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A PARTIAL ORDERING OF CONDITIONALLY POSITIVE QUADRANT DEPENDENCE

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ABSTRACT. A partial ordering is developed here among conditionally positive quadrant dependent($CPQD$) bivariate random vectors. This permits us to measure the degree of $CPQD$-ness and to compare pairs of $CPQD$ random vectors. Some properties and closure under certain statistical operations are derived.

1. Introduction

Lehmann [10] introduced the concepts of positive(negative) dependence together with some other dependent concepts. Since then, much works has been written on the subject and its extensions and numerous multivariate inequalities have been obtained. In other words, a great many papers have been devoted to various generalizations of Lehmann's concepts to finite-dimensional distributions. For references of available results, see Karlin and Rinott [8], Ebrahimi and Ghosh [6], Shaked [12], Sampson [11] and Baek [3]. Recently, Brady and Singpurwalla [5] introduced some new conditionally independent and positive(negative) quadrant dependence concepts ($CPQD(CNQD)$) of random variables. These concepts are a qualitative form of dependence(i.e., indicating simply whether the pair of random variables are mutually conditionally positive dependent or not) which has led to many applications in applied probability, reliability, and statistical inference such as analysis of variance, multivariate tests of hypothesis, sequential testing. As indicated above, since $CPQD$ is a qualitative form of dependence, it would seem difficult or impossible to compare different pairs of random variables as to their degree of $CPQD$-ness. Quite in the same spirit, we study in

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this paper the degree of $CPQD$-ness. However, for many purposes in addition to knowledge of the nature of dependence it is also important to compare pairs of $CPQD$ random vectors as to their degree of $CPQD$-ness(the exact definition is given in Section 3). Ahmed et. al [2] have studied extensively the partial ordering of positive quadrant dependence which permits us to compare pairs of positive quadrant dependent bivariate random vectors with specified marginals as to their PQD-ness. Kimelderf. G. and Sampson. A. R [9] presented a systematic basis for studying orderings of bivariate distributions according to their degree of positive dependence and introduced a general concept of a positive dependence ordering.

In this paper a partial ordering of conditionally positive quadrant dependence is developed to compare pairs of conditionally positive quadrant dependent bivariate random vectors. We present definitions and notations used throughout this paper in Section 2. The definitions and some basic properties of $CPQD$ ordering are presented in Section 3. We have also considered a family of bivariate distributions with specified marginals, the numbers of the family depending on a certain parameter, say λ. As $\lambda \uparrow$, the corresponding distribution, say H_λ, becomes increasingly $CPQD$. Certain closure properties of $CPQD$ ordering are derived in Section 4. It is shown that the ordering is preserved under convolution, mixture of a certain type, limit in distribution, and transformation of the random variables by increasing functions.

2. Preliminaries

An important principle of probability theory is that the notions of dependence and independence are conditional, the conditioning being done on some observable or unobservable quantity, say Θ. It is common to think of Θ as a parameter and this is the point of view that we adopt. Brady and Singpurwalla [5] introduced some concepts of conditional dependence between random variables. Let $\underline{X}$ and $\underline{Y}$ be two vector valued random variables, of dimension p and q, respectively.

In this section we present definitions, notations, and properties used throughout the paper. We start by stating the definitions of conditionally independence and positive(negative) dependence as per Brady and Singpurwalla [5].

DEFINITION 2.1 [5]. The random vector $\underline{X} = (X_1, \cdots, X_p)$ is $\theta \in I_1$ conditionally independent of $\underline{Y} = (Y_1, \cdots, Y_p)$ and $\theta \in I_2(I_3)$ conditionally positive(negative) quadrant dependent ($CPQD(CNQD)$) on $\underline{Y}$, denoted by $\{\underline{X} \amalg \underline{Y} | \theta \in I_1, > \theta \in I_2, < \theta \in I_3\}$ if

