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ON AN ARRAY OF WEAKLY DEPENDENT RANDOM VECTORS

TAE IL JEON

ABSTRACT. In this article we investigate the dependence between components of the random vector which is given as an asymptotic limit of an array of random vectors with interlaced mixing conditions. We discuss the cross covariance of the limiting vector process and give a stronger condition to have a central limit theorem for an array of random vectors with mixing conditions.

1. Introduction

We consider a triangular array of two-dimensional random variates $\{\xi_{ni}|1 \leq i \leq k_n\} = \{(\xi_{ni}^{(1)}, \xi_{ni}^{(2)})|1 \leq i \leq k_n\}$ such that $\xi_{ni}^{(j)}, j = 1,2$ satisfy some interlaced mixing conditions. In this article we refer to results obtained by Peligrad [4] and investigate the independence of the limiting bivariate normal distribution. Mixing sequences of random variables are sequences for which past and distant future are asymptotically independent. Let $(\Omega, \mathfrak{F}, P)$ be a probability space and let $\mathcal{A}, \mathcal{B}$ be two sub σ-algebras of $\mathfrak{F}$. Define the strong mixing coefficient by

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |P(AB) - P(A)P(B)|$$

and the supremum of the coefficients of correlation by

$$\rho(\mathcal{A}, \mathcal{B}) = \sup_{f \in L^2(\mathcal{A}), g \in L^2(\mathcal{B})} |\text{corr}(f, g)|.$$

DEFINITION 1. A strictly stationary sequence $\{X_i\}$ is called α-mixing if $\alpha(n) \to 0$, where

$$\alpha(n) = \sup_k \alpha(\sigma(X_i, i \leq k), \sigma(X_i, i \geq k+n)).$$

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Note that the mixing coefficient $\alpha(n)$ is a measure of dependence between two sub σ-algebras $\sigma(X_i, i \le k)$ and $\sigma(X_i, i \ge k+n)$. Let $\mathfrak{F}_n^m = \sigma(X_n, X_{n+1}, \dots, X_{n+m})$ and

$$\rho(n) = \sup\{|\mathrm{corr}(f,g)| : f \in L_2(\mathfrak{F}_{-\infty}^k), g \in L_2(\mathfrak{F}_{k+n}^\infty)\}.$$

DEFINITION 2. A strictly stationary sequence $\{X_i\}_{i\in\mathbb{Z}}$ is called ρ-mixing if $\rho(n) \to 0$ as $n \to \infty$.

Let $\mathfrak{F}_T = \sigma(X_i, i \in T)$, where T is a finite family of integers and $\mathrm{dist}(S,T) = \inf\{|s-t| : s \in S, T \in T\}$, where S and T are two nonempty finite subsets of $\mathbb{Z}$.

DEFINITION 3. Let $\{X_i\}$ be a strictly stationary sequence. Then define

$$\alpha^*(n) = \sup \alpha(\mathfrak{F}_T, \mathfrak{F}_S)$$

and

$$\rho^*(n) = \sup \rho(\mathfrak{F}_T, \mathfrak{F}_S),$$

where these supremums are taken over all pairs of nonempty finite subsets S, T of $\mathbb{Z}$ such that $\mathrm{dist}(S,T) \ge n$.

According to Bradley [2] we have, for $n \ge 1$,

$$\begin{aligned} &(a) \quad \alpha^*(n) \le \rho^*(n) \le 2\pi\alpha^*(n) \\ &(b) \qquad \alpha(n) \le \alpha^*(n) \\ &(c) \qquad \rho(n) \le \rho^*(n). \end{aligned}$$

Consider a triangular array of bivariate random variables $\xi_{ni} = (\xi_{ni}^{(1)}, \xi_{ni}^{(2)})$, where $1 \le i \le k_n$ and $k_n \to \infty$. Here we refer to the definitions used in Peligrad [4]. Define, for $j = 1, 2$,

$$\bar{\alpha}_{nk}^{(j)} = \sup_{s\ge 1} \alpha(\sigma(\xi_{ni}^{(j)}, i \le s), \sigma(\xi_{ni}^{(j)}, i \ge s+k))$$

and $\bar{\alpha}_k^{(j)} = \sup_n \bar{\alpha}_{nk}^{(j)}$.

