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CORRELATION DIMENSIONS OF CANTOR SETS WITH OVERLAPS

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ABSTRACT. We consider a Cantor set with overlaps Λ in $\mathbb{R}^1$. We calculate its correlation dimension with respect to the push-down measure on Λ comparing with its similarity dimension.

1. Introduction

Recently, in order to characterize fractal sets, we sometimes have used the correlation dimension instead of the Hausdorff dimension because of advantages of calculation. It is well-known that the Hausdorff dimension on a self-similar set is equal to the similarity dimension([3]) and the correlation dimension with respect to the specified probability measure on it([2]). Also, we can see that the Hausdorff dimension on a loosely self-similar set([4]) which is a generalization of self-similar sets is equal to the similarity dimension and the correlation dimension with respect to the push-down measure on it([5]). In general, the Hausdorff dimension on a set is greater than or equal to any correlation dimension([5], [7]).

In this paper, we define a Cantor set with overlaps Λ in $\mathbb{R}^1$. In general, this set is neither loosely self-similar set([4]) nor self-similar Cantor set with overlaps([7]). However, we can deal with the loosely self-similar sets in $\mathbb{R}^1$ and self-similar Cantor sets with overlaps as special cases of Cantor sets with overlaps considered in this paper(see Remark 2.1).

For the set Λ, it is not easy to find the Hausdorff dimension of Λ. In stead of the Hausdorff dimension of Λ, we apply delicate methods in [7] to calculations of the correlation dimension of Λ with respect to the defined push-down measure ν on Λ and compare its correlation dimension with its similarity dimension.

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2. Preliminaries

We define Cantor sets with overlaps in $\mathbb{R}^1$. Consider $I \equiv [0,\ 1]$. Fix an integer number $l(\geq 2)$. Suppose that a sequence of mappings $\{f_{i_1,i_2,\cdots,i_n} : (i_1,i_2,\cdots,i_n) \in \{1,2,\ \cdots,\ l\}^n$ for $n = 1,2,\cdots\}$ and numbers $0 < r_1,\ r_2,\ \cdots,\ r_l < 1$ are given such that
(i) $f_{i_1,i_2,\cdots,i_n} : I \longrightarrow I$ is defined as $f_{i_1,i_2,\cdots,i_n}(x) = r_{i_n}x + t_{i_1,i_2,\cdots,i_n}$ for some $t_{i_1,i_2,\cdots,i_n} \in \mathbb{R}$ and each $i_j \in \{1,2,\cdots,l\}(j = 1,2,\cdots,n)$,
(ii) for any $n \geq 1$, a basic set $I_{i_1,i_2,\cdots,i_n} \equiv f_{i_1} \circ f_{i_1,i_2} \circ \cdots \circ f_{i_1,i_2,\cdots,i_n}(I)$ contains l-intervals $I_{i_1,i_2,\cdots,i_n,1}$, $I_{i_1,i_2,\cdots,i_n,2}$ and $I_{i_1,i_2,\cdots,i_n l}$, so that the left-hand ends of $I_{i_1,i_2,\cdots,i_n}$ and $I_{i_1,i_2,\cdots,i_n,1}$ and the right-hand ends of $I_{i_1,i_2,\cdots,i_n}$ and $I_{i_1,i_2,\cdots,i_n,l}$ coincide, and
(iii) there exists $0 < c < \frac{1}{2}$ such that $f_{i_1,i_2,\cdots,i_n}(I) \subset [c, 1-c]$ for all $n \geq 1$ whenever $i_n \neq 1, l$.

Set

$$\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{(i_1,i_2,\cdots,i_n)\in\{1,2,\cdots,l\}^n} I_{i_1,i_2,\cdots,i_n}.$$

We call this Λ a Cantor set with overlaps.

REMARK 2.1. (1) We notice that locations of basic sets of the set Λ are free except for the first and last locations. That is, at each stage, the locations of basic sets of Λ are independent of locations of basic sets in the previous stages. In particular, if we assume the condition $f_{i_1,\cdots,i_n}(I) \cap f_{i_1,\cdots,i_{n-1},i'_n}(I) = \emptyset$ $(i_n \neq i'_n)$, then Λ becomes a loosely self-similar set([4]) in $\mathbb{R}^1$.
(2) Moreover, we notice that basic sets of Λ may overlap. If we substitute by the following conditions to (i) and (iii) in the construction of Λ, *i.e.* $f_{i_1,\cdots,i_n} = f_{i_n}$, $t_{i_1,\cdots,i_n} = t_{i_n}$ for $1 \leq i_j \leq l(j = 1,\cdots,n)$ and $c = t_2$, then the set Λ is always a self-similar Cantor set with overlaps([7]).
(3) In particular, if we add the condition $f_{i_n}(I) \cap f_{i'_n}(I) = \emptyset$ for $i_n \neq i'_n$ in (2), then Λ becomes a Cantor set which is a perfect, compact and totally disconnected set .