(a) $P(\underline{X} \in A | \underline{Y} \in B, \theta \in I_1) = P(\underline{X} \in A | \theta \in I_1)$,
(b) $P(\underline{X} \in A | \underline{Y} \in B, \theta \in I_2) \geq P(\underline{X} \in A | \theta \in I_2)(CPQD)$, and
(c) $P(\underline{X} \in A | \underline{Y} \in B, \theta \in I_3) \leq P(\underline{X} \in A | \theta \in I_3)(CNQD)$, $\forall A, B, \theta$,

where A, B are open upper sets (U is an upper set if, $\underline{a} \in U$, and $\underline{a} < \underline{b}$ implies $\underline{b} \in U$ (Shaked [12]).

Assume that $p = q = 1$. Then Definition 2.1 is equivalent to

DEFINITION 2.2 [4]. The pair (X, Y) or H is $\theta \in I_1$ conditionally independent and $\theta \in I_2(I_3)$ conditionally positive(negative) quadrant dependent ($CPQD(CNQD)$), denoted by $\{\underline{X} \amalg \underline{Y} | \theta \in I_1, > \theta \in I_2, < \theta \in I_3\}$ if

(a) $P(X \leq x, Y \leq y | \theta \in I_1) = P(X \leq x | \theta \in I_1) P(Y \leq y | \theta \in I_1)$,
(b) $P(X \leq x, Y \leq y | \theta \in I_2) \geq P(X \leq x | \theta \in I_2) P(Y \leq y | \theta \in I_2)$ $(CPQD)$, and
(c) $P(X \leq x, Y \leq y | \theta \in I_3) \leq P(X \leq x | \theta \in I_3) P(Y \leq y | \theta \in I_3)$ $(CNQD)$.

We close this section by stating the following lemma as per Brady and Singpurwalla[5].

LEMMA 2.3. *If conditions (a), (b) and (c) of Definition 2.2 hold and if the conditional expectations $E(XY|\theta)$, $E(X|\theta)$ and $E(Y|\theta)$ exist, then Definition 2.2 implies that*

(a) $E(XY|\theta \in I_1) = E(X|\theta \in I_1) E(Y|\theta \in I_1)$,
(b) $E(XY|\theta \in I_2) \geq E(X|\theta \in I_2) E(Y|\theta \in I_2)$, *and*
(c) $E(XY|\theta \in I_3) = E(X|\theta \in I_3) E(Y|\theta \in I_3)$.

A strengthening of Lemma 2.3 is

LEMMA 2.4. *Let f, g be increasing functions of X and Y, respectively. Then Definition 2.2. implies that*

(a) $Cov(f(X), g(Y)|\theta \in I_1) = 0$
(b) $Cov(f(X), g(Y)|\theta \in I_2) \geq 0$, *and*
(c) $Cov(f(X), g(Y)|\theta \in I_3) \leq 0$.

PROOF. This follows by an extension of a proof by Lehmann[10]. □

3. Ordered $CPQD$ random variables

Let $\beta = \beta(F(x|\theta), G(y|\theta))$ be the class of bivariate distribution functions H on R^2 having F and G as marginal distribution functions given θ. We consider, β^+, a subclass of β, defined by

$$\beta^+ = \Big\{H(x,y)|\ \theta \in I_2 : H \text{ is } CPQD, \quad H(x,\infty|\theta \in I_2) = F(x|\theta \in I_2), \quad H(\infty, y|\theta \in I_2) = G(y|\theta \in I_2)\Big\}.$$

DEFINITION 3.1. Let H_1 and H_2 belong to β^+. The random vector (X_1, X_2) or its distribution H_1 is more $\theta \in I_2$ conditionally positive quadrant dependent than the random vector (Y_1, Y_2) or its distribution H_2 if

$$(3.1)\quad P(X_1 \le x, X_2 \le y|\theta \in I_2) \ge P(Y_1 \le x, Y_2 \le y|\theta \in I_2)\ \forall x, y \in R^2$$

We write $H_1 > (CPQD)H_2$ or $(X_1, X_2) > (CPQD)(Y_1, Y_2)$.