The array $\{\xi_{ni} | 1 \le i \le k_n\}$ will be called strongly mixing if, for each $j = 1, 2$, $\lim_{k\to\infty} \bar{\alpha}_k^{(j)} = 0$. In order to properly define the corressponding ρ-mixing coefficients for the array we have the following definition

$$\bar{\rho}_{nk}^{(j)*} = \sup_{k\ge 1} \rho(\sigma(\xi_{ni}^{(j)}, i \in T), \sigma(\xi_{ni}^{(j)}, i \in S)),$$

where $T, S \subset \{1, 2, \dots, k_n\}$ are nonempty subsets with $\mathrm{dist}(T,S) \ge k$ and $\bar{\rho}_k^{(j)*} = \sup_n \bar{\rho}_{nk}^{(j)*}$.

2. Central limit Theorems

In this section we state some known results about CLT for an array of random variables with mixing conditions. Since we need notations and context in the proofs of the results we give the sketch of the proofs. The following theorem is the Theorem 2.1 in Peligrad [4]. Here we state the theorem,

THEOREM 1. *Let* $\{\xi_{ni} = (\xi_{ni}^{(1)}, \xi_{ni}^{(2)}) | 1 \le i \le k_n\}$ *be a triangular array of centered bivariate random variables, which is strongly mixing componentwise and* $E(\xi_{ni}^{(j)})^2 < \infty$, *for each* $j = 1, 2$. *Assume* $\lim_{k\to\infty} \bar{\rho}_k^{(j)*} < 1$, *for each* $j = 1, 2$. *Denote, for each* $j = 1, 2$, *by* $(\sigma_n^{(j)})^2 = \mathrm{var}(\sum_{i=1}^{k_n} \xi_{ni}^{(j)})$ *and assume*

$$\sup_n \frac{1}{(\sigma_n^{(j)})^2} \sum_{i=1}^{k_n} E(\xi_{ni}^{(j)})^2 < \infty$$

and for every $\varepsilon > 0$

$$\frac{1}{(\sigma_n^{(j)})^2} \sum_{i=1}^{k_n} E(\xi_{ni}^{(j)})^2 I(|\xi_{ni}^{(j)}| > \varepsilon \sigma_n^{(j)}) \to 0 \text{ as } n \to \infty.$$

Then, for each $j = 1, 2$,

$$\frac{\sum_{i=1}^{k_n} \xi_{ni}^{(j)}}{\sigma_n^{(j)}} \xrightarrow{d} \mathcal{N}(0, 1) \quad \text{as } n \to \infty,$$

where d stands for the convergence in distribution.

For the following sections we use the same notations. Therefore we need to sketch the proof of Theorem 1.(See Peligrad [4] for the detail of the proof.) For each $j = 1, 2$, let $\zeta_{ni}^{(j)} = \xi_{ni}^{(j)} / \sigma_n^{(j)}$. Then we can construct a sequence of positive numbers ε_n such that

$$\sum_{i=1}^{k_n} E(\zeta_{ni}^{(j)})^2 I(|\zeta_{ni}^{(j)}| > \varepsilon_n) \to 0 \quad \text{as } n \to \infty.$$

Truncating at the level ε_n, define, for each $j = 1, 2$,

$$\eta_{ni}^{(j)} = \zeta_{ni}^{(j)} I(|\zeta_{ni}^{(j)}| \le \varepsilon_n) - E\zeta_{ni}^{(j)} I(|\zeta_{ni}^{(j)}| \le \varepsilon_n)$$

and

$$\gamma_{ni}^{(j)} = \zeta_{ni}^{(j)} I(|\zeta_{ni}^{(j)}| > \varepsilon_n) - E\zeta_{ni}^{(j)} I(|\zeta_{ni}^{(j)}| > \varepsilon_n).$$

Note that, for each $j = 1, 2$, $\zeta_{ni}^{(j)} = \eta_{ni}^{(j)} + \gamma_{ni.}^{(j)}$ Then we can show that

$$\text{(1)} \qquad \operatorname{var}\Big(\sum_{i=1}^{k_n} \gamma_{ni}^{(j)}\Big) \to 0 \quad \text{as } n \to \infty.$$

By the argument above we can rewrite the statement in Theorem 1 into the following:

