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EXPONENTIAL DECAY OF C^1 LAGRANGE POLYNOMIAL SPLINES WITH RESPECT TO THE LOCAL CHEBYSHEV-GAUSS POINTS

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ABSTRACT. In the course of working on the preconditioning C^1 polynomial spline collocation method, one has to deal with the exponential decay of C^1 Lagrange polynomial splines. In this paper we show the exponential decay of C^1 Lagrange polynomial splines using the Chebyshev-Gauss points as the local data points.

1. Introduction

Let $I := [0,1]$ be the unit interval. Let $N > 1$ be a positive integer and $h := 1/N$. The knots are the points $t_k := kh$, $(k = 0, 1, \ldots, N)$ and k^{th} subinterval I_k of I is denoted by $I_k := (t_{k-1}, t_k)$. Let $\mathbb{P}_n$ be the set of all polynomials of degree less than or equal to n and let $\mathcal{S}_{h,n}$ be the space of C^1 polynomial splines defined on I with knots sequence $\{t_k\}_{k=0}^{N}$, i.e.,

$$\mathcal{S}_{h,n} := \{u \in C^1[0,1],\, u|_{I_k} \in \mathbb{P}_n,\ k = 1, 2, \cdots, N\}.$$

Let $\mathcal{S}_{h,n}^m$ be a particular subspace of $\mathcal{S}_{h,n}$ satisfying the mixed boundary conditions such that

$$\mathcal{S}_{h,n}^m = \{u \in \mathcal{S}_{h,n} :\, u(0) = 0,\, u'(1) = 0\}.$$

Denote by T_{n-1} the Chebyshev polynomial of degree $n-1$ and set $\eta_j := \cos\theta_j$ as zeros of T_{n-1} (see [3]), so-called the Chebyshev-Gauss points, where

$$\theta_j = \frac{(2j-1)\pi}{2(n-1)}, \quad \text{where } j = 1, 2, \ldots, n-1. \tag{1.1}$$

Setting $\eta_n := -1$ and $\eta_0 := 1$, $\{\eta_j\}_{j=0}^{n}$ can be ordered as

$$-1 =: \eta_n < \eta_{n-1} < \cdots < \eta_1 < \eta_0 := 1.$$

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Now, let us define the local Chebyshev-Gauss points $\{\xi_{i,k}\}_{i=1}^{n-1}$ as the collocation points on each subinterval I_k such that

$$(1.2) \qquad \xi_{i,k} := \frac{h}{2}\eta_{n-i} + \frac{t_{k-1}+t_k}{2}, \quad k = 1,2,\ldots,N, \; i = 1,2,\ldots,n-1.$$

For convenience, set $\xi_{0,1} := 0$ and $\xi_{n,N} := 1$.

For the basis of $\mathcal{S}_{h,n}^m$, we introduce the C^1 Lagrange polynomial splines $\{\phi_{i,k}\}_{i=1,k=1}^{n-1,N}$ with respect to the collocation points $\{\xi_{i,k}\}_{i=1,k=1}^{n-1,N}$ satisfying

$$(1.3) \qquad \phi_{i,k}(\xi_{j,l}) = \delta_{(k,i),(j,l)}, \quad i,j = 1,\cdots,n-1, \quad k,l = 1,\cdots,N$$

where

$$\delta_{(i,k),(j,l)} := \begin{cases} 1 & \text{if } (i,k) = (j,l) \\ 0 & \text{if } (i,k) \neq (j,l). \end{cases}$$

The existence and uniqueness of these splines can be verified by the Schoenberg-Whitney conditions in [8]. Then, one may easily check that $\{\phi_{i,k}\}_{i=1,k=1}^{n-1,N}$ is a basis of $\mathcal{S}_{h,n}^m$ and the construction of this basis can be done easily by the help of spline package in matlab (see [1] and [2]).