PROPERTY 1. *Let H_1, H_2, and H_3 belong to β^+. Then $H_1 > (CPQD)H_2$ and $H_2 > (CPQD)H_3$ imply $H_1 > (CPQD)H_3$.*

PROPERTY 2. *Let (X, Y) and (U, V) have distributions H_1 and H_2, respectively, where H_1 and H_2 belong to β^+. Assume that $(X, Y) > (CPQD)(U, V)$. Then $(Y, X) > (CPQD)(V, U)$.*

PROOF. Note that both $(Y, X|\theta \in I_2)$ and $(V, U|\theta \in I_2)$ have marginals $G(y|\theta \in I_2)$ and $F(x|\theta \in I_2)$ when $\theta \in I_2$. Then

$$\begin{aligned} P(Y \le y, X \le x|\theta \in I_2) &= P(X \le x, Y \le y|\theta \in I_2) \\ &\ge P(U \le x, V \le y|\theta \in I_2) \\ &= P(V \le y, U \le x|\theta \in I_2). \end{aligned}$$

We now turn our attention to a simple but important property of class β^+ □

PROPERTY 3. *The class β^+ is convex.*

PROOF. Let H_1, H_2 belong to β^+ and for $0 < \alpha < 1$,

$$(3.2)\qquad H = \alpha H_1 + (1 - \alpha)H_2;$$

i.e., a convex combination of H_1 and H_2. Since each of H_1 and $H_2 \in \beta^+$, (3.2) may be written as

$$
\begin{aligned}
&H(x, y|\theta \in I_2) \\
&= \alpha H_1(x, y|\theta \in I_2) + (1-\alpha) H_2(x, y|\theta \in I_2) \\
&\geq \alpha F(x|\theta \in I_2) G(y|\theta \in I_2) + (1-\alpha) F(x|\theta \in I_2) G(y|\theta \in I_2) \\
&= F(x|\theta \in I_2) G(y|\theta \in I_2),
\end{aligned} \tag{3.3}
$$

so that H is $CPQD$.
Moreover,

$$
\begin{aligned}
\lim_{y \to \infty} H(x, y|\theta \in I_2) &= \alpha F(x|\theta \in I_2) + (1-\alpha) F(x|\theta \in I_2) \\
&= F(x|\theta \in I_2),
\end{aligned} \tag{3.4}
$$

and

$$
\begin{aligned}
\lim_{x \to \infty} H(x, y|\theta \in I_2) &= \alpha G(y|\theta \in I_2) + (1-\alpha) G(y|\theta \in I_2) \\
&= G(y|\theta \in I_2).
\end{aligned} \tag{3.5}
$$

It follows from (3.3), (3.4), and (3.5) that $H \in \beta^+$. Thus β^+ is convex. □

PROPERTY 4. *Let H_1, H_2 belong to β^+. Assume that $H_1 > (CPQD)$ H_2. Then for $0 \leq \alpha \leq 1$,*

$$
H_1 > (CPQD) \alpha H_1 + (1-\alpha) H_2 > (CPQD) H_2. \tag{3.6}
$$

PROOF. For $\alpha = 0, 1$, it is clear that (3.6) holds. For $0 < \alpha < 1$,

$$
\begin{aligned}
H_1(x, y|\theta \in I_2) &= \alpha H_1(x, y|\theta \in I_2) + (1-\alpha) H_1(x, y|\theta \in I_2) \\
&\geq \alpha H_1(x, y|\theta \in I_2) + (1-\alpha) H_2(x, y|\theta \in I_2) \\
&\geq \alpha H_2(x, y|\theta \in I_2) + (1-\alpha) H_2(x, y|\theta \in I_2) \\
&= H_2(x, y|\theta \in I_2).
\end{aligned}
$$

Thus $H_1 > (CPQD) \alpha H_1 + (1-\alpha) H_2 > (CPQD) H_2$, for $0 < \alpha < 1$. □

DEFINITION 3.3. A family of distributions $H = \{H_\lambda(x, y|\theta \in I_2) : \lambda \in \Lambda \subset R\}$ is increasingly $CPQD$ in λ if

$$
\lambda' > \lambda \rightarrow H_{\lambda'} > (CPQD) H_\lambda.
$$

We write H is $\uparrow CPQD$ in λ.