THEOREM 2. *Let $\{\eta_{ni} | 1 \le i \le k_n\}$ be an array of bivariate random variables with zero means and finite second moments, which satisfies, for each $j = 1, 2$,*

$$|\eta_{ni}^{(j)}| \le 2\varepsilon_n \quad \text{where } \varepsilon_n > 0,$$

$$(\sigma_n^{(j)})^2 = \operatorname{var}\Big(\sum_{i=1}^{k_n} \eta_{ni}^{(j)}\Big) \to 1 \quad \text{as } n \to \infty$$

and

$$\sup_n \sum_{i=1}^{k_n} \operatorname{var}(\eta_{ni}^{(j)}) < \infty.$$

Then, for each $j = 1, 2$,

$$\sum_{i=1}^{k_n} \eta_{ni}^{(j)} \xrightarrow{d} \mathcal{N}(0, 1) \quad \text{as } n \to \infty.$$

Here we apply the blocking procedure to divide the sequence of random variables into big blocks and small blocks. Since we have constructed a sequence $\{\varepsilon_n\}$ we construct a sequence of integers $\{q_n\}$ such that the following conditions satisfied;

$$q_n \to \infty \quad \text{as } n \to \infty$$
$$q_n \varepsilon_n \to 0 \quad \text{as } n \to \infty$$
$$q_n \bar{\alpha}^{(j)}([\varepsilon_n^{-1}]) \to 0 \quad \text{as } n \to \infty,$$

where $\bar{\alpha}^{(j)}(n) = \bar{\alpha}_n^{(j)}$. For each $n \in \mathbb{N}$, define the integers by the following steps:

$$m_0 = 0$$

and, for $p = 0, 1, 2, \ldots$, let

$$m_{2p+1} = \min\left\{ m \Big| \; m > m_{2p}, \sum_{i=m_{2p}+1}^{m} \operatorname{var}(\eta_{ni}^{(j)}) \ge q_n^{-1} \right\},$$

$$m_{2p+2} = m_{2p+1} + [\varepsilon_n^{-1}].$$

We classify the groups specified by the argument above. Let

$$I_p = \{k \mid \ m_{2p} < k \leq m_{2p+1}\}, \quad J_p = \{k \mid \ m_{2p+1} < k \leq m_{2p+2}\}$$

for $p = 0, 1, 2, \ldots$. Processing the steps above we have ℓ_n blocks of indices I_p and J_p, respectively, $p = 0, 1, 2, \ldots, \ell_n$ (See [4] for the detail). Denote

$$Y_{np} = \sum_{i \in I_p} \eta_{ni}, \quad Z_{np} = \sum_{i \in J_p} \eta_{ni}, \quad 0 \leq p \leq \ell_n.$$

By an appropriate argument, we can show that

$$\text{(2)} \qquad \operatorname{var}\left(\sum_{p=1}^{\ell_n} Z_{np}\right) \to 0 \ \text{ as } \ n \to \infty,$$

which means that $\sum_{p=1}^{\ell_n} Z_{np}$ is negligible for the convergence in distribution. Moreover we get

$$\text{(3)} \qquad \lim_{n \to \infty} \operatorname{var}\left(\sum_{p=1}^{\ell_n} Y_{np}\right) = 1$$

and there exist two constants $0 < K_1 < K_2$ and $N \in \mathbb{N}$ such that

$$0 < K_1 < \sum_{p=1}^{\ell_n} \operatorname{var} Y_{np} < K_2, \quad \text{for } n \geq N.$$

Let $a_n = (\sum_{p=1}^{\ell_n} \operatorname{var} Y_{np})^{1/2}$. Then we may assume that $\{Y_{np} | 0 \leq p \leq \ell_n\}$ is an array of independent random variables. Finally we may show that $\{a_n^{-1} Y_{np} | 0 \leq p \leq \ell_n\}$ satisfies the CLT. Therefore the proof of Theorem 1 is completed.

