the problem of determining conductivity of the medium from the measurements of the electric potential on the boundary and the corresponding current flux across the boundary. We give a formula for reconstructing the conductivity and its normal derivative at the point of the boundary simultaneously from the localized Dirichlet to Neumann map around that point.

1. Introduction

Let $\Omega \in R^n$ $(n \geq 2)$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Physically Ω is considered as an isotropic, static and conductive medium with conductivity $\gamma \in L^\infty(\Omega)$. When an electric potential $f \in H^{1/2}(\partial\Omega)$ is applied to the boundary $\partial\Omega$, the potential u solves the Dirichlet problem

$$\nabla \cdot (\gamma \nabla u) = 0 \quad \text{in} \quad \Omega, \qquad u|_{\partial\Omega} = f. \tag{1}$$

Assume that there is a constant $\delta > 0$ such that $\gamma(x) \geq \delta \quad$ (a.e. $x \in \Omega$). Then, it is well known that there exists a unique weak solution $u \in H^1(\Omega)$ to (1). Define the Dirichlet to Neumann map $\Lambda_\gamma \ : \ H^{1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega)$ by

$$\langle \Lambda_\gamma f, \ g \rangle = \int_\Omega \gamma \nabla u \cdot \nabla v \ dx \quad (g \in H^{1/2}(\partial\Omega)), \tag{2}$$

where u is the solution to (1), v is any $v \in H^1(\Omega)$ satisfying $v|_{\partial\Omega} = g$ and $\langle \ , \ \rangle$ is the bilinear pairing between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$.

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Since (1) implies that (2) is independent of such v, Λ_γ in (2) is well defined. Also, when $f \in H^{3/2}(\partial\Omega)$, $\gamma \in C^1(\overline{\Omega})$ and $\partial\Omega$ is C^2, we see that $\Lambda_\gamma f = \gamma \nabla u \cdot n$, where n is the unit outer normal to $\partial\Omega$. Hence $\Lambda_\gamma f$ is the current flux across $\partial\Omega$ produced by the potential f on $\partial\Omega$.

The problem of determining conductivity of the medium from the measurements of the electric potential on the boundary and the corresponding current flux across the boundary is expressed as

Inverse Problem : "Determine $\gamma(x)$ from Λ_γ".

Since this problem was posed by A. P. Calderon, many results on uniqueness, stability, reconstruction for this inverse problem have been proved by many authors. Here we give a brief review of some of the previous works on reconstruction. When γ and $\partial\Omega$ are C^∞, using the fact that Λ_γ is a pseudodifferential operator in this case, Sylvester and Uhlmann [9] showed how to recover γ and all of its derivatives on $\partial\Omega$ from the symbol of Λ_γ. When $\partial\Omega$ is Lipschitz smooth, from Λ_γ Nachman [4] recovered γ on $\partial\Omega$ if $\gamma \in W^{1,p}(\Omega)$ with $p > n$ and recovered the first normal derivative of γ on $\partial\Omega$ if $\gamma \in W^{2,p}(\Omega)$ with $p > n/2$.

On the other hand, pointwise reconstruction of the coefficients of the equations from the localized Dirichlet to Neumann map has been studied by Brown [2] for the conductivity equation and by Robertson [8] for the elasticity equation. For $x_0 \in \partial\Omega$, they assumed some regularity conditions on $\partial\Omega$ and on the conductivity or the elastic tensor locally around x_0, and reconstructed its value at x_0. Recently, Nakamura and Tanuma [5] reconstructed the higher order derivatives of γ at $x_0 \in \partial\Omega$ *inductively* according to the regularity which γ and $\partial\Omega$ have around x_0.

In this article we give a formula for reconstructing γ and its normal derivative at $x_0 \in \partial\Omega$ *simultaneously* from the localized Λ_γ around x_0. Our formula is straightforward. In fact, in Nakamura and Tanuma [5] (and in Nachman [4]), to recover the normal derivative of γ at $x_0 \in \partial\Omega$, one needs to know not only the value $\gamma(x_0)$ but also all the values of γ in a neighborhood of x_0 on $\partial\Omega$. Our new formula needs not any information of γ but only some regularity assumption on γ around x_0.