Next we present an example of a family which is increasingly $CPQD$ in the indexing parameter.

EXAMPLE 3.4. A bivariate family of $H_\lambda(x, y|\ \theta \in I_2)$, $0 < \lambda < 1 \uparrow CPQD$ in λ, where $H_\lambda\ (x, y|\ \theta \in I_2) = \lambda\ H(x, y|\ \theta \in I_2) + (1-\lambda)F(x|\theta \in I_2)G(y|\theta \in I_2)$ and $H \in \beta^+$. It is clear that $H_\lambda \subset \beta^+$ by Property 4. For $0 < \lambda_1 < \lambda_2 < 1$,

$$\begin{aligned}
&\lambda_2 H(x, y|\theta \in I_2) + (1-\lambda_2)F(x|\theta \in I_2)G(y|\theta \in I_2) \\
&\quad - F(x|\theta \in I_2)G(y|\theta \in I_2) \\
&= \lambda_2[H(x, y|\theta \in I_2) - F(x|\theta \in I_2)G(y|\theta \in I_2)] \\
&\geq \lambda_1[H(x, y|\theta \in I_2) + (1-\lambda_1)F(x|\theta \in I_2)G(y|\theta \in I_2) \\
&\quad - F(x|\theta \in I_2)G(y|\theta \in I_2)]
\end{aligned}$$

which yields

$$\begin{aligned}
&\lambda_2[H(x, y|\theta \in I_2) + (1-\lambda_2)F(x|\theta \in I_2)G(y|\theta \in I_2)] \\
\geq &\lambda_1[H(x, y|\theta \in I_2) + (1-\lambda_1)F(x|\theta \in I_2)G(y|\theta \in I_2)].
\end{aligned}$$

Thus $H_\lambda(x, y|\theta \in I_2)$ is $\uparrow CPQD$ in λ.

4. Close properties of $(\beta^+, > (CPQD))$

In this section we establish the preservation of the $CPQD$ ordering under convolutions, mixtures, limit in distribution and transformations of the random variables by increasing functions. Below, we show that the conditional ordering is preserved under convolution. We need the following lemma which is of independent interest given θ.

LEMMA 4.1. *Let*

(a) $\underline{X} = (X_1, X_2)$ *and* $\underline{Y} = (Y_1, Y_2)$ *have distributions* H_1 *and* H_2 *respectively, where* H_1 *and* H_2 *belong to* β^+ *such that* $H_1 > (CPQD)H_2$ *and let*

(b) $\underline{Z} = (Z_1, Z_2)$ *with an arbitrary* $CPQD$ *distribution function* H *conditionally independent of both* $\underline{X}$ *and* $\underline{Y}$ *given* θ. *Then* $\underline{X} + \underline{Z} > (CPQD)\underline{Y} + \underline{Z}$.

PROOF. First we will show that $\underline{X}+\underline{Z}$ and $\underline{Y}+\underline{Z}$ are $CPQD$.

$$\begin{aligned}&Cov(f(X_1+Z_1), g(X_2+Z_2)|\theta \in I_2)\\ &= Cov(E(f(X_1+Z_1)|\theta \in I_2, \underline{Z}),\ E(g(X_2+Z_2)|\theta \in I_2, \underline{Z})\\ &\quad + E(Cov(f(X_1+Z_1), g(X_2+Z_2)|\theta \in I_2, \underline{Z})) \geq 0.\end{aligned}$$