3. Cross covariance of vector process and CLT

Throughout this section we assume that the conditions in Theorem 1 in section 2 hold. Note that the results stated in the previous section guarantee the componentwise CLT for the given array of vector process with componentwise mixing conditions. What we are interested in is to investigate the limiting behavior of the array of vector process $\{\xi_{ni} | 1 \leq i \leq k_n\}$. To investigate the limiting behavior it is necessary to see the limiting behavior of the cross covariance of the sum of sequence of normalized vector process $\{\zeta_{ni} | 1 \leq i \leq k_n\}$. It is natural to compute the cross covariance of the vector process $\{\zeta_{ni} | 1 \leq i \leq k_n\}$. We study whether the sequence of vector process

$$\left(\sum_{i=1}^{k_n} \zeta_{ni}^{(1)}, \sum_{i=1}^{k_n} \zeta_{ni}^{(2)} \right) \tag{4}$$

converges in distribution to a bivariate normal vector process (Z_1, Z_2), where $Z_1 \sim \mathcal{N}(0,1), Z_2 \sim \mathcal{N}(0,1)$. As mentioned in the proof of Theorem 1 in section 2, we have known how the big blocks are constructed and that they act like an array of independent random variables. We will see what the big blocks say about the dependence between the components in the limiting distribution. So it is worth investigating the limit of the following sequence of cross covariances

$$\operatorname{Cov}\left(\sum_{i=1}^{k_n} \zeta_{ni}^{(1)}, \sum_{i=1}^{k_n} \zeta_{ni}^{(2)} \right) \tag{5}$$

for the given array of random vectors $\{\xi_{ni} = (\xi_{ni}^{(1)}, \xi_{ni}^{(2)}) | 1 \leq i \leq k_n\}$. By a simple argument we have the following fact, which states the given mixing conditions on each component of the sequence of vector process give at least the uniform boundedness of the sequence of cross covariances.

THEOREM 3. *Let $\{\eta_{ni} | 1 \leq i \leq k_n\}$ be an array of bivariate random variables with zero means and finite second moments, which satisfies, for each $j = 1, 2$,*

$$|\eta_{ni}^{(j)}| \leq 2\varepsilon_n \quad \text{where } \varepsilon_n > 0,$$

$$(\sigma_n^{(j)})^2 = \operatorname{var}\left(\sum_{i=1}^{k_n} \eta_{ni}^{(j)}\right) \to 1 \quad \text{as } n \to \infty$$

and

$$\sup_n \sum_{i=1}^{k_n} \operatorname{var} \eta_{ni}^{(j)} < \infty.$$

Then there is a number M such that

$$\left| \operatorname{Cov}\left(\sum_{i=1}^{k_n} \eta_{ni}^{(1)}, \sum_{i=1}^{k_n} \eta_{ni}^{(2)} \right) \right| \leq M, \quad \text{for all } n.$$

If the sequence in (5) converges to a number ρ, then we have that the array of random vectors in (4) converges in distribution to a bivariate normal distribution (Z_1, Z_2) with covariance matrix

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

To investigate the convergence of (5), for $j = 1, 2$, let $A_n^{(j)} = \sum_{i=1}^{k_n} \xi_{ni}^{(j)}$. Then $A_n^{(j)} = \sigma_n^{(j)} \zeta_n^{(j)}, j = 1, 2$. Consider the following sequence

$$E\left[\left(\sum_{i=1}^{k_n} \zeta_{ni}^{(1)}\right)\left(\sum_{i=1}^{k_n} \zeta_{ni}^{(2)}\right)\right]. \tag{6}$$

Then

$$\left| E\left\{\left(\sum_{i=1}^{k_n} \zeta_{ni}^{(1)}\right)\left(\sum_{i=1}^{k_n} \zeta_{ni}^{(2)}\right)\right\} - E\left\{\left(\sum_{i=1}^{k_m} \zeta_{mi}^{(1)}\right)\left(\sum_{i=1}^{k_m} \zeta_{mi}^{(2)}\right)\right\}\right|$$

$$\begin{aligned}
&= \left| \frac{1}{\sigma_n^{(1)}\sigma_n^{(2)}} E(A_n^{(1)} A_n^{(2)}) - \frac{1}{\sigma_m^{(1)}\sigma_m^{(2)}} E(A_m^{(1)} A_m^{(2)}) \right| \\
&\leq \left| \frac{1}{\sigma_n^{(1)}\sigma_n^{(2)}} E(A_n^{(1)} A_n^{(2)}) - \frac{1}{\sigma_n^{(1)}\sigma_m^{(2)}} E(A_n^{(1)} A_m^{(2)}) \right| \\
&\qquad + \left| \frac{1}{\sigma_n^{(1)}\sigma_m^{(2)}} E(A_n^{(1)} A_m^{(2)}) - \frac{1}{\sigma_m^{(1)}\sigma_m^{(2)}} E(A_m^{(1)} A_m^{(2)}) \right| \\
&= \frac{1}{\sigma_n^{(1)}} E\left\{ A_n^{(1)} \left(\frac{A_n^{(2)}}{\sigma_n^{(2)}} - \frac{A_m^{(2)}}{\sigma_m^{(2)}} \right) \right\} + \frac{1}{\sigma_n^{(1)}} E\left\{ A_m^{(2)} \left(\frac{A_n^{(1)}}{\sigma_n^{(1)}} - \frac{A_m^{(1)}}{\sigma_m^{(1)}} \right) \right\}.
\end{aligned} \tag{7}$$