Using these splines, one can apply to polynomial spline collocation method as one of $h-p$ version of the Chebyshev spectral collocation method. For example, we consider a simple model problem $-u'' + u = f$ in $(0,1)$ with the mixed boundary conditions $u(0) = 0 = u'(1)$. The corresponding scheme is given by

$$\sum \big[-\phi''_{j,l}(\xi_{i,k}) + \phi_{j,l}(\xi_{i,k})\big]\, u_{j,l} = f(\xi_{i,k}),$$

where $u_{j,l} = u(\xi_{j,l})$ is the coefficient vector. One may try to investigate a preconditioner constructed by the finite element method using the continuous piecewise linear functions with respect to the collocation points $\{\xi_{i,k}\}$. Here, we need to show the exponential decay of the splines $\{\phi_{i,k}\}$ in order to prove the equivalence of the Chebyshev continuous L^2-norm and discrete L^2-norm (for more details see [3] and [7]).

The polynomial spline collocation method using the Chebyshev-Gauss points locally has the same fashion as the Chebyshev spectral collocation method. Many mathematicians have studied the Chebyshev spectral collocation method together with the Legendre spectral collocation method and the methods have been applied to solve various numerical problems. The authors have studied the preconditioning C^1 polynomial spline collocation method for elliptic equations using the Legendre-Gauss points in [7]. The closed approaches of the exponential decay for C^1 Lagrange cubic splines can be found in [4] and [6], and the numerical approach using C^1 Lagrange cubic splines was given in [5].

The purpose of this paper is to show the exponential decay of C^1 Lagrange polynomial splines with respect to the local Chebyshev-Gauss points $\{\xi_{i,k}\}$ which allow us to investigate the preconditioning polynomial spline collocation method using the local Chebyshev-Gauss points.

2. Exponential decay

Let

$$q_i(t) := \prod_{j\neq i, j=0}^{n} (t-\eta_j).$$

Then the i^{th} *Lagrange interpolating polynomial* $r_i(t)$ with knots at $\{\eta_i\}_{i=0}^{n}$ can be written as

$$r_i(t) = \frac{q_i(t)}{q_i(\eta_i)} \quad \text{and} \quad r_i(t) = \frac{1-t^2}{1-\eta_i^2}\tilde{r}_i(t) \quad \text{for } i \neq 0, n,$$

where

$$\tilde{r}_i(t) = \prod_{j\neq i, j=1}^{n-1} \frac{t-\eta_j}{\eta_i-\eta_j}.$$

LEMMA 2.1. *The followings hold.*

(2.1) $$|r_0(t)| \leq 1, \quad |r_n(t)| \leq 1, \quad (-1 \leq t \leq 1),$$

(2.2) $$r_0'(1) = (n-1)^2 + \frac{1}{2},$$

(2.3) $$\max_{1\leq i\leq n-1} |r_i(t)| \leq 2\,(n-1)^2,$$

and

(2.4) $$\max_{1\leq i\leq n-1} \left\{ |r_i'(1)|, \left|\frac{1-\eta_i}{1+\eta_i}\right| \cdot |r_i'(1)| \right\} \leq 4\,(n-1)^2.$$

PROOF. Since $T_{n-1}(t) = \kappa_{n-1}\prod_{j=1}^{n-1}(t-\eta_j)$ where $\kappa_{n-1} := 2^{n-2}(n-1 \geq 1)$ is the leading coefficient of T_{n-1}, we have

(2.5) $$\kappa_{n-1}\, q_0(t) = (t+1)T_{n-1}(t), \quad \kappa_{n-1}\, q_n(t) = (t-1)T_{n-1}(t),$$

and

$$r_0 = \frac{\kappa_{n-1}q_0(t)}{\kappa_{n-1}q_0(\eta_0)} = \frac{(t+1)T_{n-1}(t)}{2}, \quad r_n = \frac{\kappa_{n-1}q_n(t)}{\kappa_{n-1}q_n(\eta_n)} = \frac{(t-1)T_{n-1}(t)}{2(-1)^n}.$$

These yield the conclusions (2.1), (2.2), and (2.3).