Note that the first and second terms are greater than of equal to zero for any increasing functions f and g. So $\underline{X}+\underline{Z}$ is $CPQD$, similarly we can show that $\underline{Y}+\underline{Z}$ is also $CPQD$.
Next we need to show that for each $(a_1, a_2) \in R^2$,

$$\begin{aligned}&P(X_1+Z_1 \leq a_1, X_2+Z_2 \leq a_2|\theta \in I_2)\\ &\geq P(Y_1+Z_1 \leq a_1, Y_2+Z_2 \leq a_2|\theta \in I_2).\end{aligned} \tag{4.1}$$

Note that the left side of (4.1)

$$\begin{aligned}&= \iint P(X_1 \leq a_1 - z_1, X_2 \leq a_2 - z_2|\theta \in I_2) dH_{Z_1,Z_2|\theta\in I_2}(Z_1, Z_2|\theta \in I_2)\\ &\geq \iint P(Y_1 \leq a_1 - z_1, Y_2 \leq a_2 - z_2|\theta \in I_2) dH_{Z_1,Z_2|\theta\in I_2}(Z_1, Z_2|\theta \in I_2)\\ &= P(Y_1+Z_1 \leq a_1, Y_2+Z_2 \leq a_2|\theta \in I_2).\end{aligned}$$

The above inequality follows from the assumption that $\underline{X} > (CPQD)$ $\underline{Y}$. □

THEOREM 4.2. *Suppose (X_i, Y_i) and (U_i, V_i) are such that $(X_i, Y_i) > (CPQD)(U_i, V_i)$ for $i = 1, 2$. Further, let (X_2, Y_2) be conditionally independent of both (X_1, Y_1) and (U_1, V_1) given θ, and (U_1, V_1) be conditionally independent of (U_2, V_2) given θ. Then $(X_1+X_2, Y_1+Y_2) > (CPQD)(U_1+U_2, V_1+V_2)$.*

PROOF. By assumption $(X_1, Y_1) > (CPQD)(U_1, V_1)$. Specifying $\underline{Z}$ to be (X_2, Y_2), we apply Lemma 4.1 to obtain

$$(X_1+X_2, Y_1+Y_2) > (CPQD)(U_1+X_2, V_1+Y_2). \tag{4.2}$$

Next, we use the assumption $(X_2, Y_2) > (CPQD)(U_2, V_2)$, specify $\underline{Z}$ to be (U_1, V_1), and again use Lemma 4.1 yielding

$$(U_1+X_2, V_1+Y_2) > (CPQD)(U_1+U_2, V_1+V_2). \tag{4.3}$$

By combining (4.2) and (4.3), $(X_1+X_2, Y_1+Y_2) > (CPQD)(U_1+U_2, V_1+V_2)$.

From Definition 2.2, Lemma 2.4 and Definition 3.1 it follows that if and only if

$$Cov_{H_1}(f(X_1), g(X_2)) > (CPQD)Cov_{H_2}(f(Y_1), g(Y_2)) \tag{4.4}$$

for all increasing functions f and g, where H_1 and H_2 are distributions of (X_1, X_2) and (Y_1, Y_2), respectively and H_1 and H_2 belong to β^+. □

THEOREM 4.3. *Let $\underline{X} = (X_1, X_2)$ and $\underline{Y} = (Y_1, Y_2)$ have distributions H_1 and H_2, where H_1 and H_2 belong to β^+ such that $(X_1, X_2) > (CPQD)(Y_1, Y_2)$. Then $(f(X_1), X_2) > (CPQD)(f(Y_1), Y_2)$ for all increasing functions f.*

PROOF. Let h and g be increasing functions. Since hf is an increasing function for all increasing function f, $Cov_{H_1}(h(f(X_1)), g(X_2)) > (CPQD)Cov_{H_2}(h(f(Y_1)), g(Y_2))$ according to (4.4). Hence $(f(X_1), X_2) > (CPQD)(f(Y_1), Y_2)$. □

COROLLARY 4.4. *Let $\underline{X} = (X_1, X_2)$ and $\underline{Y} = (Y_1, Y_2)$ have distributions H_1 and H_2 respectively, where H_1 and H_2 belong to β^+, such that $(X_1, X_2) > (CPQD)(Y_1, Y_2)$. Then $(f(X_1), g(X_2)) > (CPQD)$ $(f(Y_1), g(Y_2))$ for all increasing functions f and g.*

Our next result deals with the preservation of the $CPQD$ ordering under mixture. In order to motivate our definition of a subclass of β^+ in which the $CPQD$ is preserved under mixture we need a definition and a result.

