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APPLICATIONS OF THE REPRODUCING KERNEL THEORY TO INVERSE PROBLEMS

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ABSTRACT. In this survey article, we shall introduce the applications of the theory of reproducing kernels to inverse problems. At the same time, we shall present some operator versions of our fundametal general theory for linear transforms in the framework of Hilbert spaces.

1. Reproducing kernels

We consider any positive matrix $K(p,q)$ on E; that is, for an abstract set E and for a complex–valued function $K(p,q)$ on $E \times E$, it satisfies that for any finite points $\{p_j\}$ of E and for any complex numbers $\{C_j\}$,

$$\sum_j \sum_{j'} C_j \overline{C_{j'}} K(p_{j'}, p_j) \geq 0.$$

Then, by the fundamental theorem by Moore–Aronszajn we have:

PROPOSITION 1.1. *For any positive matrix $K(p,q)$ on E, there exists a uniquely determined functional Hilbert space H_K comprising functions $\{f\}$ on E and admitting the reproducing kernel $K(p,q)$* (RKHS H_K) *satisfying and characterized by*

$$K(\cdot, q) \in H_K \ \text{ for any } \ q \in E \tag{1.1}$$

and, for any $q \in E$ and for any $f \in H_K$

$$f(q) = (f(\cdot), K(\cdot, q))_{H_K}. \tag{1.2}$$

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For some general properties for reproducing kernel Hilbert spaces and for various constructions of the RKHS H_K from a positive matrix $K(p,q)$, see the recent book [24] and its Chapter 2, Section 5, respectively.

2. Connection with linear transforms

We shall connect linear transforms in the framework of Hilbert spaces with reproducing kernels.

For an abstract set E and for any Hilbert (possibly finite–dimensional) space $\mathcal{H}$, we shall consider an $\mathcal{H}$–valued function $\boldsymbol{h}$ on E

$$\boldsymbol{h}: \quad E \longrightarrow \mathcal{H} \tag{2.1}$$

and the linear transform for $\mathcal{H}$

$$f(p) = (\boldsymbol{f}, \boldsymbol{h}(p))_{\mathcal{H}} \quad \text{for} \quad \boldsymbol{f} \in \mathcal{H} \tag{2.2}$$

into a linear space comprising functions $\{f(p)\}$ on E. For this linear transform (2.2), we form the positive matrix $K(p,q)$ on E defined by

$$K(p,q) = (\boldsymbol{h}(q), \boldsymbol{h}(p))_{\mathcal{H}} \quad \text{on} \quad E \times E. \tag{2.3}$$

Then, we have the following fundamental results:

(I) For the RKHS H_K admitting the reproducing kernel $K(p,q)$ defined by (2.3), the images $\{f(p)\}$ by (2.2) for $\mathcal{H}$ are characterized as the members of the RKHS H_K.

(II) In general, we have the inequality in (2.2)

$$\|f\|_{H_K} \le \|\boldsymbol{f}\|_{\mathcal{H}}, \tag{2.4}$$

however, for any $f \in H_K$ there exists a uniquely determined $\boldsymbol{f}^* \in \mathcal{H}$ satisfying

$$f(p) = (\boldsymbol{f}^*, \boldsymbol{h}(p))_{\mathcal{H}} \quad \text{on } E \tag{2.5}$$

and

$$\|f\|_{H_K} = \|\boldsymbol{f}^*\|_{\mathcal{H}}. \tag{2.6}$$

In (2.4), the isometry holds if and only if $\{\boldsymbol{h}(p); p \in E\}$ is complete in $\mathcal{H}$.

(III) We can obtain the inversion formula for (2.2) in the form

$$f \longrightarrow \boldsymbol{f}^*, \tag{2.7}$$

by using the RKHS H_K.

However, this inversion formula will depend on, case by case, the realizations of the RKHS H_K.

(IV) Conversely, if we have an isometrical mapping $\tilde{L}$ from a RKHS H_K admitting a reproducing kernel $K(p,q)$ on E onto a Hilbert space $\mathcal{H}$, then the mapping $\tilde{L}$ is linear and the isometrical inversion $\tilde{L}^{-1}$ is represented in the form (2.2) by

$$h(p) = \tilde{L}K(\cdot, p) \text{ on } E. \tag{2.8}$$

Further, then $\{h(p); p \in E\}$ is complete in $\mathcal{H}$.