Applying Chebyshev polynomial expansion to $\tilde{r}_i(t)$, we have

$$\tilde{r}_i(t) = \frac{1}{n-1} + \frac{2}{n-1}\sum_{k=1}^{n-2} T_k(x_i)T_k(t).$$

Using the fact that $|T_k(t)| \leq 1$, we have

$$|\tilde{r}_i(t)| \leq 2.$$

Since $\frac{2}{\pi}\theta \leq \sin\theta$ for $0 \leq \theta \leq \frac{\pi}{2}$, we have

$$|r_i(t)| = \frac{|1-t^2|}{|1-\eta_i^2|}|\tilde{r}_i(t)| \leq 2(n-1)^2.$$

And since for $i = 1, 2, \cdots, n-1$,

$$\kappa_{n-1}\ (t-\eta_i)q_i(t) = (t-1)(t+1)T_{n-1}(t),$$

by differentiating and using the fact $T_{n-1}(\eta_i) = 0$, it follows that

$$r_i'(1) = \frac{-2}{(1-\eta_i)(1-\eta_i^2)T_{n-1}'(\eta_i)}.$$

Using

$$T_{n-1}'(t) = \frac{\sin(n-1)\theta}{\sin\theta} \quad \text{where } \theta = \arccos t,$$

we have

$$\left(\frac{1-\eta_i}{1+\eta_i}\right) r_i'(1) = \frac{2(-1)^i}{(n-1)\big(1-\cos(\pi-\theta_i)\big)\sin(\pi-\theta_i)}.$$

Applying $\frac{2}{\pi}\theta \leq \sin\theta$ for $0 \leq \theta \leq \frac{\pi}{2}$ to get

$$\big(1-\cos(\pi-\theta_i)\big)\sin(\pi-\theta_i) \geq \frac{1}{2}\left(\frac{2\theta_1}{\pi}\right)^3 = \frac{1}{2(n-1)^3},$$

we have the conclusion (2.4). □

Throughout this section we use the notation $\vec{\Phi}(x) := (\Phi(x), \Phi'(x))^t$.

LEMMA 2.2. *Let Φ_n and $\Psi_{n,i}$ be polynomials of degree n such that*

$$\Phi_n(t) := \Phi_n(1)r_0(t) + \Phi_n(-1)r_n(t), \tag{2.6}$$

vanishes at knots $\{\eta_j\}_{j=1}^{n-1}$ and

$$\Psi_{n,i}(t) := \Psi_{n,i}(1)r_0(t) + \Psi_{n,i}(-1)r_n(t) + r_i(t) \tag{2.7}$$

satisfies $\Psi_{n,i}(\eta_j) = \delta_{ij}$ for $i, j = 1, 2, \cdots, n-1$. Then we have

$$\vec{\Phi}_n(1) = D_n \vec{\Phi}_n(-1), \quad \vec{\Psi}_{n,i}(1) = D_n \vec{\Psi}_{n,i}(-1) + E_{n,i} \tag{2.8}$$

where $D_n := (d_{ij})$ *is the matrix whose entries are*

$$d_{11} = d_{22} = 2r_0'(1)(-1)^{n-1}$$
$$d_{12} = 2(-1)^{n-1}, d_{21} = (-\frac{1}{2} + 2(r_0'(1))^2)(-1)^{n-1}$$

and

$$E_{n,i} := \begin{pmatrix} \frac{1-\eta_i}{1+\eta_i}\, 2\, r_i'(1) \\ \left(1 - \frac{1-\eta_i}{1+\eta_i}\, 2\, r_0'(1)\right) r_i'(1) \end{pmatrix}.$$

Proof. Since

$$\Phi_n'(1) = \Phi_n(1)r_0'(1) + \Phi_n(-1)r_n'(1),$$
$$\Phi_n'(-1) = \Phi_n(1)r_0'(-1) + \Phi_n(-1)r_n'(-1),$$

the conclusion (2.8) follows from (2.5). □

Let $\rho = \rho(n) := |D_n(1,1)| = 2r_0'(1) = 2(n-1)^2 + 1 \geq 3 \quad (n \geq 2)$.