We note that a reconstruction formula for the shape of the inclusion in Ω from Λ_γ was given by, for example, Ikehata [3].

For the elasticity equation, there are other works by Akamatsu, Nakamura and Steinberg [1], Nakamura, Tanuma and Uhlmann [6], Nakamura and Uhlmann [7].

In this article, to make the essential part clear we restrict our arguments to the case where the boundary is flat around $x_0 \in \partial\Omega$.

2. Result

We assume that $\partial\Omega$ is flat around $x = 0 \in \partial\Omega$ and that $\Omega, \partial\Omega$ are given by

$$\Omega = \{x_n > 0\},\ \partial\Omega = \{x_n = 0\}$$

locally around $x = 0$, where $x = (x', x_n) = (x_1, \cdots, x_{n-1}, x_n)$.

Let $t = (t', 0) = (t_1, \cdots, t_{n-1}, 0)$ be any unit tangent to $\partial\Omega$ at $x = 0$. Let $\eta(x') \in C_0^2(R^{n-1})$ satisfy

$$0 \le \eta \le 1, \quad \int_{R^{n-1}} \eta^2\, dx' = 1, \quad \operatorname{supp} \eta \subset \{|x'| < 1\}. \tag{3}$$

For any positive integer N, put

$$\phi_N(x') = e^{\sqrt{-1}\, N x' \cdot t'} \eta(\sqrt{N} x'). \tag{4}$$

Assuming that γ is continuous around $x = 0$, Brown [2] and Robertson [8] proved that

$$\lim_{N\to\infty} N^{\frac{n-3}{2}} \langle \Lambda_\gamma \phi_N, \overline{\phi_N} \rangle = \gamma(0). \tag{5}$$

Our main result is the following.

Theorem. *Let $\eta(x') \in C_0^4(R^{n-1})$ satisfy (3) and let $\phi_N(x')$ be given by (4). Put*

$$\psi_N(x') = e^{\sqrt{-1}\, \frac{N}{2} x' \cdot t'} \eta(\sqrt{N} x').$$

Assume that $D_{x'}^{\alpha'} D_{x_n}^{\alpha_n} \gamma$ is continuous around $x = 0$ for any multi-index (α', α_n) such that $|\alpha'| + 2\alpha_n \le 2$. Then,

$$\begin{aligned} &\lim_{N\to\infty} N^{\frac{n-1}{2}} \Big[4 \langle \Lambda_\gamma \psi_N, \overline{\psi_N} \rangle - 2 \langle \Lambda_\gamma \phi_N, \overline{\phi_N} \rangle \Big] \\ &= \frac{\partial}{\partial x_n} \gamma(0) + 3\,\gamma(0) \int_{R^{n-1}} \big(|\nabla_{x'} \eta|^2 - (t' \cdot \nabla_{x'} \eta)^2 \big)\, dx'. \end{aligned} \tag{6}$$

Remarks.

1. In our inverse problem, the left hand side of (6) is observable. On the other hand, the integral $\int_{R^{n-1}}\left(|\nabla\eta|^2-(t'\cdot\nabla\eta)^2\right)dx'$ in the right hand side is controllable, that is, this integral is determined explicitly from the sequences of Dirichlet data $\{\phi_N\}$ and $\{\psi_N\}$ and we can choose additional sequences $\{\tilde{\phi}_N\}$ and $\{\tilde{\psi}_N\}$ so that this integral has another value. Therefore, we obtain a 2×2 system of equations which can be solved for $\gamma(0)$ and $\frac{\partial}{\partial x_n}\gamma(0)$ simultaneously.
2. By (5) and $\lim_{N\to\infty} N^{\frac{n-3}{2}} < \Lambda_\gamma\psi_N, \overline{\psi_N} >= \gamma(0)/2$, $N^{\frac{n-3}{2}}$ times the quantity in $[\cdot]$ of the left hand side of (6) tends to zero as $N\to\infty$.
3. Since the suppots of ϕ_N and ψ_N are in $\{|x'|\le 1/\sqrt{N}\}$, the Dirichlet to Neumann map Λ_γ in (6) is localized around $x=0$ more closely as $N\to\infty$.
4. In the case where $\partial\Omega$ is curved around $x=0$, assuming that $\partial\Omega$ is locally C^3 at $x=0$ we obtain the analogous reconstruction formula.
5. Analogous results will hold for reconstruction of higher order derivatives of γ at the boundary, for $n(\ge 3)$ dimensional anisotropic conductivity equations and for $n(\ge 2)$ dimensional elasticity equations.

