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LIFTING PROPERTIES ON $L^1(\mu)$

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ABSTRACT. In this paper we show that some operators defined on $L^1(\mu)$ and on $C(K)$ into a Banach space with the RNP have the lifting property.

1. Introduction

The main purpose of this paper is to establish some conditions under which linear operators on $L^1(\mu)$ have the lifting property. These results are then applied to obtain a lifting theorem for $C(K)$ where K is a compact Hausdorff space.

We begin our discussion with a summary of known results concerning the corresponding problems for continuous linear operators between Banach spaces. Suppose that E, F and G are Banach spaces and ϕ is a surjective map of F onto G which maps the closed unit ball in G onto the closed unit ball in F and that T is a bounded linear operator of E into F. When does T have a norm preserving lifting $\tilde{T} : E \to G$, that is, when does there exist a continuous linear mapping $\tilde{T} : E \to G$ such that $\|\tilde{T}\| = \|T\|$ and such that the following diagram commutes?

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ \tilde{T}\downarrow & \nearrow\phi & \\ G & & \end{array} \tag{1.1}$$

Grothendieck [3], Pelczynski [6] and Köthe [4] have established the following results concerning this question.

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THEOREM 1.1. *For a fixed Banach space E, such a lifting $\tilde{T}$ exists for arbitrary F, G, ϕ, and T as above diagram if and only if $E = l^1(\Gamma)$ for some index set Γ.*

Moreover, if we add the restrictions that G, F are dual spaces and that ϕ is an adjoint mapping, then $l^1(\Gamma)$ can be replaced by an AL-space.

We now give the notion of the Radon-Nikodym property which is applied to our main results. The following definitions are useful to our study.

DEFINITION 1.2 [1]. A Banach space E has the *Radon-Nikodym property* with respect to (Ω, Σ, μ) if for each μ-continuous vector measure $G : \Sigma \to E$ of the bounded variation there exists $g \in L^1(\mu, E)$ such that $G(X) = \int_X g d\mu$ for all $X \in \Sigma$. Also a Banach space E has the *Radon-Nikodym property (RNP)* if E has the Radon-Nikodym property with respect to every finite measure space.

A bounded linear operator $T : L^1(\mu) \to E$ is *representable* if there exists $g \in L^\infty(\mu, E)$ such that $Tf = \int_\Omega f g d\mu$ for all $f \in L^1(\mu)$.

The next theorem gives the connection between the Radon-Nikodym theorem and the Riesz Representation theorem.

THEOREM 1.3 [1]. *Let E be a Banach space and (Ω, Σ, μ) be a finite measure space. Then E has the RNP with respect to (Ω, Σ, μ) if and only if each $T \in \mathcal{L}(L^1(\mu), E)$ is representable.*

For more details about the Radon-Nikodym property, the reader can refer to the book [1]. Also we need the definition of the absolutely summing operator on a Banach space E.

DEFINITION 1.4 [2]. A bounded linear operator $T : E \to F$ is called *absolutely summing* if T maps weakly unconditionally Cauchy series in E into absolutely convergent series in F.

The following theorem gives some equivalent descriptions of absolutely summing operators.

THEOREM 1.5 [2]. *Any one of the following statements about a bounded linear operator $T : E \to F$ implies all others.*

a) T is absolutely summing.

b) T maps unconditionally convergent series in E into absolutely convergent series in F.

c) There exists a constant $K > 0$ such that for any finite set $x_1, x_2, \cdots, x_n \in E$ the following inequality holds;

$$\sum_{i=1}^{n} \|Tx_i\| \leq K \sup\{\sum_{i=1}^{n} |x^* x_i| : x^* \in B_{E^*}\}. \tag{1.2}$$

2. Main results

In this paper, our main questions are based on Theorem 1.1, that is, which bounded linear operators on $L^1(\mu)$ into a Banach space F can have the lifting property? Due to Theorem 1.1, $E = l^1(\Gamma)$ for some index set Γ only has the lifting property. But if we impose more conditions, we can find a lifting on $L^1(\mu)$ and on $C(K)$, which may not be the unique lifting. The main idea of the theorem is based on Theorem 1.1 and the Lewis and Stegall's Theorem in [5].