Lemma 2.3. (i) *For a function* $\phi_{i,k}$ *on the interval* I_l, $(l \neq k)$, *there is a* 2×2 *matrix* $E_{n,h}$ *satisfying*

$$\begin{pmatrix} \phi_{i,k}(t_l) \\ \phi_{i,k}'(t_l) \end{pmatrix} = D_{n,h} \begin{pmatrix} \phi_{i,k}(t_{l-1}) \\ \phi_{i,k}'(t_{l-1}) \end{pmatrix} \tag{2.9}$$

where

$$D_{n,h} := (-1)^{n+1} \begin{bmatrix} \rho & h \\ \frac{1}{h}(\rho^2 - 1) & \rho \end{bmatrix}.$$

(ii) *For* $\phi_{i,k}$ *satisfying* $\phi_{i,k}(\xi_{i,k}) = 1$ *and* $\phi_{i,k}(\xi_{j,k}) = 0$ *if* $i \neq j$ *on the interval* $I_k = [t_{k-1}, t_k]$, *we have*

$$\begin{pmatrix} \phi_{i,k}(t_k) \\ \phi_{i,k}'(t_k) \end{pmatrix} = D_{n,h} \begin{pmatrix} \phi_{i,k}(t_{k-1}) \\ \phi_{i,k}'(t_{k-1}) \end{pmatrix} + E_{n,i,h} \tag{2.10}$$

where

$$E_{n,i,h} := \begin{pmatrix} \frac{1-\eta_i}{1+\eta_i}\, 2\, r_i'(1) \\ \frac{2}{h}\left(1 - \frac{1-\eta_i}{1+\eta_i}\, \rho\right) r_i'(1) \end{pmatrix}$$

Proof. The linear change of variables $t = \frac{h}{2}(s+1) + t_{k-1}(-1 \leq s \leq 1)$ applied to (2.8) yields the conclusion. □

COROLLARY 2.4. *For $r = 0,\ 1$, we have*

$$\left|\phi_{i,k}^{(r)}(t_l)\right| \le \left(\frac{1}{\rho}\right)^{j-l} \left|\phi_{i,k}^{(r)}(t_j)\right|, \quad (0 \le l \le j < k) \tag{2.11}$$

and

$$\left|\phi_{i,k}^{(r)}(t_l)\right| \le \left(\frac{1}{\rho}\right)^{l-j} \left|\phi_{i,k}^{(r)}(t_j)\right|. \quad (k \le j \le l \le N) \tag{2.12}$$

PROOF. We may have

$$\phi_{i,k}(t_l)\phi'_{i,k}(t_l) \ge 0, \quad (0 \le l < k),$$
$$\phi_{i,k}(t_l)\phi'_{i,k}(t_l) \le 0, \quad (k \le l < N),$$

hence, with $r = 0,\ 1$, it follows from (2.9) that

$$\left|\phi_{i,k}^{(r)}(t_{l-1})\right| \le \frac{1}{\rho}\left|\phi_{i,k}^{(r)}(t_l)\right|, \qquad (1 \le l < k) \tag{2.13}$$

$$\left|\phi_{i,k}^{(r)}(t_l)\right| \le \frac{1}{\rho}\left|\phi_{i,k}^{(r)}(t_{l-1})\right|. \qquad (k \le l \le N) \tag{2.14}$$

Then this complete the proof. □

The eigenvalues of $D_{n,h}$ are given by

$$\lambda_1 = (-1)^{n+1}\left(\rho + \sqrt{\rho^2 - 1}\right) \quad \text{and} \quad \lambda_2 = (-1)^{n+1}\left(\rho - \sqrt{\rho^2 - 1}\right)$$

which satisfy

$$|\lambda_1| \ge 3 + 2\sqrt{2}, \quad \lambda_1\lambda_2 = 1 \quad \text{and} \quad (-1)^{n+1}\lambda_1 > (-1)^{n+1}\lambda_2 > 0.$$