3. Outline of proof

Let $\zeta(x_n)\in C^\infty([0,\infty))$ satisfy $0\le\zeta\le 1, \zeta(x_n)=1$ for $0\le x_n\le 1/2$ and 0 for $1\le x_n$ and put

$$\zeta_N(x_n)=\zeta(\sqrt{N}x_n).$$

From the definition (2) it follows that

$$\begin{aligned}&4\langle\Lambda_\gamma\psi_N,\overline{\psi_N}\rangle-2\langle\Lambda_\gamma\phi_N,\overline{\phi_N}\rangle\\ &=4\int_\Omega\gamma\nabla v_N\cdot\nabla(\zeta_N\overline{\Psi_N})\,dx-2\int_\Omega\gamma\nabla u_N\cdot\nabla(\zeta_N\overline{\Phi_N})\,dx,\end{aligned}\tag{7}$$

where $v_N\in H^1(\Omega)$ satisfies

$$\nabla_x\cdot(\gamma\,\nabla_x v_N)=0\quad\text{in}\quad\Omega,\qquad v_N|_{\partial\Omega}=\psi_N,\tag{8}$$

$u_N\in H^1(\Omega)$ satisfy

$$\nabla_x\cdot(\gamma\,\nabla u_N)=0\quad\text{in}\quad\Omega,\qquad u_N|_{\partial\Omega}=\phi_N,\tag{9}$$

$\Psi_N(x)$ and $\Phi_N(x)$ are $H^1(\Omega)$ extensions of ψ_N and $\phi_N \in H^{1/2}(\partial\Omega)$ respectively, and these are given below by (11) and (12).

Introducing the scaling transformation

$$y_i = \sqrt{N}\, x_i \quad (i = 1, 2, \cdots, n-1), \qquad y_n = N\, x_n, \tag{10}$$

let

$$\Psi_N(x) = e^{\sqrt{-1}\frac{N}{2}x'\cdot t'}\, e^{-\frac{y_n}{2}} \sum_{l=0}^{2} N^{\frac{-l}{2}}\, V_l(y', y_n) \tag{11}$$

be an approximate solution to (8) such that

$$\nabla_x \cdot \big(\gamma\, \nabla_x \Psi_N(x)\big) = o(N) \qquad (N \longrightarrow +\infty)$$

uniformly on the set $\{|y'| < 1, 0 \leq y_n < +\infty\}$ and

$$\Psi_N(x)|_{\partial\Omega} = \psi_N,$$

where $y' = (y_1, \cdots, y_{n-1}), V_0(y', y_n) = \eta(y')$, and $V_l(y', y_n)$ $(l \geq 1)$ are polynomials of y_n with $V_l(y', 0) = 0$ whose coefficients are C^∞ functions of y' compactly supported in $\{|y'| < 1\}$. Similarly, let

$$\Phi_N(x) = e^{\sqrt{-1}Nx'\cdot t'}\, e^{-y_n} \sum_{l=0}^{2} N^{\frac{-l}{2}}\, U_l(y', y_n) \tag{12}$$

be an approximate solution to (9) such that

$$\nabla_x \cdot \big(\gamma\, \nabla_x \Phi_N(x)\big) = o(N) \qquad (N \longrightarrow +\infty)$$

uniformly on the set $\{|y'| < 1, 0 \leq y_n < +\infty\}$ and

$$\Phi_N(x)|_{\partial\Omega} = \phi_N,$$

where $U_0(y', y_n) = \eta(y')$, and $U_l(y', y_n)$ $(l \geq 1)$ are polynomials of y_n with $U_l(y', 0) = 0$ whose coefficients are C^∞ functions of y' compactly supported in $\{|y'| < 1\}$. Note that we use the regularity condition on γ when constructing $\Psi_N(x)$ and $\Phi_N(x)$. In fact,