Consider the variance of terms in the last line in (7). For each $j = 1, 2$, the variance is

$$E\left(\frac{A_n^{(j)}}{\sigma_n^{(j)}} - \frac{A_m^{(j)}}{\sigma_m^{(j)}} \right)^2. \tag{8}$$

Then (8) is equal to the following

$$\frac{1}{(\sigma_n^{(j)})^2} E\left(A_n^{(j)}\right)^2 - \frac{2}{\sigma_n^{(j)}\sigma_m^{(j)}} E\left\{\left(A_n^{(j)}\right)\left(A_m^{(j)}\right)\right\} + \frac{1}{(\sigma_n^{(j)})^2} E\left(A_n^{(j)}\right)^2. \tag{9}$$

Note that the first and the third term in (9) tends to 1, respectively, as $n \to \infty$. If the mid term in (9) approaches to 2 as $n, m \to \infty$, then (6) is a Cauchy sequence and hence converges. We state the result as the following;

PROPOSITION 1. *Suppose that, for each $j = 1, 2$,*

$$E\left[\left(\sum_{i=1}^{k_n} \zeta_{ni}^{(j)}\right)\left(\sum_{i=1}^{k_m} \zeta_{mi}^{(j)}\right)\right] \to 1 \quad \text{as } n, m \to \infty. \tag{10}$$

Then there is a number ρ such that

$$E\left[\left(\sum_{i=1}^{k_n} \zeta_{ni}^{(1)}\right)\left(\sum_{i=1}^{k_n} \zeta_{ni}^{(2)}\right)\right] \to \rho \quad \text{as } n \to \infty.$$

This indicates that the limiting behavior of the auto covariance of the sum of each array of $\{\zeta_{ni}^{(j)} | 1 \leq i \leq k_n\}, j = 1, 2$ determines the limiting behavior of the cross covariance of the normalized vector process $\{\zeta_{ni} | 1 \leq i \leq k_n\}$.

THEOREM 4. *The necessary and sufficient condition for which (10) holds is, for each $j = 1, 2$,*

$$E\left(\sum_{p=1}^{\ell_n} Y_{np}^{(j)}\right)\left(\sum_{p=1}^{\ell_m} Y_{mp}^{(j)}\right) \to 1 \quad \text{as } n, m \to \infty. \tag{11}$$

PROOF. Since, for each $j = 1, 2$, $\zeta_{ni}^{(j)} = \eta_{ni}^{(j)} + \gamma_{ni}^{(j)}$,

$$\begin{aligned}
&E\Big[\Big(\sum_{i=1}^{k_n} \zeta_{n}^{(j)} i\Big)\Big(\sum_{i=1}^{k_m} \zeta_{mi}^{(j)}\Big)\Big] \\
&= E\Big[\Big(\sum_{i=1}^{k_n} \eta_{ni}^{(j)} + \gamma_{ni}^{(j)}\Big)\Big(\sum_{i=1}^{k_m} \eta_{mi}^{(j)} + \gamma_{mi}^{(j)}\Big)\Big] \\
&= E\Big[\Big(\sum_{i=1}^{k_n} \eta_{ni}^{(j)}\Big)\Big(\sum_{i=1}^{k_m} \eta_{mi}^{(j)}\Big)\Big] + E\Big[\Big(\sum_{i=1}^{k_m} \eta_{mi}^{(j)}\Big)\Big(\sum_{i=1}^{k_n} \gamma_{ni}^{(j)}\Big)\Big] \\
&\quad + E\Big[\Big(\sum_{i=1}^{k_n} \eta_{ni}^{(j)}\Big)\Big(\sum_{i=1}^{k_m} \gamma_{mi}^{(j)}\Big)\Big] + E\Big[\Big(\sum_{i=1}^{k_n} \gamma_{ni}^{(j)}\Big)\Big(\sum_{i=1}^{k_m} \gamma_{mi}^{(j)}\Big)\Big].
\end{aligned} \tag{12}$$