Let, for integer $k \ge 1$,

$$q_k := \lambda_1^k + \lambda_2^k = \lambda_1^k\left(1 + \sigma_n^k\right) \quad \text{and} \quad p_k := \lambda_1^k - \lambda_2^k = \lambda_1^k\left(1 - \sigma_n^k\right)$$

where

$$0 < \sigma_n = \frac{\lambda_2}{\lambda_1} = \frac{1}{\lambda_1^2} \le \frac{1}{(3 + 2\sqrt{2})^2}.$$

Then it is easy to verify that, for any $k = 1, 2, \cdots, N$,

$$0 < q_{N-k}q_k, \quad q_{N-k}p_k, \quad p_{k-1}p_{N-k}, \quad p_{k-1}q_{N-k} \le C\, q_N, \tag{2.15}$$

where C is an absolute constant independent of n and h.
From Lemma 2.3 with $\phi_{i,k}(t_0) = 0 = \phi'_{i,k}(t_N)$,

$$\begin{pmatrix} \phi_{i,k}(t_{k-1}) \\ \phi'_{i,k}(t_{k-1}) \end{pmatrix} = D_{n,h}^{k-1} \begin{pmatrix} 0 \\ \phi'_{i,k}(t_0) \end{pmatrix} \tag{2.16}$$

and

$$\begin{pmatrix} \phi_{i,k}(t_k) \\ \phi'_{i,k}(t_k) \end{pmatrix} = D_{n,h}^{k-N} \begin{pmatrix} \phi_{i,k}(t_N) \\ 0 \end{pmatrix}. \tag{2.17}$$

We combine (2.10), (2.16), and (2.17) to obtain

$$D_{n,h}^{k-N} \begin{pmatrix} \phi_{i,k}(t_N) \\ 0 \end{pmatrix} = D_{n,h}^{k} \begin{pmatrix} 0 \\ \phi'_{i,k}(t_0) \end{pmatrix} + E_{n,i,h}. \tag{2.18}$$

Note that, for positive integer m, since $D_{n,h} D_{n,-h} = I$,

$$D_{n,h}^{m} = \frac{1}{2} \begin{bmatrix} q_m & \frac{h}{\sqrt{\rho^2-1}} p_m \\ \frac{p_m\sqrt{\rho^2-1}}{h} & q_m \end{bmatrix}$$

and

$$D_{n,h}^{-m} = \frac{1}{2} \begin{bmatrix} q_m & \frac{-h}{\sqrt{\rho^2-1}} p_m \\ \frac{p_m\sqrt{\rho^2-1}}{-h} & q_m \end{bmatrix}.$$

LEMMA 2.5. *Let $\phi_{i,k}(t), (i = 1, \cdots, n-1, k = 1, \cdots, N)$ be the basis function for $\mathcal{S}_{h,n}^m$. Then there is a constant C, independent of n, h, i, and k, such that*

$$|\phi_{i,k}(t_k)| \le C\,(n-1)^2 \quad \text{and} \quad |\phi_{i,k}(t_{k-1})| \le C\,(n-1)^2. \tag{2.19}$$

PROOF. The system of equations (2.18) yields

$$\begin{bmatrix} q_{N-k} & \frac{-h}{\sqrt{\rho^2-1}}{}_k \\ \frac{p_{N-k}\sqrt{\rho^2-1}}{-h} & -q_k \end{bmatrix} \begin{pmatrix} \phi_{i,k}(t_N) \\ \phi'_{i,k}(t_0) \end{pmatrix} = 2E_{n,i,h}.$$

Combining (2.17), (2.15), and (2.16) with the above system of equations, one can estimate at $a = t_k$ or t_{k-1}

$$|\,\phi_{i,k}(a)\,| \le C \left(1 + \left|\frac{1-\eta_i}{1+\eta_i}\right|\right) |r'_i(1)|. \tag{2.20}$$

The conclusions come from combining Lemma 2.1 with (2.20). □

COROLLARY 2.6. *For the basis function $\phi_{i,k}(t)$ for $\mathcal{S}_{h,n}^m$, we have the upper bounds such that*

(2.21)

$$|\phi_{i,k}(t)| \le |\phi_{i,k}(t_{k-1})| + |\phi_{i,k}(t_k)| + C\,(n-1)^2 \le C\,(n-1)^2 \quad \text{on} \quad I_k,$$

and

$$|\phi_{i,k}(t)| \leq \max\{|\phi_{i,k}(t_{m-1})|, |\phi_{i,k}(t_m)|\} \quad on \quad I_m \tag{2.22}$$

where C is a constant independent of n, i, k, and h.