$$\begin{aligned} V_1(y', y_n) &= \sqrt{-1}\, (t' \cdot \nabla\eta)(y')\, y_n, \\ V_2(y', y_n) &= (g_0(y') + g_1(y'))y_n + \frac{1}{2}g_1(y')y_n^2, \end{aligned} \tag{13}$$

where

$$g_0(y') = \sum_{i=1}^{n-1} \frac{\partial^2 \eta}{\partial y_i^2}(y') + \frac{\eta(y')}{2\gamma(0)}\Big(\sqrt{-1}\sum_{i=1}^{n-1} t_i \frac{\partial\gamma}{\partial x_i}(0) - \frac{\partial\gamma}{\partial x_n}(0)\Big),$$

$$g_1(y') = -\sum_{i,j=1}^{n-1} t_i\, t_j \frac{\partial^2 \eta}{\partial y_i \partial y_j}(y')$$

and

$$\begin{aligned} U_1(y', y_n) &= \sqrt{-1}\,(t' \cdot \nabla\eta)(y')\, y_n, \\ U_2(y', y_n) &= (h_0(y') + h_1(y'))y_n + h_1(y')y_n^2, \end{aligned} \tag{14}$$

where

$$h_0(y') = \frac{1}{2}\sum_{i=1}^{n-1} \frac{\partial^2 \eta}{\partial y_i^2}(y') + \frac{\eta(y')}{2\gamma(0)}\Big(\sqrt{-1}\sum_{i=1}^{n-1} t_i \frac{\partial\gamma}{\partial x_i}(0) - \frac{\partial\gamma}{\partial x_n}(0)\Big)$$

$$h_1(y') = -\frac{1}{2}\sum_{i,j=1}^{n-1} t_i\, t_j \frac{\partial^2 \eta}{\partial y_i \partial y_j}(y').$$

For the details of construction of them, we refer to [5].

Put

$$v_N = \Psi_N + s_N, \quad u_N = \Phi_N + r_N.$$

Substituting them into (7) we have

$$\begin{aligned} &4\langle \Lambda_\gamma \psi_N, \overline{\psi_N}\,\rangle - 2\langle \Lambda_\gamma \phi_N, \overline{\phi_N}\,\rangle \\ &= \int_\Omega \gamma\big(\, 4\nabla\Psi_N \cdot \nabla(\zeta_N \overline{\Psi_N}) - 2\nabla\Phi_N \cdot \nabla(\zeta_N \overline{\Phi_N})\big)\; dx \\ &\quad + 4\int_\Omega \gamma\nabla s_N \cdot \nabla(\zeta_N \overline{\Psi_N})\; dx - 2\int_\Omega \gamma\nabla r_N \cdot \nabla(\zeta_N \overline{\Phi_N})\; dx \\ &= I + II + III, \end{aligned}$$

where

$$\begin{aligned} I &= \int_\Omega \gamma\big(\, 4\nabla\Psi_N \cdot \nabla(\zeta_N \overline{\Psi_N}) - 2\nabla\Phi_N \cdot \nabla(\zeta_N \overline{\Phi_N})\big)\; dx, \\ II &= 4\int_\Omega \gamma\nabla s_N \cdot \nabla(\zeta_N \overline{\Psi_N})\; dx, \\ III &= -2\int_\Omega \gamma\nabla r_N \cdot \nabla(\zeta_N \overline{\Phi_N})\; dx. \end{aligned}$$

From the arguments in [5, 8], the estimates for II and III can be given as

$$II = o(N^{-\frac{n-1}{2}}), \qquad III = o(N^{-\frac{n-1}{2}}) \qquad (N \longrightarrow +\infty).$$