When (2.2) is isometrical, sometimes we can use the isometrical mapping for a realization of the RKHS H_K, conversely—that is, if the inverse L^{-1} of the linear transform (2.2) is known, then we have $\|f\|_{H_K} = \|L^{-1}f\|_{\mathcal{H}}$.

We shall state some general applications of the results (I)~(IV) to several wide subjects and their basic references:

(1) Linear transforms ([14], [21]).
(2) Integral transforms among smooth functions ([28]).
(3) Nonharmonic integral transforms ([17]).
(4) Various norm inequalities ([17], [22]).
(5) Nonlinear transforms ([22], [25]).
(6) Linear integral equations ([29]).
(7) Linear differential equations with variable coefficients ([29]).
(8) Approximation theory ([7]).
(9) Representations of inverse functions ([23]).
(10) Various operators among Hilbert spaces ([26]).
(11) Sampling theorems ([24], Chapter 4, Section 2).
(12) Interpolation problems of Pick-Nevanlinna type ([17], [18]).
(13) Analytic extension formulas and their applications ([24]).

In this survey article, we shall refer to the applications to inverse problems. Furthermore, as our original results, we shall present some operator versions of the fundamental theory (I)~(IV), which may be expected to have wide applications, similarly.

In connection with inverse problems, (II) gives a general method constructing the general inversion formula (2.7) in the linear transform (2.2). As typical examples, we shall refer to the Weierstrass transform and the Laplace transform. Meanwhile, (IV) gives a general method determining the linear system (2.2) from an isometrical mapping $\tilde{L}$ from a RKHS

H_K onto a Hilbert space H. Sometimes, the system vector $\boldsymbol{h}(p)$ represents a Green's function. Then, the input and the output in (2.2) can be interpreted by the law induced from the Green's function.

3. Weierstrass transform

As a typical example, we shall consider the Weierstrass transform

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\boldsymbol{R}} F(\xi) \exp\left[-\frac{(x-\xi)^2}{4t}\right] d\xi \tag{3.1}$$

for functions $F \in L_2(\boldsymbol{R}, d\xi)$. Then, by using (I) and (II) we obtained in [15] simply and naturally the isometrical identity

$$\int_{\boldsymbol{R}} |F(\xi)|^2 d\xi = \frac{1}{\sqrt{2\pi t}} \int\int_{\boldsymbol{R}^2} |u(z,t)|^2 \exp\left[-\frac{y^2}{2t}\right] dxdy \tag{3.2}$$

for the analytic extension $u(z,t)$ of $u(x,t)$ to the entire complex $z = x+iy$ plane.

Of course, the image $u(x,t)$ of (3.1) is the solution of the heat equation

$$u_{xx}(x,t) = u_t(x,t) \quad \text{on } \boldsymbol{R} \times \{t > 0\} \tag{3.3}$$

satisfying the initial condition

$$\lim_{t \to +0} \|u(x,t) - F(x)\|_{L_2(\boldsymbol{R},dx)} = 0.$$

On the other hand, by using the properties of the solution $u(x,t)$ of (3.3), N. Hayashi derived the identity

$$\int_{\boldsymbol{R}} |F(\xi)|^2 d\xi = \sum_{j=0}^{\infty} \frac{(2t)^j}{j!} \int_{\boldsymbol{R}} |\partial_x^j u(x,t)|^2 dx. \tag{3.4}$$

The two identities (3.2) and (3.4) were a starting point for obtaining our various analytic extension formulas and their applications. We can easily obtain the inversion formulas for the Weierstrass transform from the isometrical identities (3.2) and (3.4).

As to the equality of the right hand sides of (3.2) and (3.4), we obtained directly

THEOREM 3.1. ([9]) *For any analytic function $f(z)$ on the strip $S_r = \{|\mathrm{Im} z| < r\}$ with a finite integral*

$$\int\int_{S_r} |f(z)|^2 dxdy < \infty,$$

we have the identity

$$(3.5) \qquad \int\int_{S_r} |f(z)|^2 dxdy = \sum_{j=0}^{\infty} \frac{(2r)^{2j+1}}{(2j+1)!} \int_{\mathbf{R}} |\partial_x^j f(x)|^2 dx.$$

Conversely, for a smooth function $f(x)$ with a convergence sum (3.5) on $\mathbf{R}$, there exists an analytic extension $f(z)$ onto S_r satisfying (3.5).