PROOF. The linear change of variables $s = \frac{2}{h}(t - t_{k-1}) - 1$ converts $\phi_{i,k}$ on I_k to a polynomial $\Psi_{n,i}(s)$ on $[-1, 1]$ defined in (2.7), and on I_m, $(m \neq k)$ to a polynomial Φ_n defined in (2.6). Using Lemma 2.1, (2.6), and (2.7),we have

$$|\Phi_n(s)| \leq \max\{|\Phi_n(-1)|, |\Phi_n(1)|\} \quad \text{on} \quad [-1, 1],$$

and

$$|\Psi_{n,i}(s)| \leq |\Psi_{n,i}(-1)| + |\Psi_{n,i}(1)| + C\,(n-1)^2 \quad \text{on} \quad [-1, 1]$$

where C is a constant independent of n and i. Applying a transformation $t = \frac{h}{2}(s+1) + t_{k-1}$ to $\Psi_{n,i}$ on the interval I_k and to Φ_n on each interval I_m with $m \neq k$ and then using (2.19), we have the conclusion. □

THEOREM 2.7. *For the basis functions $\phi_{i,k}(t)$, there exists a positive constant C, independent of h, i and k, such that*

$$|\phi_{i,k}(t)| \leq C \left(\frac{1}{\rho}\right)^{|k-m|} \quad on \quad I_m.$$

PROOF. For the case of $m = k$, using (2.21) and the fact that $(n-1)^2 \leq \rho \leq \rho^2$ yield the conclusion. On the other hand, if $m > k$, then using (2.22), (2.12), and (2.19) yields

$$|\phi_{i,k}(t)| \leq |\phi_{i,k}(t_{m-1})| \leq \left(\frac{1}{\rho}\right)^{|m-k|-1} |\phi_{i,k}(t_k)| \leq c\,\rho^2 \left(\frac{1}{\rho}\right)^{|m-k|},$$

which implies the conclusion. For the case $m < k$, it can be similarly shown. □

This theorem shows that $\phi_{i,k}$ satisfying (1.3) decays exponentially as t moves toward end points of $[0, 1]$ with the exponential decay factor $\rho(n) = \rho = 2(n-1)^2 + 1$ for a fixed n.

References

[1] C. de Boor, *A practical guide to splines*, Applied Mathematical Sciences, Springer-Verlag **27** (1978).

[2] ———, *Spline tool box for use with Matlab*, The Mathworks Inc., 1996.

[3] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zand, *Spectral Methods in Fluid Dynamics*, Springer-Verlag, Berlin, New York, 1987.

[4] S. D. Kim and S. C. Kim, *Exponential decay of C^1−cubic splines vanishing at two symmetric points in each knot interval*, Numer. Math. **76** (1997), 479–488.

[5] S. D. Kim and S. V. Parter, *Preconditioning cubic spline collocation discretizations of elliptic equations*, Numer. Math. **72** (1995), 39–72.

[6] S. D. Kim and B. C. Shin, *On the Exponential decay of C^1−cubic Lagrange splines on non-uniform meshes and for non-uniform data points*, Houston J. Math. **24** (1998), no. 1, 173–183.

[7] ———, *Preconditioning C^1-Lagrange Polynomial Spline Collocation Method of Elliptic Equations*, submitted.

[8] J. Schoenberg and A. Whitney, *On Pólya frequency functions, III: The Positivity of Translation Determinants with Application to The Interpolation Problem by Spline Curves*, Trans. Amer. Math. Soc. **74** (1953), 246–259.

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