Moreover, noting the supports of $\zeta_N \Psi_N(y)$ and $\zeta_N \Phi_N(y)$, we put

$$D_N = \Big\{ |x'| \le \frac{1}{\sqrt{N}}, 0 \le x_n \le \frac{1}{2\sqrt{N}} \Big\},$$
$$D'_N = \Big\{ |x'| \le \frac{1}{\sqrt{N}}, \frac{1}{2\sqrt{N}} \le x_n \Big\}$$

and rewrite I as

$$\begin{aligned} I &= \int_{D_N} \gamma\big(4\nabla\Psi_N \cdot \nabla\overline{\Psi_N} - 2\nabla\Phi_N \cdot \nabla\overline{\Phi_N}\big)\, dx, \\ &\quad + \int_{D'_N} \gamma\big(4\nabla\Psi_N \cdot \nabla(\zeta_N\overline{\Psi_N}) - 2\nabla\Phi_N \cdot \nabla(\zeta_N\overline{\Phi_N})\big)\, dx \\ &= I_1 + I_2, \end{aligned}$$

where

$$I_1 = \int_{D_N} \gamma\big(4\nabla\Psi_N \cdot \nabla\overline{\Psi_N} - 2\nabla\Phi_N \cdot \nabla\overline{\Phi_N}\big)\, dx,$$
$$I_2 = \int_{D'_N} \gamma\big(4\nabla\Psi_N \cdot \nabla(\zeta_N\overline{\Psi_N}) - 2\nabla\Phi_N \cdot \nabla(\zeta_N\overline{\Phi_N})\big)\, dx.$$

From (11) and (12) we see that

$$I_2 = O(e^{-\sqrt{N}/2}) \quad (N \longrightarrow +\infty).$$

Therefore,

$$\lim_{N\to\infty} N^{\frac{n-1}{2}} \Big[4 < \Lambda_\gamma \psi_N, \overline{\psi_N} > -2 < \Lambda_\gamma \phi_N, \overline{\phi_N} > \Big] = \lim_{N\to\infty} N^{\frac{n-1}{2}} I_1.$$

Now we rewrite (11) and (12) as

$$\Psi_N(x) = \sum_{l=0}^{2} \Psi_N^l(x), \qquad \Phi_N(x) = \sum_{l=0}^{2} \Phi_N^l(x)$$

where

$$\Psi_N^l(x) = e^{\sqrt{-1}\frac{N}{2}x'\cdot t'}\, e^{-\frac{N}{2}x_n}\, V_l(\sqrt{N}x', Nx_n)\, N^{-\frac{l}{2}},$$
$$\Phi_N^l(x) = e^{\sqrt{-1}Nx'\cdot t'}\, e^{-Nx_n}\, U_l(\sqrt{N}x', Nx_n)\, N^{-\frac{l}{2}}, \quad l = 0, 1, 2. \tag{15}$$

The leading terms of Ψ_N and Φ_N, that is, Ψ_N^0 and Φ_N^0 are

$$\Psi_N^0(x) = e^{\sqrt{-1}\frac{N}{2}x'\cdot t'}\, e^{-\frac{N}{2}x_n}\eta(\sqrt{N}x'),$$
$$\Phi_N^0(x) = e^{\sqrt{-1}Nx'\cdot t'}\, e^{-Nx_n}\eta(\sqrt{N}x').$$

Note that

$$\nabla\Psi_N^0(x) = \left[\frac{N}{2}\begin{pmatrix}\sqrt{-1}\,t' \\ -1\end{pmatrix}\eta(\sqrt{N}x') + \sqrt{N}\begin{pmatrix}(\nabla_{y'}\eta)(\sqrt{N}x') \\ 0\end{pmatrix}\right] \times e^{\sqrt{-1}\frac{N}{2}x'\cdot t'}\, e^{-\frac{N}{2}x_n}$$

and

$$\nabla\Phi_N^0(x) = \left[N\begin{pmatrix}\sqrt{-1}\,t' \\ -1\end{pmatrix}\eta(\sqrt{N}x') + \sqrt{N}\begin{pmatrix}(\nabla_{y'}\eta)(\sqrt{N}x') \\ 0\end{pmatrix}\right] \times e^{\sqrt{-1}Nx'\cdot t'}\, e^{-Nx_n}.$$

