SOME PROBLEMS ON 3-DIMENSIONAL REAL HYPERSURFACES IN COMPLEX SPACE FORMS

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ABSTRACT. Niebergall and Ryan posed some open questions on 3-dimensional real hypersurfaces in complex space forms. In this paper we give affirmative answers to such kind of questions.

1. Introduction

Recently, R. Niebergall and P. J. Ryan([5]) gave the necessary background material to access the study of real hypersurfaces in complex space forms and gave a survey of this field of the study. Also they posed some questions and problems which were not solved until now. Now let us introduce one of them:

"*Question 9.10. Many results have been proved for $n \geq 3$ but questions remain concerning the case $n = 2$. For example, Theorems 5.5, 6.18, 6.19, 6.20, 6.21, 6.23, and 6.30 can be considered from this point of view.*"

The first and fourth theorems of the above question are as follows:

THEOREM A [5, Theorem 5.5]. *Let M^{2n-1}, where $n \geq 3$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Assume that*

$$(\nabla_{X'}A)Y' = -c\langle \phi X', Y'\rangle W$$

and

$$\langle (A\phi - \phi A)X', Y'\rangle = 0$$

for all X' and Y' in $W^{\perp}$. Then M is an open subset of a Type A hypersurface from Takagi's list or Montiel's list.

THEOREM B [5, Theorem 6.20]. *Let M^{2n-1}, where $n \geq 3$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Then M cannot have harmonic curvature, that is, $(\nabla_Y S)Z - (\nabla_Z S)Y$ cannot vanish identically.*

Theorem A was proved by Ki and Suh ([2]). Relevant references to Theorem B are [Ki [1]; Kwon and Nakagawa [4]; Kim [3]].

The purpose of the present paper is to prove the above two theorems in the case of $n = 2$. In this paper we use the same terminologies and notations as in [5].

2. Preliminaries

Let $\widetilde{M}(c)$ be a space of constant holomorphic sectional curvature $4c$ with real dimension $2n$ and Levi-Civita connection $\widetilde{\nabla}$. For an immersed manifold $i : M^{2n-1} \longrightarrow \widetilde{M}$, the Levi-Civita connection ∇ of the induced metric and the shape operator A of the immersion are characterized respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y \rangle \xi$$

and

$$\widetilde{\nabla}_X \xi = -AX$$

where ξ is a local choice of unit normal. Let $J : T\widetilde{M} \longrightarrow T\widetilde{M}$ be the complex structure with properties $J^2 = -I$, $\widetilde{\nabla} J = 0$, and $\langle JX, JY \rangle = \langle X, Y \rangle$. Define the structure vector $W = -J\xi$. Then $W \in TM$ and $|W| = 1$.

Define a skew-symmetric (1,1)-tensor ϕ from the tangential projection of J by

$$JX = \phi X + \langle X, W \rangle \xi.$$

Then we have

(2.1) $\phi^2 X = -X + \langle X, W \rangle W, \quad \phi W = 0,$

$$\langle \phi X, \phi Y \rangle = \langle X, Y \rangle - \langle X, W \rangle \langle Y, W \rangle,$$

that is, $(\phi, W, \langle, \rangle)$ determines an almost contact metric structure. We denote $W^\perp = \{X \in TM | \langle X, W \rangle = 0\}$. The Gauss and Codazzi equations are given by

(2.2) $R(X, Y) = AX \wedge AY + c(X \wedge Y + \phi X \wedge \phi Y + 2\langle X, \phi Y \rangle \phi),$

(2.3) $(\nabla_X A)Y - (\nabla_Y A)X = c(\langle X, W \rangle \phi Y - \langle Y, W \rangle \phi X + 2\langle X, \phi Y \rangle W),$

where $X \wedge Y$ denotes the linear transformation satisfying

$$(X \wedge Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y.$$

From equation (2.2) we get the Ricci tensor S of type (1,1) as

(2.4) $SX = (2n+1)cX - 3c\langle X, W \rangle W + mAX - A^2 X,$

where m denotes the trace of A.