DEFINITION 4.5. A random variable Y is θ conditionally stochastically increasing (CSI) in the random variable X if $E(f(Y)|X = x, \theta)$ is increasing in for all real valued increasing function f given θ.

THEOREM 4.6. *Let (a) (X_1, X_2) given λ, a random variable be $CPQD$, (b) X_i be θ conditionally stochastically increasing in λ for $i = 1, 2$. Then (X_1, X_2) is $CPQD$.*

PROOF. Let f, g be increasing functions. In view of Lemma 2.4 (b), it is enough to show that

$$Cov(f(X_1), g(X_2)|\theta \in I_2) \geq 0.$$

Note that

$$\begin{aligned} &Cov(f(X_1), g(X_2)|\theta \in I_2, \lambda) \\ &= Cov_\lambda(E(f(X_1)|\theta \in I_2, \lambda), E(g(X_2)|\theta \in I_2, \lambda)) \\ &\quad + E_\lambda(Cov(f(X_1), g(X_2)|\theta \in I_2, \lambda)). \end{aligned}$$

The first term on the right is nonnegative when $\theta \in I_2$ by (b) for increasing f and g. For such f and g the second term is nonnegative when

$\theta \in I_2$ using assumption (a). Since X_1 and X_2 are $CPQD$ if and only if $Cov(f(X_1), g(X_2)|\theta \in I_2) \geq 0$ for all increasing f and g, by Lemma 2.4 (b), (X_1, X_2) is $CPQD$. □

We may now define the class β_λ^+ by

$$\beta_\lambda^+ = \Big\{H_\lambda : H(x, \infty|\theta \in I_2, \lambda) = F(x|\theta \in I_2, \lambda),$$
$$H(\infty, y|\theta \in I_2, \lambda) = G(y|\theta \in I_2, \lambda),$$
$$H_\lambda|\lambda \text{ is } CPQD, \text{ both } F \text{ and } G \text{ are } CSI \text{ in } \lambda\Big\}.$$

Now consider $(\beta_\lambda^+, > (CPQD))$. The following theorem shows that if two elements of β^+ are ordered according to $> (CPQD)$, then after mixing on λ when $\theta \in I_2$, the resulting elements in β_λ^+ preserve the same order.

THEOREM 4.7. *Let $(X_1, X_2|\lambda)$ and $(Y_1, Y_2|\lambda)$ belong to β_λ^+. Assume $(X_1, X_2|\lambda) > (CPQD)(Y_1, Y_2|\lambda)$. Then unconditionally $(X_1, X_2), (Y_1, Y_2)$ belong to β^+ and $(X_1, X_2|\lambda) > (CPQD)(Y_1, Y_2|\lambda)$.*

PROOF. From Theorem 4.6, (X_1, X_2) and (Y_1, Y_2) are $CPQD$. Now,

$$\begin{aligned} E(f(X_1), g(X_2)|\theta \in I_2) &= E_\lambda(E(f(X_1)g(X_2)|\theta \in I_2, \lambda)) \\ &\geq E_\lambda(E(f(Y_1)g(Y_2)|\theta \in I_2, \lambda)) \\ &= E(f(Y_1)g(Y_2)|\theta \in I_2). \end{aligned}$$ □

In the following Theorem 4.8, we show that the conditional ordering is preserved under limits in distributions.

THEOREM 4.8. *Let*

(a) *$\underline{X}_n = (X_{1n}, X_{2n})$ and $\underline{Y}_n = (Y_{1n}, Y_{2n})$ have distributions H_n and H_n' such that $H_n > (CPQD)H_n'$ for every n,*
(b) *(X_1, X_2) and (Y_1, Y_2) have distributions H_1 and H_1' and,*
(c) *H_n, H_n', converge weakly to H_1, H_1', respectively.*

Then $H_1 > (CPQD)H_1'$.