Hence we see that in $N^{\frac{n-1}{2}}I_1$, the contribution only from the leading terms Ψ_N^0 and Φ_N^0 is

$$\begin{aligned}
&N^{\frac{n-1}{2}}\int_{D_N}\gamma\big(\,4\nabla\Psi_N^0\cdot\nabla\overline{\Psi_N^0} - 2\nabla\Phi_N^0\cdot\nabla\overline{\Phi_N^0}\big)\,dx \\
&= N^{\frac{n-1}{2}}N^2\int_{D_N}\gamma(x', x_n)\,\eta^2(\sqrt{N}x')\,2(e^{-Nx_n} - 2e^{-2Nx_n})\,dx \\
&\quad + N^{\frac{n-1}{2}}N\int_{D_N}\gamma(x', x_n)\,\big|(\nabla_{y'}\eta)(\sqrt{N}x')\big|^2\,2(2e^{-Nx_n} - e^{-2Nx_n})\,dx,
\end{aligned}$$

and after the change of variables (10),

$$\begin{aligned}
&= N\int_0^{\sqrt{N}/2}\int_{|y'|\le 1}\gamma\big(\frac{y'}{\sqrt{N}}, \frac{y_n}{N}\big)\,\eta^2(y')\,2(e^{-y_n} - 2e^{-2y_n})\,dy'dy_n \\
&\quad + \int_0^{\sqrt{N}/2}\int_{|y'|\le 1}\gamma\big(\frac{y'}{\sqrt{N}}, \frac{y_n}{N}\big)\big|(\nabla_{y'}\eta)(y')\big|^2\,2(2e^{-y_n} - e^{-2y_n})\,dy'dy_n.
\end{aligned} \tag{16}$$

On the other hand, from the condition on the regularity of γ we can expand it as

$$\gamma(\frac{y'}{\sqrt{N}}, \frac{y_n}{N}) = \gamma(0',0) + \frac{1}{\sqrt{N}} \sum_{|\alpha'|=1} \frac{\partial^{|\alpha'|}}{\partial {x'}^{\alpha'}} \gamma(0',0) {y'}^{\alpha'}$$
$$+ \frac{1}{N} \sum_{|\alpha'|=2} \frac{1}{\alpha'!} \frac{\partial^{|\alpha'|}}{\partial {x'}^{\alpha'}} \gamma(0',0) {y'}^{\alpha'}$$
$$+ \frac{1}{N} \frac{\partial}{\partial x_n} \gamma(0',0) y_n + o(\frac{1}{N}) \qquad (N \longrightarrow +\infty).$$

Substituiting this into (16), and using (3) and the equalities

$$\int_0^{\sqrt{N}/2} e^{-y_n} - 2e^{-2y_n} \; dy_n = O(e^{-\sqrt{N}/2}),$$
$$\int_0^{\sqrt{N}/2} y_n (e^{-y_n} - 2e^{-2y_n}) \; dy_n = \frac{1}{2} + O(\sqrt{N} e^{-\sqrt{N}/2}),$$
$$\int_0^{\sqrt{N}/2} 2e^{-y_n} - e^{-2y_n} \; dy_n = \frac{3}{2} + O(e^{-\sqrt{N}/2}) \quad (N \longrightarrow +\infty),$$

we obtain

$$N^{\frac{n-1}{2}} \int_{D_N} \gamma\big(4\nabla\Psi_N^0 \cdot \nabla\overline{\Psi_N^0} \; dx - 2\nabla\Phi_N^0 \cdot \nabla\overline{\Phi_N^0}\big) \; dx$$
$$= \frac{\partial}{\partial x_n} \gamma(0',0) + 3\,\gamma(0',0) \int_{R^{n-1}} |\nabla_{x'} \eta|^2 \, dx' + o(1) \quad (N \longrightarrow +\infty).$$