THEOREM 3.2. ([9]) *For any $\alpha > 0$ and for any entire function $f(z)$ with a finite integral*

$$\int\int_{\mathbf{R}^2} |f(z)|^2 \exp\left[-\frac{y^2}{\alpha}\right] dxdy < \infty,$$

we have the identity

$$(3.6) \quad \frac{1}{\sqrt{\alpha\pi}} \int\int_{\mathbf{R}^2} |f(z)|^2 \exp\left[-\frac{y^2}{\alpha}\right] dxdy = \sum_{j=0}^{\infty} \frac{\alpha^j}{j!} \int_{\mathbf{R}} |\partial_x^j f(x)|^2 dx.$$

Conversely, for a smooth function $f(x)$ with a convergence sum (3.6) on $\mathbf{R}$, there exists an analytic extension $f(z)$ onto $\mathbf{C}$ satisfying the identity (3.6).

Our typical results of another type were obtained from the integral transform

$$(3.7) \qquad v(x,t) = \frac{1}{t} \int_0^t F(\xi) \frac{x \exp\left\{\frac{-x^2}{4(t-\xi)}\right\}}{2\sqrt{\pi}\,(t-\xi)^{\frac{3}{2}}} \xi d\xi$$

in connection with the heat equation (3.3) for $x > 0$ satisfying the conditions, for $u(x,t) = tv(x,t)$

$$u(0,t) = tF(t) \quad \text{for} \quad t \geq 0$$

and

$$u(x,0) = 0 \quad \text{on} \quad x \geq 0.$$

Then, we obtained

THEOREM 3.3. ([1] and [19]) *Let $\Delta(\frac{\pi}{4})$ denote the sector $\left\{|\arg z| < \frac{\pi}{4}\right\}$. Then, for any analytic function $f(z)$ on $\Delta\left(\frac{\pi}{4}\right)$ with a finite integral*

$$\int\int_{\Delta(\frac{\pi}{4})} |f(z)|^2 dxdy < \infty,$$

we have the identity

$$\int\int_{\Delta(\frac{\pi}{4})} |f(z)|^2 dxdy = \sum_{j=0}^{\infty} \frac{2^j}{(2j+1)!} \int_0^{\infty} x^{2j+1} \left|\partial_x^j f(x)\right|^2 dx. \tag{3.8}$$

Conversely, for any smooth function $f(x)$ *on* $\{x > 0\}$ *with a convergence sum in* (3.8), *there exists an analytic extension* $f(z)$ *onto* $\Delta\left(\frac{\pi}{4}\right)$ *satisfying* (3.8).

Let $\Delta(\alpha)$ be the sector $\{|\arg z| < \alpha\}$. Then, by using the conformal mapping e^z, H. Aikawa examined the relation between Theorem 3.1 and Theorem 3.3. Then, he used the Mellin transform and some expansion of Gauss' hypergeometric series $F(\alpha, \beta; \gamma; z)$ and we obtained a general version of Theorem 3.3 and a version for the Szegö space:

THEOREM 3.4. ([2]) *Let* $0 < \alpha < \frac{\pi}{2}$. *Then, for any analytic function* $f(z)$ *on* $\Delta(\alpha)$ *with a finite integral*

$$\int\int_{\Delta(\alpha)} |f(z)|^2 dxdy < \infty,$$

we have the identity

$$\int\int_{\Delta(\alpha)} |f(z)|^2 dxdy = \sin(2\alpha) \sum_{j=0}^{\infty} \frac{(2\sin\alpha)^{2j}}{(2j+1)!} \int_0^{\infty} x^{2j+1} |\partial_x^j f(x)|^2 dx. \tag{3.9}$$

Conversely, for a smooth function $f(x)$ *with a convergence sum on* $x > 0$ *in* (3.9), *there exists an analytic extension* $f(z)$ *onto* $\Delta(\alpha)$ *satisfying the identity* (3.9).