By (1) and Hölder's inequality, the last three terms in (12) tend to 0 as $n \to \infty$. Therefore it is sufficient to show that, for each $j = 1, 2$,

$$\lim_{m,n\to\infty} E\Big[\Big(\sum_{i=1}^{k_n} \eta_{ni}^{(j)}\Big)\Big(\sum_{i=1}^{k_m} \eta_{mi}^{(j)}\Big)\Big] = \lim_{m,n\to\infty} E\Big[\Big(\sum_{i=1}^{\ell_n} Y_{ni}^{(j)}\Big)\Big(\sum_{i=1}^{\ell_m} Y_{mi}^{(j)}\Big)\Big].$$

Since, for each $j = 1, 2$,

$$\sum_{i=1}^{k_m} \eta_{m_i}^{(j)} = \sum_{p=1}^{\ell_m} \left(Y_{mp}^{(j)} + Z_{mp}^{(j)} \right), \quad \sum_{i=1}^{k_n} \eta_{n_i}^{(j)} = \sum_{p=1}^{\ell_n} \left(Y_{np}^{(j)} + Z_{np}^{(j)} \right),$$

we have

$$\begin{aligned}
&E\Big[\Big(\sum_{i=1}^{k_n} \eta_{ni}^{(j)}\Big)\Big(\sum_{i=1}^{k_m} \eta_{mi}^{(j)}\Big)\Big] \\
(13) \quad &= E\Big[\Big(\sum_{p=1}^{\ell_m} Y_{mp}^{(j)}\Big)\Big(\sum_{p=1}^{\ell_n} Y_{np}^{(j)}\Big)\Big] + E\Big[\Big(\sum_{p=1}^{\ell_m} Y_{mp}^{(j)}\Big)\Big(\sum_{p=1}^{\ell_n} Z_{np}^{(j)}\Big)\Big] \\
&\quad + E\Big[\Big(\sum_{p=1}^{\ell_n} Y_{np}^{(j)}\Big)\Big(\sum_{p=1}^{\ell_m} Z_{mp}^{(j)}\Big)\Big] + E\Big[\Big(\sum_{p=1}^{\ell_m} Z_{mp}^{(j)}\Big)\Big(\sum_{p=1}^{\ell_n} Z_{np}^{(j)}\Big)\Big].
\end{aligned}$$

By (2), the last three terms in (13) tend to 0 as $n, m \to \infty$. Thus we have completed the proof. □

Theorem 4 indicates that the limiting behavior of the auto covariance of the sequence of sum of big blocks determines the limiting behavior of the cross covariance of the normalized vector process $\{\zeta_{ni} | 1 \le i \le k_n\}$. Note that we have only the uniform boundedness of the sequence in (10) or (11). But it does not guarantee the convergence of the sequence in (11). It does not seem to be easy to find some sufficient conditions for convergence of (10). It might help finding sufficient conditions if we add some mixing conditions between components of an array of vector process. We leave it as an open question. To have the convergence we may assume a strong condition; for instance, convergence in probability. Here we assume that for each $j = 1, 2$, $\{\sum_{i=1}^{k_n} \zeta_{ni}^{(j)}\}$ is Cauchy in probability. Then (10) holds true since the sequence of second moments of each component converges to the second moment of the normal distribution, that is, 1. By those arguments we have the following.

THEOREM 5. *Suppose, for each $j = 1, 2$, $\{\sum_{i=1}^{k_n} \zeta_{ni}^{(j)}\}$ is Cauchy in probability. Then there exists a number $0 \le \rho \le 1$ such that*

$$\lim_{n\to\infty} \mathrm{Cov}\left(\sum_{i=1}^{k_n} \zeta_{ni}^{(1)}, \sum_{i=1}^{k_n} \zeta_{ni}^{(2)} \right) = \rho$$

and hence

$$\left(\sum_{i=1}^{k_n} \zeta_{ni}^{(1)}, \sum_{i=1}^{k_n} \zeta_{ni}^{(2)} \right) \xrightarrow{d} (Z_1, Z_2),$$

where (Z_1, Z_2) *is a bivariate normal distribution* $\mathcal{N}_2(\mu, \Sigma)$. *Here the expectation and the covariance matrix are*

$$\mu = (EZ_1, EZ_2) = (0,0) \text{ and } \Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