THEOREM 2.1. *Let (Ω, Σ, μ) be any finite measure space and E be a* Banach *space with the* RNP. *Then for any $\epsilon > 0$ and for any* Banach *space F if $T : L^1(\mu) \to E$ is a bounded linear operator and $\phi : F \to E$ is any surjective map, then T has a lifting $\tilde{T} : L^1(\mu) \to F$ such that $\|\tilde{T}\| \leq (1+\epsilon)\|T\|$, $\phi \circ \tilde{T} = T$ and the following diagram commutes;*

$$\begin{array}{ccc} L^1(\mu) & \xrightarrow{T} & E \\ \tilde{T}\downarrow & \nearrow\phi & \\ F & & \end{array} \tag{2.1}$$

PROOF. Let $T : L^1(\mu) \to E$ be a bounded linear operator and E have the RNP. Then by Theorem 1.3, there exists $g \in L^\infty(\mu, E)$ such that $T(f) = \int_\Omega fgd\mu$ for all $f \in L^1(\mu)$. Since g is μ-measurable, there is a sequence (f_n) of countably valued $\mu-$measurable functions such that for each n, $\|g - f_n\|_\infty < \epsilon\|T\|/2^{n+1}$.

Let $g_1 = f_1$ and $g_n = f_n - f_{n-1}$ for all $n \geq 2$. Then we have

$$\begin{aligned} \|g - \sum_{i=1}^{n} g_i\|_\infty &= \|g - f_n\| \\ &< \frac{\epsilon\|T\|}{2^{n+1}} \quad \text{for all} \quad n. \end{aligned}$$

For each n, g_n is also countably valued $\mu-$measurable function. Hence we can say $g_n = \sum_{k=1}^\infty x_{n,k}\chi_{E_{n,k}}$ where $(E_{n,k})_{k=1}^\infty$ is a disjoint member of Σ and $\|x_{n,k}\| < \frac{\epsilon\|T\|}{2^n}$ for all $n \geq 2$.

Define $U : L^1(\mu) \to l^1(N \times N)$ by

$$U(f)(n,k) = \|x_{n,k}\| \int_{E_{n,k}} fd\mu.$$

Then we have

$$\|U(f)\| \leq \sum_{n,k} \|x_{n,k}\| \left| \int_{E_{n,k}} f d\mu \right|$$
$$\leq \sum_{k=1}^{\infty} \|x_{1,k}\| \int_{E_{1,k}} |f| d\mu + \sum_{n=2}^{\infty}\sum_{k=1}^{\infty} \|x_{n,k}\| \int_{E_{n,k}} |f| d\mu.$$

Since $\|g\|_{\infty} = \|T\|$ and $\|g - g_1\| < \frac{\epsilon\|T\|}{2}$, we have $\|x_{1,k}\| < (1+\epsilon/2)\|T\|$ for all k and for all $n \geq 2$, $\|x_{n,k}\| < \frac{\epsilon}{2^n}\|T\|$.

$$\|U(f)\| \leq (1+\epsilon/2)\|T\| \sum_{k=1}^{\infty} \int_{E_{1,k}} |f| d\mu + \sum_{n=2}^{\infty}\sum_{k=1}^{\infty} \frac{\epsilon}{2^n}\|T\| \int_{E_{n,k}} |f| d\mu$$

(2.2) $$\leq (1+\epsilon)\|T\|\|f\|_1.$$

Hence, $\|U\| \leq (1+\epsilon)\|T\|$.

Without loss of generality, we can assume $\|x_{n,k}\| \neq 0$, for all n and k. Define $V : l^1(N \times N) \to E$ by $V((a_{n,k})) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{n,k}\frac{x_{n,k}}{\|x_{n,k}\|}$. Then $\|V\| \leq 1$. Now, for $f \in L^1(\mu)$

$$VU(f) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \|x_{n,k}\| \cdot \int_{E_{n,k}} f d\mu \cdot \frac{x_{n,k}}{\|x_{n,k}\|}$$
$$= \sum_{n=1}^{\infty} \int_{\Omega} f g_n d\mu \qquad \text{by the dominated convergent theorem}$$

(2.3) $$= \int_{\Omega} f g d\mu = T(f).$$

Also by Theorem 1.1, $l^1(N \times N)$ has the lifting property, and we can find the lifting $\tilde{V} : l^1 \to F$ such that $V = \phi \circ \tilde{V}$ and $\|\tilde{V}\| = \|V\|$.