THEOREM 3.5. ([2]) *Let* $0 < \alpha < \frac{\pi}{2}$. *Then, for any analytic function* $f(z)$ *on* $\Delta(\alpha)$ *satisfying*

$$\int_{|\theta|<\alpha} |f(re^{i\theta})|^2 dr < \infty,$$

we have the identity

$$\int_{\partial\Delta(\alpha)} |f(z)|^2 |dz| = 2\cos\alpha \sum_{j=0}^{\infty} \frac{(2\sin\alpha)^{2j}}{(2j)!} \int_0^{\infty} x^{2j} |\partial_x^j f(x)|^2 dx \tag{3.10}$$

where $f(z)$ *mean Fatou's nontangential boundary values of* f *on* $\partial\Delta(\alpha)$.

Conversely, for a smooth function $f(x)$ on $x > 0$ with a convergence sum in (3.10), *there exists an analytic extension $f(z)$ onto $\Delta(\alpha)$ satisfying the identity* (3.10).

4. Real inversion formulas for the Laplace transform

As another typical and important example, we shall consider real inversion formulas of the Laplace transform. The inversion formulas of the Laplace transform are, in general, given by complex forms. The observation in many cases however gives us real data only and so, it is important to establish the real inversion formula of the Laplace transform, because we have to extend the real data analytically onto a half complex plane. The analytic extension formula is, in general, very involved and makes the stability unclear. In particular, in the Reznitskaya transform combining the solutions of hyperbolic and parabolic partial differential equations, we need the real inversion formula, because the observation data of the solutions of hyperbolic partial differential equations are real-valued. See [27].

Since the image functions of the Laplace transform are, in general, analytic on a half–plane on the complex plane, in order to obtain the real inversion formula, we need half plane versions $\Delta\left(\frac{\pi}{2}\right)$ of Theorem 3.4 and Theorem 3.5, which are a crucial case $\alpha = \frac{\pi}{2}$ in those theorems. By using the famous Gauss summation formula and transformation properties in the Mellin transform we obtained, in a very general version containing the Bergman and the Szegö spaces:

THEOREM 4.1. ([20]) *For any $q > 0$, let $H_{K_q}(R^+)$ denote the Bergman-Selberg space admitting the reproducing kernel*

$$K_q(z, \bar{u}) = \frac{\Gamma(2q)}{(z + \bar{u})^{2q}}$$

on the right half plane $R^+ = \{z; \mathrm{Re}\, z > 0\}$. Then, we have the identity

$$\|f\|^2_{H_{K_q}(R^+)} = \left(\frac{1}{\Gamma(2q-1)\pi} \in t \int_{R^+} |f(z)|^2 (2x)^{2q-2} dx dy \ , \ q > \frac{1}{2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+2q+1)}$$

$$(4.1) \qquad \cdot \int_0^{\infty} |\partial_x^n \left(x f'(x)\right)|^2 x^{2n+2q-1} dx.$$

Conversely, any smooth function $f(x)$ on $\{x > 0\}$ with a convergence summation in (4.1) can be extended analytically onto R^+ and the analytic extension $f(z)$ satisfying $\lim_{x\to\infty} f(x) = 0$ belongs to $H_{K_q}(R^+)$ and the identity (4.1) is valid.

For the Laplace transform

$$f(z) = \int_0^\infty F(t)e^{-zt}dt, \tag{4.2}$$

we have, immediately, the isometrical identity, for any $q > 0$

$$\begin{aligned} \|f\|^2_{H_{K_q}(R^+)} &= \int_0^\infty |F(t)|^2 t^{1-2q} dt \\ (&:= \|F\|^2_{L^2_q}) \end{aligned} \tag{4.3}$$

from (I) and (II). By using (4.3) and (4.1), we obtain

THEOREM 4.2. ([5]) *For the Laplace transform (4.2), we have the inversion formula*

$$F(t) = s - \lim_{N\to\infty} \int_0^\infty f(x)e^{-xt} P_{N,q}(xt)dx \quad (t > 0) \tag{4.4}$$

where the limit is taken in the space L^2_q and the polynomials $P_{N,q}$ are given by

$$\begin{aligned} P_{N,q}(\xi) &= \sum_{0\le\nu\le n\le N} \frac{(-1)^{\nu+1}\Gamma(2n+2q)}{\nu!(n-\nu)!\Gamma(n+2q+1)\Gamma(n+\nu+2q)} \xi^{n+\nu+2q-1} \\ &\cdot\Big\{\frac{2(n+q)}{n+\nu+2q}\xi^2 - \Big(\frac{2(n+q)}{n+\nu+2q} + 3n + 2q\Big)\xi \\ &+ (n+\nu+2q)\Big\}. \end{aligned} \tag{4.5}$$