4. Independence between components

In this section we state the independence between the limiting vector processes when each componentwise sequence is independent of the other. We have known that a central limit theorem holds for a centered strictly stationary strong mixing sequence of random variables which has the polynomial mixing rate $\alpha(n) \le Kn^{-\theta}, \theta > 0$ and $EX_1^{2+\delta} < \infty, \delta > 0$. Bradley [2] showed a CLT for a centered strictly stationary sequence with mixing condition $\alpha^*(n) \to 0$. Note that there are no other moment conditions. We state the theorem.

THEOREM 6. *If* $(X_k, k \in \mathbb{Z})$ *is a strictly stationary sequence of real centered square integrable random variables such that* $\sigma_n^2 = \text{var}(\sum_{i=1}^n X_i) \to \infty$, *and* $\alpha^*(n) \to 0$ *as* $n \to \infty$, *then*

$$\frac{\sum_{i=1}^n X_i}{\sigma_n} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as } n \to \infty.$$

Peligrad [4] has shown the following theorem:

THEOREM 7. *Suppose* $\{X_k, k \in \mathbb{N}\}$ *is a strongly mixing strictly stationary sequence of random variables which are centered and have finite second moments. Assume that* $\lim_{n\to\infty} \rho^*(n) < 1$ *and* $\sigma_n^2 \to \infty$. *Then*

$$\liminf \frac{\sigma_n^2}{n} > 0$$

and

$$\frac{\sum_{k=1}^n X_k}{\sigma_n} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as } n \to \infty.$$

Consider a centered stationary random vectors $\mathbf{X}_n = (X_n^{(1)}, X_n^{(2)})$ with independence between $\{X_n^{(1)}\}$ and $\{X_n^{(2)}\}$. Define the partial sums $S_n^{(1)} = \sum_{k=1}^n X_k^{(1)}$, $S_n^{(2)} = \sum_{k=1}^n X_k^{(2)}$. Analogous to the definitions (1) and (3) of mixing rates $\alpha(n)$ and $\alpha^*(n)$ we can define $\alpha_1(n), \alpha_2(n), \alpha_1^*(n), \alpha_2^*(n)$ for the sequences of random variables $\{X_n^{(1)}\}$ and $\{X_n^{(2)}\}$, respectively. Then we have the following theorem. The proof of componentwise CLT is quite similar to that of [1]. But to have CLT for vector process and independence between the resulting normal processes we have to use the method of moments in Gaussian process. Since the computation of the limiting behavior of the cross covariance is lengthy but routine we skip the computation and state the result only. So we have CLT for vector process and independence between the components of the limiting vector process (Z_1, Z_2).

THEOREM 8. *Let* $\mathbf{X}_n = (X_n^{(1)}, X_n^{(2)})$ *be a centered stationary random vectors with independence between* $\{X_n^{(1)}\}$ *and* $\{X_n^{(2)}\}$. *Suppose* $E[X_n^{(1)}]^2 < \infty, E[X_n^{(2)}]^2 < \infty$, $E(S_n^{(1)})^2 = E[\sum_{k=1}^n X_k^{(1)}]^2 = \sigma_{1n}^{(2)} \to \infty$, $E(S_n^2)^2 = E[\sum_{k=1}^n X_k^{(2)}]^2 = \sigma_{2n}^2 \to \infty$, $\alpha_1^*(n) \to 0$, *and* $\alpha_2^*(n) \to 0$ *as* $n \to \infty$. *Then*

$$\left(\frac{S_n^{(1)}}{\sigma_{1n}}, \frac{S_n^{(2)}}{\sigma_{2n}} \right) \xrightarrow{d} (Z_1, Z_2),$$

where $Z_1 \sim \mathcal{N}(0,1)$ *and* $Z_2 \sim \mathcal{N}(0,1)$. *Moreover* Z_1 *and* Z_2 *are independent, that is, the expectation and covariance matrix satisfy*

$$\mu = (EZ_1, EZ_2) = (0,0) \text{ and } \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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Department of Mathematics
Taejon University
Taejon 300-716, Korea
E-mail: tijeon@dragon.taejon.ac.kr