It is known ([5]) that

(2.5) $\nabla_X W = \phi AX$

and

(2.6) $(\nabla_X \phi)Y = \langle Y, W \rangle AX - \langle AX, Y \rangle W.$

If W is a principal vector, then M is called a Hopf hypersurface. A fundamental fact about Hopf hypersurfaces is that the principal curvature α corresponding to W is constant for complex space forms of nonzero holomorphic sectional curvature.

Now we list standard examples of hypersurfaces in complex space forms. These examples are so prevalent in the subject that they have acquired a standard nomenclature. In complex projective space CP^n, the hypersurfaces divided into five types, A-E, while the complex hyperbolic space CH^n has just two types. Types are further subdivided, e.g., A1, A2. The list is as follows([5]). In complex projective space, CP^n:

(A1) Geodesic spheres.
(A2) Tubes over totally geodesic complex projective spaces CP^k, where $1 \le k \le n-2$.
(B) Tubes over complex quadrics and RP^n.
(C) Tubes over the Segre embedding of $CP^1 \times CP^m$ where $2m+1 = n$ $n \ge 5$.
(D) Tubes over the Plücker embedding of the complex Grassmann manifold $G_{2,5}$. Occur only for $n = 9$.
(E) Tubes over the cannonical embedding of the Hermitian symmetric space $SO(10)/U(5)$. Occur only for $n = 15$.

This list is often referred to as "Takagi's list".

In complex hyperbolic space CH^n the list is as follows:

(A0) Horosphere.
(A1) Geodesic spheres and tubes over totally geodesic complex hyperbolic hyperplanes.
(A2) Tubes over totally geodesic CH^k, where $1 \le k \le n-2$.
(B) Tubes over totally real hyperbolic space RH^n.

We refer to the list as "Montiel's list".

Here we refer the following theorem for later use.

THEOREM 2.1 [5, Theorem 4.1]. *Let M^{2n-1}, where $n \geq 2$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Then $\phi A = A\phi$ if and only if M is an open subset of a Type A hypersurface.*

3. Extension of Theorem A to $n = 2$

In this section we consider 3-dimensional real hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. We assume that

$$(3.1) \qquad (\nabla_{X'} A)Y' = -c\langle \phi X', Y' \rangle W$$

and

$$(3.2) \qquad \langle (A\phi - \phi A)X', Y' \rangle = 0$$

for all X' and Y' in $W^\perp$. We choose a local frame field W, U, ϕU of M and put

$$(3.3) \qquad \begin{aligned} AW &= \alpha W + bU + e\phi U, \\ AU &= bW + \beta U + \delta\phi U, \\ A\phi U &= eW + \delta U + \gamma\phi U, \end{aligned}$$

where we have used the property $\langle AY, Z \rangle = \langle Y, AZ \rangle$. From the second equation of (3.3) we get

$$\phi AU = \beta\phi U - \delta U,$$

which and the third equation of (3.3), and (3.2) imply

$$\langle \phi AU - A\phi U, U \rangle = -2\delta = 0$$

and

$$\langle \phi AU - A\phi U, \phi U \rangle = \beta - \gamma = 0.$$

Hence (3.3) is reduced to

$$(3.4) \qquad \begin{aligned} AW &= \alpha W + bU + e\phi U, \\ AU &= bW + \beta U, \\ A\phi U &= eW \qquad\quad + \beta\phi U. \end{aligned}$$

From the second equation of (3.4) we obtain, by the help of (2.5) and (3.1),

$$(3.5) \qquad A\nabla_U U = (Ub)W + b\beta\phi U + (U\beta)U + \beta\nabla_U U.$$