PROOF.

$$\begin{aligned} P(X_1 \leq x_1, X_2 \leq x_2|\theta \in I_2) &= \lim_{n\to\infty} P(X_{1n} \leq x_1, X_{2n} \leq x_2|\theta \in I_2) \\ &\geq \lim_{n\to\infty} P(Y_{1n} \leq x_1, Y_{2n} \leq x_2|\theta \in I_2) \\ &= P(Y_1 \leq x_1, Y_2 \leq x_2|\theta \in I_2). \end{aligned}$$

Thus $H_1 > (CPQD)H_1'$. □

Finally, we show that $CPQD$ ordering is invariant under transformations of increasing functions. Before stating the theorem, we introduce the following definition.

DEFINITION 4.9. $f, g : R^n \to R$ are concordant for the i^{th} coordinate if, with all other coordinates held fixed, f, g are either both increasing or both decreasing $i = 1, 2, \cdots, n$.

THEOREM 4.10. *Let* $\{(X_i, Y_i)^{H_j},\ i = 1, 2, \cdots, n\}$ *be n-independent pairs from a bivariate distribution* $H_j,\ i = 1, 2$. *Suppose* H_1 *and* H_2 *belong to* β^+ *such that* $H_1 > (CPQD)H_2$. *Then for every pair* (f, g) *of concordant functions,* $Cov_{H_1}(f(X_1, \cdots, X_n), g(Y_1, \cdots, Y_n)) > (CPQD)Cov_{H_2}(f(X_1, \cdots, X_n), g(Y_1, \cdots, Y_n))$.

PROOF. First observe that in view of Definition 2.2 and Lemma 2.4, it is sufficient to consider the case where all pairs (X_i, Y_i) are $CPQD$. The result follows if we prove that for any functions h_1 and h_2 having the properties of f and g respectively,

$$Cov(h_1(X_1, \cdots, X_n), h_2(Y_1, \cdots, Y_n)|\theta \in I_2) \geq 0. \tag{4.5}$$

This is so since for any non-negative concordant functions k_1 and k_2 the functions $k_1 f$ and $k_2 g$ have the same properties as do f and g given θ. To show that (4.5) is valid, we follow an iteration argument. We have

$$\begin{aligned} &Cov(h_1(X_1, \cdots, X_n), h_2(Y_1, \cdots, Y_n)|\theta \in I_2) \\ &= Cov(h_1^*(X_2, \cdots, X_n)|\theta \in I_2, h_2^*(Y_2, \cdots, Y_n)|\theta \in I_2) \\ &\quad + E(Cov(h_1(X_1, \cdots, X_n), h_2(Y_1, \cdots, Y_n)| \\ &\qquad \theta \in I_2, X_2, \cdots, X_n, Y_2, \cdots, Y_n)) \end{aligned}$$

where

$$(h_1^*(X_2, \cdots, X_n)|\theta \in I_2) = E(h_1(X_1, \cdots, X_n)|\theta \in I_2, X_2, \cdots, X_n)$$
$$(h_2^*(Y_2, \cdots, Y_n)|\theta \in I_2) = E(h_2(Y_1, \cdots, Y_n)|\theta \in I_2, Y_2, \cdots, Y_n)$$

Observe that the second term of the functions side of the above equation is non-negative when $\theta \in I_2$ while the functions h_1^* and h_2^* in the first term have same properties in $x_2, \cdots, x_n$ and $y_2, \cdots, y_n$ as do the functions h_1 and h_2 given $\theta \in I_2$. The result now follows by proceeding with the iteration argument used above. Thus $(f(X_1, \cdots, X_n), g(Y_1, \cdots, Y_n))$ is $CPQD$. Thus (4.5) holds according to (4.4). □

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