By (13) and (14), the second terms of Ψ_N and Φ_N, that is, Ψ_N^1 and Φ_N^1 are

$$\Psi_N^1(x) = e^{\sqrt{-1}\frac{N}{2}x'\cdot t'} \; e^{-\frac{N}{2}x_n} \sqrt{-1}(t' \cdot \nabla_{y'}\eta)(\sqrt{N}x') N x_n \, N^{\frac{-1}{2}},$$
$$\Phi_N^1(x) = e^{\sqrt{-1}Nx'\cdot t'} \; e^{-Nx_n} \sqrt{-1}(t' \cdot \nabla_{y'}\eta)(\sqrt{N}x') N x_n \, N^{\frac{-1}{2}}$$

and then

$$\nabla\Psi_N^1(x) = \Bigg[\frac{\sqrt{N}}{2} \begin{pmatrix} \sqrt{-1}\, t' \\ -1 \end{pmatrix} \sqrt{-1}(t' \cdot \nabla_{y'}\eta)(\sqrt{N}x') N x_n$$
$$+ \sqrt{N} \begin{pmatrix} 0' \\ 1 \end{pmatrix} \sqrt{-1}(t' \cdot \nabla_{y'}\eta)(\sqrt{N}x')$$
$$+ \sqrt{-1} \begin{pmatrix} \nabla_{y'}(t' \cdot \nabla_{y'}\eta)(\sqrt{N}x') \\ 0 \end{pmatrix} N x_n \Bigg]$$
$$\times \, e^{\sqrt{-1}\frac{N}{2}x'\cdot t'} \; e^{-\frac{N}{2}x_n},$$

$$\nabla\Phi_N^1(x) = \Big[\sqrt{N}\begin{pmatrix}\sqrt{-1}\,t'\\ -1\end{pmatrix}\sqrt{-1}(t'\cdot\nabla_{y'}\eta)(\sqrt{N}x')Nx_n + \sqrt{N}\begin{pmatrix}0'\\ 1\end{pmatrix}\sqrt{-1}(t'\cdot\nabla_{y'}\eta)(\sqrt{N}x') + \sqrt{-1}\begin{pmatrix}\nabla_{y'}(t'\cdot\nabla_{y'}\eta)(\sqrt{N}x')\\ 0\end{pmatrix}Nx_n\Big] \times e^{\sqrt{-1}Nx'\cdot t'}\,e^{-Nx_n}.$$

Hence we see that in $N^{\frac{n-1}{2}}I_1$,

$$N^{\frac{n-1}{2}}\int_{D_N}\gamma\Big(4(\nabla\Psi_N^0\cdot\nabla\overline{\Psi_N^1}+\nabla\Psi_N^1\cdot\nabla\overline{\Psi_N^0}) - 2(\nabla\Psi_N^0\cdot\nabla\overline{\Psi_N^1}+\nabla\Psi_N^1\cdot\nabla\overline{\Psi_N^0})\Big)\,dx$$

becomes after the change of variables (10),

$$\int_0^{\sqrt{N}/2}\int_{|y'|\le 1}\gamma\big(\frac{y'}{\sqrt{N}},\frac{y_n}{N}\big)\,\Big(t'\cdot\nabla_{y'}(t'\cdot\nabla_{y'}\eta)\eta(y')-(t'\cdot\nabla_{y'}\eta)^2(y')\Big) \times 4y_n(e^{-y_n}-e^{-2y_n})\,dy'dy_n$$

which tends to

$$3\gamma(0',0)\int_{R^{n-1}}t'\cdot\nabla_{y'}(t'\cdot\nabla_{y'}\eta)\eta(y')-(t'\cdot\nabla_{y'}\eta)^2(y')\,dy'$$

as $N\longrightarrow+\infty$.

In the same way, we see that in $N^{\frac{n-1}{2}}I_1$,

$$N^{\frac{n-1}{2}}\int_{D_N}\gamma\big(4\nabla\Psi_N^1\cdot\nabla\overline{\Psi_N^1}-2\nabla\Phi_N^1\cdot\nabla\overline{\Phi_N^1}\big)\,dx$$

tends to

$$3\gamma(0',0)\int_{R^{n-1}}(t'\cdot\nabla_{y'}\eta)^2(y')\,dy'$$

as $N\longrightarrow+\infty$. Moreover, it can be easily checked that in $N^{\frac{n-1}{2}}I_1$, the other contributions from (15) tend to zero as $N\longrightarrow+\infty$.

Finally, integrating $t'\cdot\nabla_{y'}(t'\cdot\nabla_{y'}\eta)\eta(y')$ by parts, we obtain the theorem.

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