By the orthonormal expansion of $\nabla_U U$, we have

$$(3.6) \qquad \nabla_U U = -\langle\phi\nabla_U U, U\rangle\phi U,$$

which gives

$$(3.7) \qquad A\nabla_U U = -\langle\phi\nabla_U U, U\rangle(eW + \beta\phi U).$$

Substituting (3.6) and (3.7) into (3.5), we find

$$(3.8) \qquad e\langle\phi\nabla_U U, U\rangle + Ub = 0, \quad U\beta = 0, \quad b\beta = 0.$$

Diffferentiating the third equation of (3.4) by U, we obtain, by the help of (2.5), (2.6), (3.1), (3.4), (3.6), and (3.8),

$$(3.9) \qquad -cW - \beta(\alpha W + e\phi U) + \langle\phi\nabla_U U, U\rangle bW$$
$$= (Ue)W + e\beta\phi U - \beta^2 W.$$

Hence we obtain

$$(3.10) \qquad -c - \beta\alpha + \langle\phi\nabla_U U, U\rangle b = Ue - \beta^2, \quad \beta e = 0.$$

Assume that $e \neq 0$ at a point p of M. Then $e \neq 0$ in an open neighborhood O of p. In O, we have $\beta = 0$ by the help of (3.10). Hence (3.4) is reduced to

$$(3.11) \qquad AW = \alpha W + bU + e\phi U, \quad AU = bW, \quad A\phi U = eW$$

in O. From the first equation of (3.8), we obtain

$$(3.12) \qquad \langle\phi\nabla_U U, U\rangle = -\frac{1}{e}Ub,$$

which and (3.10) imply

$$(3.13) \qquad bUb + eUe = -ce.$$

Differentiating the first equation and the second equation of (3.11) covariantly, we obtain

$$(3.14) \quad (\nabla_U A)W = (U\alpha)W + (Ub)U + b\nabla_U U + (Ue)\phi U + e\phi\nabla_U U$$

and

$$(3.15) \qquad (\nabla_W A)U + A\nabla_W U = (Wb)W + b^2\phi U - beU,$$

where we have used (2.5), (2.6), and (3.11). From (2.3) we get

$$(\nabla_W A)U - (\nabla_U A)W = c\phi U,$$

which and (3.14), and (3.15) imply

(3.16)
$$c\phi U + A\nabla_W U = (Wb)W + b^2\phi U - beU - (U\alpha)W - (Ub)U - b\nabla_U U$$
$$- (Ue)\phi U - e\phi\nabla_U U.$$

By taking the inner product (3.16) with ϕU, we have

(3.17)
$$c + e^2 = b^2 + b\langle\phi\nabla_U U, U\rangle - Ue,$$

where we have used
$$\langle A\nabla_W U, \phi U\rangle = e\langle\nabla_W U, W\rangle = -e\langle U, \nabla_W W\rangle$$
$$= -e\langle U, \phi AW\rangle = e^2.$$

Substituting (3.12) into (3.17) and taking account of (3.13), we obtain

(3.18)
$$b^2 - e^2 = 0.$$

By taking the inner product (3.16) with U and using
$$\langle A\nabla_W U, U\rangle = \langle\nabla_W U, bW\rangle = -b\langle U, \nabla_W W\rangle = -b\langle U, \phi AW\rangle = be,$$

we obtain

(3.19)
$$2be + Ub + e\langle\phi\nabla_U U, U\rangle = 0.$$

Substituting (3.12) into (3.19), we have $be = 0$. Hence we get $b = 0$. Since $b^2 - e^2 = 0$ from (3.18), we have $e = 0$ on O. This contradicts to our assumption. Therefore, we have $e = 0$ on all of M.