The truncation error is estimated by the inequality

$$\begin{aligned} &\left\|F(t) - \int_0^\infty f(x)e^{-xt}P_{N,q}(xt)dx\right\|^2_{L^2_q} \\ &\le \sum_{n=N+1}^\infty \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty \left|\partial_x^n \left[xf'(x)\right]\right|^2 x^{2n+2q-1} dx. \end{aligned} \tag{4.6}$$

In order to obtain an inversion formula which converges pointwisely in (4.4), we considered an inversion formula of the Laplace transform for the Sobolev space satisfying

$$\int_0^\infty \left(|F(t)|^2 + |F'(t)|^2\right) dt < \infty,$$

in [3]. In some subspaces of $H_{K_q}(R^+)$ and L_q^2, we established an error estimate for the inversion formula (4.4). Some characteristics of the strong singularity of the polynomials $P_{N,q}(\xi)$ and some effective algorithms for the real inversion formula (4.4) are examined by J. Kajiwara and M. Tsuji [11, 12]. Furthermore, they gave numerical experiments by using computers.

5. Representation of initial heat distributions by means of their heat distributions as functions of time

In the Weierstrass transform (3.1), we obtained the isometrical identity, for any fixed $x \in \boldsymbol{R}$,

$$\begin{aligned} &\int_{-\infty}^\infty |F(\xi)|^2 d\xi \\ = &\; 2\pi \sum_{j=0}^\infty \frac{1}{j!\Gamma\left(j+\frac{3}{2}\right)} \int_0^\infty \left|\partial_t^j \left[t\partial_t u(x,t)\right]\right|^2 t^{2j-\frac{1}{2}} dt \\ &+ 2\pi \sum_{j=0}^\infty \frac{1}{j!\Gamma\left(j+\frac{5}{2}\right)} \int_0^\infty \left|\partial_t^j \left[t\partial_t \partial_x u(x,t)\right]\right|^2 t^{2j+\frac{1}{2}} dt. \end{aligned} \tag{5.1}$$

From this identity, we can obtain the inversion formula

$$u(x,t) \longrightarrow F(\xi) \quad \text{for any fixed } x. \tag{5.2}$$

We, in general, in the multi–dimensional Weierstrass transform, established an exact and analytical representation formula of the initial heat distribution F by means of the observations

$$u(x_1, x', t) \;\text{ and }\; \frac{\partial u(x_1, x', t)}{\partial x_1} \tag{5.3}$$

for $x' = (x_2, x_3, \cdots, x_n) \in \boldsymbol{R}^{n-1}$ and $t > 0$, at any fixed point x_1, in [13].

We set

$$\sigma_F = \{\sup |x|,\; x \in \mathrm{supp} F\} \tag{5.4}$$

and supp$F$ denotes the smallest closed set outside which F vanishes almost everywhere. By using the isometrical identities (3.2), (3.4) and (5.1), we can solve the inverse source problem of determining the size σ_F of the initial heat distribution F from the heat flow $u(x,t)$ observed either at any fixed time t or at any fixed position x. See [30].

In this way, by using the theory of reproducing kernels, we derived many inversion formulas of integral transforms containing the Cauchy integral representation formula and the Poisson integral formula ([16, 17]).

6. Operator versions

We shall give operator versions of the fundamental theory (I)–(IV) which may be expected to have many concrete applications.

For an abstract set $\mathbf{\Lambda}$, we shall consider an operator-valued function L_λ on $\mathbf{\Lambda}$,

$$\mathbf{\Lambda} \longrightarrow L_\lambda \tag{6.1}$$

where L_λ are bounded linear operators from a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{H}$,

$$L_\lambda : \mathcal{H} \longrightarrow \mathcal{H}. \tag{6.2}$$

In particular, we are interested in the inversion formula

$$L_\lambda x \longrightarrow x, \quad x \in \mathcal{H}. \tag{6.3}$$

We shall fix an element $\mathbf{b} \in \mathcal{H}$ and consider the linear mapping