We adopt notations used in [7]. Put $\Sigma = \{1,2,\cdots,l\}^{\mathbb{N}}$. For $\tau = (\tau_1,\tau_2,\cdots,\tau_n,\cdots)$, $w = (w_1,w_2,\cdots,\ w_n,\ \cdots) \in \Sigma$, write $\tau \wedge w = \tau_1,\ \tau_2,\ \cdots,\ \tau_n$, where $n = \min\{k : \tau_{k+1} \neq w_{k+1}$ for $k \geq 1\}$. If $\tau_1 \neq w_1$ then $\tau \wedge w = 0$. Define a metric ρ on Σ as $\rho(\tau,w) = r^{\tau\wedge w}$, where $r^{\tau_1,\cdots,\tau_n} = r_{\tau_1} \cdot r_{\tau_2} \cdots r_{\tau_n}$. Write $[\tau|_n]$ for a cylinder set, $[\tau_1,\cdots,\tau_n] = \{w \in \Sigma :$

$\tau_i = w_i$ for $1 \le i \le n$, $\tau_{n+1} \ne w_{n+1}\}$. The metric ρ_2 on Σ^2 is defined as $\rho_2((\tau, w),\ (\tau', w')) = \max\{\rho(\tau, \tau'),\ \rho(w, w')\}$.

For $\tau = (\tau_1, \tau_2, \cdots) \in \Sigma$, we define an onto map Π from Σ to K as

$$\Pi(\tau) = \bigcap_{n=1}^{\infty} f_{\tau_1} \circ f_{\tau_1,\tau_2} \circ \cdots \circ f_{\tau_1,\tau_2,\cdots,\tau_n}(I).$$

The number $s > 0$ with $\sum_{i=1}^{l} r_i^s = 1$ is called the similarity dimension([3]). Consider the probability measure μ on Σ with weights $(r_1^s, r_2^s, \cdots, r_l^s)$, i.e. $\mu([\tau|_n]) = r_{\tau_1}^s \cdot r_{\tau_2}^s \cdots r_{\tau_n}^s$ for any $n \ge 1$. Define the push-down measure $\nu = \mu \circ \Pi^{-1}$ on Λ and let $\mu_2 = \mu \times \mu$. Then ν and μ_2 are probability measures on Λ and $\Sigma^2 = \Sigma \times \Sigma$ respectively.

We recall the following definition of the correlation dimension([5], [7]) of $A(\subset \mathbb{R}^d)$ with respect to a probability measure η on A ;

$$D_2(A, \eta) = \sup\{\alpha \ge 0 : I_\alpha(\eta) < \infty\},$$

where $I_\alpha(\eta) = \int_A \int_A |x - y|^{-\alpha} d\eta(x) d\eta(y)$ is the α-energy of A with respect to η. In particular, if Λ is a Cantor set with overlaps and ν is the push-down probability measure on Λ, we write $D_2(\Lambda)$ for $D_2(\Lambda, \nu)$ and $I_\alpha(\nu)$ for $I_\alpha(\nu) = \int_{\Sigma^2} |\ \Pi(\tau) - \Pi(w)\ |^{-\alpha} d\mu_2$.

Denote the diameter of a set A by $|A|$. For any $\epsilon > 0$, we say that $[\tau|_n]$ is an ϵ-cylinder if $|[\tau|_n]| \le \epsilon < |[\tau|_{n-1}]|$. The set $[\tau|_n, w|_m] \equiv [\tau|_n] \times [w|_m]$ is an ϵ-cylinder in Σ^2 if both $[\tau|_n]$ and $[w|_m]$ are ϵ-cylinders in Σ. The set of ϵ-cylinders in Σ is denoted by $\mathcal{C}_\epsilon$. The collection of ϵ-cylinders $\mathcal{C}_\epsilon^2 = \mathcal{C}_\epsilon \times \mathcal{C}_\epsilon$ provides a disjoint cover of Σ^2 by sets of diameter ϵ([7]).

REMARK 2.2. (1) Σ is the only 1-cylinder.
(2) For an ϵ-cylinder $[\tau|_n]$, $r_0\epsilon \le |[\tau|_n]| \le \epsilon$ where $r_0 = \min\{r_1, \cdots, r_l\}$.
(3) The measures of ϵ-cylinders $[\tau|_n]$ and $[\tau|_n,\ w|_m]$ satisfy $(r_0)^s\epsilon^s < \mu(\ [\tau|_n]\ ) \le \epsilon^s$ and $(r_0)^{2s}\epsilon^{2s} \le \mu_2(\ [\tau|_n, w|_m]\ ) \le \epsilon^{2s}$.

Recall the upper box dimension([1]) of a bounded set K in a metric space which is denoted by $\overline{\dim}_B K$. i.e., $\overline{\dim}_B K = \limsup_{\epsilon \to 0} \frac{\log N(K,\epsilon)}{-\log \epsilon}$, where $N(K, \epsilon)$ is the smallest number of balls of diameter ϵ needed to cover K. From easy calculations(cf. [1]), we get the following result.

PROPOSITION 2.3. *For $A \subset \Sigma^2$, let $N_\epsilon(A)$ be the number of ϵ-cylinders intersecting A. Then*

$$\overline{\dim}_B(A) = \limsup_{\epsilon \to 0} \frac{\log N_\epsilon(A)}{-\log \epsilon}.$$

On account of the Proposition 2.3 and remark 2.2, we obtain the following results from standard arguments.