Now suppose that $b \neq 0$ at a point p' of M. Then there exists an open neighborhood O' of p' in M such that $b \neq 0$ in O'. Therefore we have $\beta = 0$ in O' by the help of (3.8). Hence (3.4) is reduced to

(3.20)
$$AW = \alpha W + bU, \quad AU = bW, \quad A\phi U = 0.$$

From the first equation of (3.8) and (3.10) we have respectively

(3.21)
$$Ub = 0, \quad \langle\phi\nabla_U U, U\rangle = \frac{c}{b}.$$

Differentiating the first equation and the second equation of (3.20) respectively, we obtain

$$(\nabla_U A)W = (U\alpha)W + b\nabla_U U, \quad (\nabla_W A)U + A\nabla_W U = (Wb)W + b^2\phi U,$$

where we have used (2.5), (3.20), and (3.21).

Since $(\nabla_W A)U - (\nabla_U A)W = c\phi U$ by (2.3), we have

(3.22)
$$c\phi U + A\nabla_W U = (Wb)W - (U\alpha)W + b^2\phi U - b\nabla_U U.$$

By taking the inner product (3.22) with ϕU and using $A\phi U = 0$, we have

$$c = b^2 + b\langle \phi \nabla_U U, U \rangle = b^2 + c$$

because of (3.21). This contradicts to the assumption $b \neq 0$. So we have $b = 0$ on all of M.

Hence M is a Hopf hypersurface and we obtain from (3.4)

$$AW = \alpha W, \quad AU = \beta U, \quad A\phi U = \beta \phi U,$$

which implies

$$AZ = \beta Z + (\alpha - \beta)\langle Z, W \rangle W$$

for every vector field Z on M. Hence we have $\phi A = A\phi$. By the Theorem 2.1 we have the following.

THEOREM 3.1. *Let M be a 3-dimensional real hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Assume that*

$$(\nabla_{X'}A)Y' = -c\langle \phi X', Y' \rangle W$$

and

$$\langle (A\phi - \phi A)X', Y' \rangle = 0$$

for all X' and Y' in $W^\perp$. Then M is an open subset of a Type A hypersurface from Takagi's list or Montiel's list.

4. Extension of Theorem B to $n = 2$

Let M be a 3-dimensional Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$ and let $(\nabla_Y S)Z - (\nabla_Z S)Y$ vanishes identically.

We choose a local frame field $W, X', \phi X'$ of M and put

(4.1)
$$AW = \alpha W,$$
$$AX' = \alpha_1 W + \beta_1 X' + \gamma_1 \phi X',$$
$$A\phi X' = \alpha_1' W + \beta_1' X' + \gamma_1' \phi X'.$$

Then we have $\alpha_1 = 0$ because of

$$\alpha_1 = \langle AX', W \rangle = \langle X', AW \rangle = \langle X', \alpha W \rangle = \alpha\langle X', W \rangle = 0.$$

Similarly, we have $\alpha_1' = 0$ and $\beta_1' = \gamma_1$. So (4.1) is rewritten as

$$
\text{(4.2)} \qquad\qquad
\begin{aligned}
AW &= \alpha W, \\
AX' &= \beta_1 X' + \gamma_1 \phi X', \\
A\phi X' &= \gamma_1 X' + \gamma_1' \phi X'.
\end{aligned}
$$

Therefore $A : W^\perp \longrightarrow W^\perp$ is also a linear transformation and A satisfies $\langle AZ, Y \rangle = \langle Z, AY \rangle$ for every Y and Z in $W^\perp$. Hence A is diagonalizable. So we can take a principal vector $U \in W^\perp$ of A with corresponding principal curvature β. Since $W^\perp$ is a two dimensional space, another principal vector must be of the form $\pm \phi U$. Hence we have another local frame field W, U, ϕU of M. So we can put

$$
\text{(4.3)} \qquad\qquad AW = \alpha W, \quad AU = \beta U, \quad A\phi U = \gamma \phi U.
$$

These equations show that the trace of A is given by $m = \alpha + \beta + \gamma$.