$$\begin{aligned} X_{\mathbf{b}}(\lambda) &= (L_\lambda x, \mathbf{b})_{\mathcal{H}} \\ &= (x, L_\lambda^* \mathbf{b})_H, \quad x \in H \end{aligned} \tag{6.4}$$

into a linear space comprising functions on $\mathbf{\Lambda}$. For this linear transform (6.4), we form the positive matrix $K_{\mathbf{b}}(\lambda,\mu)$ on $\mathbf{\Lambda}$ defined by

$$\begin{aligned} K_{\mathbf{b}}(\lambda,\mu) &= (L_\mu^* \mathbf{b}, L_\lambda^* \mathbf{b})_{\mathcal{H}} \\ &= (L_\lambda L_\mu^* \mathbf{b}, \mathbf{b})_{\mathcal{H}} \quad \text{on} \quad \mathbf{\Lambda} \times \mathbf{\Lambda}. \end{aligned} \tag{6.5}$$

Then, as in (I) $\sim$ (IV), we have the following fundamental results:

(I$'$) For the RKHS $H_{K_{\mathbf{b}}}$ admitting the reproducing kernel $K_{\mathbf{b}}(\lambda,\mu)$ defined by (6.5), the images $\{X_{\mathbf{b}}(\lambda)\}$ by (6.4) for $\mathcal{H}$ are characterized as the members of the RKHS $H_{K_{\mathbf{b}}}$.

(II$'$) In general, we have the inequality in (6.4)

$$\|X_{\mathbf{b}}\|_{H_{K_{\mathbf{b}}}} \leq \|x\|_{\mathcal{H}}, \tag{6.6}$$

however, for any $X_{\mathbf{b}} \in H_{K_{\mathbf{b}}}$ there exists a uniquely determined $x' \in \mathcal{H}$ satisfying

$$X_{\mathbf{b}}(\lambda) = (x', L_\lambda^* \mathbf{b})_{\mathcal{H}} \quad \text{on} \quad \mathbf{\Lambda} \tag{6.7}$$

and

$$\|X_{\mathbf{b}}\|_{H_{K_{\mathbf{b}}}} = \|x'\|_{\mathcal{H}}. \tag{6.8}$$

In (6.6), the isometry holds if and only if $\{L_\lambda^* \mathbf{b}; \lambda \in \mathbf{\Lambda}\}$ is complete in $\mathcal{H}$.

(III$'$) We can obtain the inversion formula for (6.4) and so, for the mapping (6.3) as in (III), in the form

$$L_\lambda x \longrightarrow (L_\lambda x, \mathbf{b})_{\mathcal{H}} = X_{\mathbf{b}}(\lambda) \longrightarrow x, \tag{6.9}$$

by using the RKHS $H_{K_{\mathbf{b}}}$.

(IV$'$) Conversely, if we have an isometrical mapping $\tilde{L}$ from a RKHS $H_{K_{\mathbf{b}}}$ admitting a reproducing kernel $K_{\mathbf{b}}(\lambda, \mu)$ on $\mathbf{\Lambda}$ onto a Hilbert space $\mathcal{H}$, then the mapping $\tilde{L}$ is linear and the isometrical inversion $\tilde{L}^{-1}$ is represented in the form (6.4) by using

$$L_\lambda^* \mathbf{b} = \tilde{L} K_{\mathbf{b}}(\cdot, \lambda) \ \text{ on } \ \mathbf{\Lambda}. \tag{6.10}$$

Further, then $\{L_\lambda^* \mathbf{b}; \lambda \in \mathbf{\Lambda}\}$ is complete in $\mathcal{H}$.

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References

[1] H. Aikawa, N. Hayashi, and S. Saitoh, *The Bergman space on a sector and the heat equation*, Complex Variables **15** (1990), 27–36.

[2] ———, *Isometrical identities for the Bergman and the Szegö spaces on a sector*, J. Math. Soc. Japan **43** (1991), 196–201.