Finally, if we take a lifting $\tilde{T} = \tilde{V} \circ U$, then

$$\phi \circ \tilde{T} = \phi \circ (\tilde{V} \circ U)$$
$$= (\phi \circ \tilde{V}) \circ U$$

(2.4) $$= V \circ U = T$$

and

$$\|\tilde{T}\| = \|\tilde{V} \circ U\|$$
$$\leq \|\tilde{V}\|\|U\| \quad (\text{since} \quad \|U\| \leq (1+\epsilon)\|T\|, \quad \|V\| \leq 1)$$

(2.5) $$= \|V\|\|U\| \leq (1+\epsilon)\|T\|.$$

Hence $\tilde{T}$ is a desired lifting of T on $L^1(\mu)$. □

COROLLARY 2.2. *Let for* $1 \leq p < \infty$, $E = l^p$ *spaces or for* $1 < p < \infty$, $E = L^p(\mu)$ *spaces, if for any* Banach *space* F, $T : L^1(\mu) \to E$ *is a bounded linear operator and* $\phi : F \to E$ *is a surjective map. Then* T *has a lifting property, that is, the following diagram commutes*

$$\begin{array}{ccc} L^1(\mu) & \xrightarrow{T} & E \\ \tilde{T}\downarrow & \nearrow \phi & \\ F & & \end{array} \tag{2.6}$$

PROOF. For $1 \leq p < \infty$, $E = l^p$ spaces have the Radon-Nikodym property and for $1 < p < \infty$, $E = L^p(\mu)$ spaces also have the Radon-Nykodym property. Then by Theorem 2.1, any bounded linear operator on $L^1(\mu)$ to a Banach space E with the RNP has the lifting property. □

COROLLARY 2.3. *Every representable linear operator on* $L^1(\mu)$ *has a lifting operator.*

PROOF. Let for any Banach space E and F, $T : L^1(\mu) \to E$ be a bounded linear operator and $\phi : F \to E$ be a surjective map. Then by Theorem 1.3, E has the RNP with respect to the measure space (Ω, Σ, μ). Hence by Theorem 2.1, we can find a operator $\tilde{T}$ such that $\phi \circ \tilde{T} = T$ and $\|\tilde{T}\| \leq (1 + \epsilon)\|T\|$. □

Next, we can find a lifting property on $C(K)$ where K is a compact Hausdorff space. In this case, we impose more conditions on operators and target Banach spaces.

THEOREM 2.4. *Let* E *be a* Banach *space with the* RNP *and* F *be any* Banach *space. If* $T : C(K) \to E$ *is an absolutely summing operator and* $\phi : F \to E$ *is any onto map, then there is* $\tilde{T} : C(K) \to F$ *such that* $\phi \circ \tilde{T} = T$ *and* $\|\tilde{T}\| \leq \lambda \|T\|$, *for some* $\lambda > 0$.

PROOF. Let E be a Banach space with the RNP and $T : C(K) \to E$ be an absolutely summing operator. Then by the Pietch factorization theorem [1, p.164], we can say

$$\begin{array}{ccc} C(K) & \xrightarrow{T} & E \\ & J \searrow \quad \nearrow S & \\ & L^1(\mu) & \end{array} \tag{2.7}$$

where $J : C(K) \to L^1(\mu)$ is just natural inclusion with $\|J\| = \mu(K) = \|T\|_{as}$ and $\|S\| \leq 1$.

Then by Theorem 2.1, S has a lifting $\tilde{S} : L^1(\mu) \to F$ such that $\|\tilde{S}\| \leq (1+\epsilon)\|S\|$ and $\phi \circ \tilde{S} = S$.

Finally, if we take $\tilde{T} = \tilde{S} \circ J$ as a lifting operator, then we have

$$\begin{aligned} \phi \circ \tilde{T} &= \phi \circ (\tilde{S} \circ J) \\ &= (\phi \circ \tilde{S}) \circ J = S \circ J = T \end{aligned} \tag{2.8}$$

and $\|\tilde{T}\| \leq \|\tilde{S}\|\|J\| \leq (1+\epsilon)\|T\|_{as}$. □

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