It is known ([5, Lemma 2.2]) that if M^{2n-1}, where $n \geq 2$, is a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c$, then we have

$$
\text{(4.4)} \qquad\qquad A\phi A - \frac{\alpha}{2}(A\phi + \phi A) - c\phi = 0.
$$

Using (4.3) and (4.4), we obtain

$$
\text{(4.5)} \qquad\qquad \alpha(\beta + \gamma) = 2\beta\gamma - 2c.
$$

Using (2.4) and taking account of (4.3), we obtain

$$
\text{(4.6)} \qquad SW = fW, \ SU = hU, \ S\phi U = d\phi U, \ \phi SU = h\phi U,
$$

where f, h, and d are given by

$$
f = 2c + m\alpha - \alpha^2, \ h = 5c + m\beta - \beta^2, \ d = 5c + m\gamma - \gamma^2.
$$

For later use we prepare the following equations.

$$
\text{(4.7)} \quad
\begin{aligned}
&\nabla_W W = 0, \ \nabla_W U = -\nu\phi U, \ \nabla_W(\phi U) = \nu U, \ \nabla_U W = \beta\phi U, \\
&\quad \nabla_U U = -\sigma\phi U, \ \nabla_U(\phi U) = -\beta W + \sigma U, \ \nabla_{\phi U} W = -\gamma U, \\
&\quad \nabla_{\phi U} U = \gamma W - \mu\phi U, \ \nabla_{\phi U}\phi U = \mu U,
\end{aligned}
$$

where we have put

$$
\sigma = \langle \phi\nabla_U U, U \rangle, \ \mu = \langle \phi\nabla_{\phi U} U, U \rangle, \ \text{and} \ \nu = \langle \phi\nabla_W U, U \rangle.
$$

To obtain (4.7) we have used (2.5), (2.6), and (4.3). For example,

$$\nabla_W(\phi U)$$
$$= \langle \nabla_W(\phi U), W \rangle W + \langle \nabla_W(\phi U), U \rangle U + \langle \nabla_W(\phi U), \phi U \rangle \phi U$$
$$= - \langle \phi U, \nabla_W W \rangle W - \langle \phi U, \nabla_W U \rangle U$$
$$= - \langle \phi U, \phi A W \rangle W + \langle \phi \nabla_W U, U \rangle U = \nu U.$$

Since

$$(\nabla_W S)U - (\nabla_U S)W = \nabla_W(SU) - S\nabla_W U - [\nabla_U(SW) - S\nabla_U W] = 0,$$

we have, by the help of (4.6) and (4.7),

$$-(Uf)W + (Wh)U + (-h\nu - f\beta + d\nu + d\beta)\phi U = 0,$$

which gives

(4.8) $Uf = 0, \quad Wh = 0, \quad -h\nu - f\beta + d\nu + d\beta = 0.$

Similarly, from $(\nabla_W S)\phi U - (\nabla_{\phi U} S)W = 0$ and $(\nabla_U S)\phi U - (\nabla_{\phi U} S)U = 0$, we have

(4.9) $(d - h)\nu + (f - h)\gamma = 0, \quad Wd = 0, \quad (\phi U)f = 0,$

(4.10) $-d\beta + \beta f - h\gamma + f\gamma = 0, -\sigma h + \sigma d - (\phi U)h = 0,$
$$Ud - \mu d + h\mu = 0.$$

From the first equation of (4.10), $m = \alpha + \beta + \gamma$, and (4.6) we obtain

(4.11) $3c(\beta + \gamma) + \beta\gamma(\beta + \gamma) - \alpha(\beta^2 + \gamma^2) = 0.$

Since

$$\nabla_U \nabla_{\phi U} \phi U - \nabla_{\phi U} \nabla_U \phi U - \nabla_{[U,\phi U]} \phi U = R(U, \phi U)\phi U,$$

we have, by the help of (2.2) and (4.7),

(4.12) $U\mu + \beta\nu - \sigma^2 + \gamma\nu - \mu^2 - 2\beta\gamma - (\phi U)\sigma - 4c = 0.$

Similarly, for the triples $(\phi U, U, U), (U, \phi U, W), (\phi U, W, \phi U)$, and (U, W, U) we obtain