[3] K. Amano, S. Saitoh, and A. Syarif, *A real inversion formula for the Laplace transform in a Sobolev space*, Zeitschrift für Analysis und ihre Anwendungen **18** (1999), 1031–1038.
[4] K. Amano, S. Saitoh, and M. Yamamoto, *Error estimates of the real inversion formulas of the Laplace transform*, Graduate School of Math. Sci. The University of Tokyo, Preprint Series (1998), 98–29, *Integral Transforms and Special Functions*, **10** (2000), 1–14.
[5] D.-W. Byun and S. Saitoh, *A real inversion formula for the Laplace transform*, Zeitschrift für Analysis und ihre Anwendungen **12** (1993), 597–603.
[6] ———, *Approximation by the solutions of the heat equation*, J. Approximation Theory **78** (1994), 226–238.
[7] ———, *Best approximation in reproducing kernel Hilbert spaces*, Proc. of the 2th International Colloquium on Numerical Analysis, VSP-Holland, (1994), 55–61.
[8] ———, *Analytic extensions of functions on the real line to entire functions*, Complex Variables **26** (1995), 277–281.
[9] N. Hayashi and S. Saitoh, *Analyticity and smoothing effect for the Schrödinger equation*, Ann. Inst. Henri Poincaré **52** (1990), 163–173.
[10] ———, *Analyticity and global existence of small solutions to some nonlinear Schrödinger equation*, Commun. Math. Phys. **139** (1990), 27–41.
[11] J. Kajiwara and M. Tsuji, *Program for the numerical analysis of inverse formula for the Laplace transform*, Proceedings of the Second Korean-Japanese Colloquium on Finite or Infinite Dimensional Complex Analysis, (1994), 93–107.
[12] ———, *Inverse formula for Laplace transform*, Proceedings of the 5th International Colloquium on Differential Equations, VHP-Holland, (1995), 163–172.
[13] G. Nakamura, S. Saitoh, and A. Syarif, *Representations of initial heat distributions by means of their heat distributions as functions of time*, Inverse Problems **15** (1999), 1255–1261.
[14] S. Saitoh, *Hilbert spaces induced by Hilbert space valued functions*, Proc. Amer. Math. Soc. **89** (1983), 74–78.
[15] ———, *The Weierstrass transform and an isometry in the heat equation*, Applicable Analysis **16** (1983), 1–6.
[16] ———, *Some fundamental interpolation problems for analytic and harmonic functions of class L_2*, Applicable Analysis **17** (1984), 87–106.
[17] ———, *Theory of Reproducing Kernels and its Applications*, Pitman Research Notes in Mathematics Series, Longman Scientific & Technical, UK **189** (1988),
[18] ———, *Interpolation problems of Pick-Nevanlinna type*, Pitman Research Notes in Mathematics Series **212** (1989), 253–262.
[19] ———, *Isometrical identities and inverse formulas in the one-dimensional heat equation*, Applicable Analysis **40** (1991), 139–149.
[20] ———, *Representations of the norms in Bergman-Selberg spaces on strips and half planes*, Complex Variables **19** (1992), 231–241.
[21] ———, *One approach to some general integral transforms and its applications*, Integral Transforms and Special Functions **3** (1995), 49–84.
[22] ———, *Natural norm inequalities in nonlinear transforms*, General Inequalities, Birkhäuser Verlag, Basel, Boston **7** (1997), 39–52.
[23] ———, *Representations of inverse functions*, Proc. Amer. Math. Soc. **125** (1997), 3633–3639.

[24] ———, *Integral Transforms, Reproducing Kernels and their Applications*, Pitman Research Notes in Mathematics Series, Addison Wesley Longman, UK **369** (1997).

[25] ———, *Nonlinear transforms and analyticity of functions*, Nonlinear Mathematical Analysis and Applications, Hadronic Press, Palm Harbor, (1998), 223–234.

[26] ———, *Various operators in Hilbert space induced by transforms*, International J. of Applied Math. **1** (1999), 111–126.

[27] S. Saitoh and M. Yamamoto. *Stability of Lipschitz type in determination of initial heat distribution*, J. of Inequa. & Appl. 1 (1997), 73–83.

[28] ———, *Integral transforms involving smooth functions*, Reproducing Kernels and their Applications, Kluwer Academic Publishers, 1999.

[29] S. Saitoh, *Linear integro-differential equations and the theory of reproducing kernels*, Volterra Equations and Applications, C. Corduneanu and I.W. Sandberg (eds), Gordon and Breach Science Publishers (2000), Amsterdam.

[30] V. K. Tuan, S. Saitoh, and M. Saigo, *Size of support of initial heat distribution in the 1D heat equation*, Applicable Analysis **74** (2000), 439–446.

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