(4.13) $$-U\gamma - \mu\beta + \mu\gamma = 0,$$

(4.14) $$-(\phi U)\beta + \gamma\sigma - \beta\sigma = 0,$$

(4.15) $$\nu\gamma - \beta\gamma - \beta\nu + \alpha\gamma + c = 0,$$

(4.16) $$\nu\beta - \nu\gamma - \beta\gamma + \beta\alpha + c = 0.$$

It is known([5, Lemma 2.13]) that $Wm = 0$ in a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Since $m = \alpha + \beta + \gamma$, we have $W(\beta + \gamma) = 0$. Hence we get

$$Wf = W(2c + \alpha(\beta + \gamma)) = 0.$$

From (4.8) and (4.9), we have $Uf = 0$ and $(\phi U)f = 0$. So, f is constant. From $f = 2c + \alpha(\beta + \gamma)$ and (4.5), we have $f = 2\beta\gamma$ and $\beta\gamma$ is constant.

CASE 1. $\alpha = 0$.

In this case we have $c = \beta\gamma$ from (4.5). Hence we get $4c(\beta + \gamma) = 0$ from (4.11) and we find $\beta = -\gamma$. Since $m = \alpha + \beta + \gamma$, we obtain $h = d$ from (4.6). Therefore we have $h = f$ from the first equation of (4.9) and hence we get $3c = \beta^2$ from (4.6). Since $c = \beta\gamma = -\beta^2$, we find $c = 0$. This contradicts to our hypothesis $c \neq 0$.

CASE 2. $\alpha \neq 0$.

Since $f = 2c + \alpha(\beta + \gamma) = 2\beta\gamma$ from (4.5) and (4.6) and f is constant, β and γ are constants.
   (i) If $\beta = \gamma$, then we obtain $\beta = \gamma = 0$ from (4.5) and (4.11). Substituting $\beta = \gamma = 0$ into (4.5), we find $c = 0$. This contradicts to our hypothesis $c \neq 0$.
   (ii) If $\beta \neq \gamma$, then we obtain $\mu = \sigma = 0$ from (4.13) and (4.14). Substituting $\mu = \sigma = 0$ into (4.12), we get

(4.17) $(\beta + \gamma)\nu = 2\beta\gamma + 4c.$

From (4.12) and (4.15) we obtain

(4.18) $2\beta\nu - \gamma(\alpha + \beta) - 5c = 0,$

where we have used $\mu = \sigma = 0$.
   Similarly, from (4.12) and (4.16) we have

(4.19) $2\gamma\nu - \beta(\alpha + \gamma) - 5c = 0.$

From (4.18) and (4.19) we find $\nu = -\dfrac{\alpha}{2}$.

   Substituting $\nu = -\dfrac{\alpha}{2}$ into (4.17) and taking account of (4.5), we find $\beta\gamma = -c$.

   Substituting $\nu = -\dfrac{\alpha}{2}$ and $\beta\gamma = -c$ into (4.15), we get $\beta + \gamma = -\dfrac{4c}{\alpha}$.
   Thus β and γ are roots of the quadratic equation

$$x^2 + \frac{4c}{\alpha}x - c = 0.$$

Hence we have

$$(4.20) \qquad\qquad 4c^2 + c\alpha^2 \geq 0.$$

On the other hand, if we substitute $\beta\gamma = -c$ and $\beta + \gamma = -\dfrac{4c}{\alpha}$ into (4.11), then we have

$$(4.21) \qquad\qquad 12c^2 + c\alpha^2 = 0.$$

From (4.20) and (4.21) we find $c = 0$, which contradicts to our hypothesis.

Therefore we have the following.

THEOREM 4.1. *Let M be a 3-dimensional Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Then M cannot have harmonic curvature, that is, $(\nabla_Y S)Z - (\nabla_Z S)Y$ cannot vanish identically.*

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