PROPOSITION 2.4. *$\overline{\dim}_B \Sigma^2 = 2s$ and if $A \subset \Sigma^2$ and $\overline{\dim}_B A < 2s$, then $\mu_2(A) = 0$.*

3. Results

We recall the notion of thickness([6], [7]) needed in our result.

Let $K \subset \mathbb{R}$ be a compact set and let $\widehat{K}$ be its convex hull. Then $\widehat{K} \setminus K = \cup_{i=1}^{l} E_i$, $l \leq \infty$, where E_i are complementary intervals(gaps). Enumerate the gaps so that $|E_1| \geq |E_2| \geq \cdots$. For $k \geq 1$, let F_k be the component of $\widehat{K} \setminus \cup_{i=1}^{k-1} E_i$ containing E_k. Then $F_k = F_k^l \cup E_k \cup F_k^r$ where F_k^l and F_k^r are the closed intervals adjacent to E_k.
Define

$$\theta_k = \min\left\{ \frac{|F_k^l|}{|E_k|}, \ \frac{|F_k^r|}{|E_k|} \right\}.$$

The thickness of K is defined as $\theta(K) = \inf\{\theta_k \ : \ k \geq 1\}$.

Throughout this paper, let Λ, Π, μ, ν and s be as in the Section 2.

Set $Z = \{(\tau, \ w) \in \Sigma^2 : \Pi(\tau) = \Pi(w)\}$ and $H_\epsilon = \{[\tau|_n, w|_m] \in \mathcal{C}_\epsilon^2 : [\tau|_n, w|_m] \cap Z \neq \emptyset\}$. Denote by $N_\epsilon \equiv N_\epsilon(Z)$ the cardinality of H_ϵ.
Write $\Lambda_{\tau_1,\tau_2,\cdots,\tau_n} = f_{\tau_1} \circ f_{\tau_1,\tau_2} \circ \cdots \circ f_{\tau_1,\tau_2,\cdots,\tau_n}(\Lambda)$ for $\tau = (\tau_1, \tau_2, \cdots, \tau_n, \cdots) \in \Sigma$ and $n \geq 1$.

PROPOSITION 3.1. *$D_2(\Lambda) \leq 2s - \overline{\dim}_B Z$*

PROOF. Suppose $\alpha > 2s - \overline{\dim}_B Z$. Let $[\tau|_n, \ w|_m] \in H_\epsilon$. Then $[\tau|_n, \ w|_m] \cap Z \neq \emptyset$, and so $\Lambda_{\tau_1,\tau_2,\cdots,\tau_n} \cap \Lambda_{w_1,w_2,\cdots,w_m} \neq \emptyset$. Hence for any $\tau \in [\tau|_n]$ and $w \in [w|_m]$,

$$\begin{aligned} | \ \Pi(\tau) - \Pi(w) \ | &\leq |\Lambda_{\tau_1,\tau_2,\cdots,\tau_n}| \ + \ |\Lambda_{w_1,w_2,\cdots,w_m}| \\ &= \left(r^{\tau|_n} + r^{w|_m}\right) |\Lambda| \\ &\leq 2\epsilon. \end{aligned}$$

Therefore, by Remark 2.2,

$$\begin{aligned} \int_{[\tau|_n, w|_m]} | \ \Pi(\tau) - \Pi(w) \ |^{-\alpha} \, d\mu_2 &\geq 2^{-\alpha} \epsilon^{-\alpha} \mu_2([\tau|_n, w|_m]) \\ &\geq 2^{-\alpha} \ r_0^{2s} \ \epsilon^{2s-\alpha}. \end{aligned}$$

We have

$$\begin{aligned} I_\alpha(\nu) &= \int_{\sum^2} \mid \Pi(\tau) - \Pi(w) \mid^{-\alpha} d\mu_2 \\ &\geq \sum_{H_\epsilon} \int_{[\tau|_n, w|_m]} \mid \Pi(\tau) - \Pi(w) \mid^{-\alpha} d\mu_2 \\ &\geq C_1 N_\epsilon \ \epsilon^{2s-\alpha} \end{aligned}$$

where $C_1 = 2^{-\alpha}(r_0)^{2s}$. Thus, if $\overline{\dim}_B Z = \limsup_{\epsilon \to 0} \frac{\log N_\epsilon}{-\log \epsilon} > 2s - \alpha$, then $I_\alpha(\nu) = \infty$. Hence $D_2(\Lambda) \leq 2s - \overline{\dim}_B Z$. □

LEMMA 3.2. ([6] Gap Lemma) *Let K_1 and K_2 be Cantor sets with thickness η_1 and η_2 respectively. If $\eta_1 \cdot \eta_2 > 1$, then one of the following three alternatives occurs: K_1 is contained in a gap of K_2; K_2 is contained in a gap of K_1; $K_1 \cap K_2 \neq \emptyset$.*

LEMMA 3.3. *Let $\theta(\Lambda) > 1$ and let $[\tau|_n,\ w|_m] \in \mathcal{C}_\epsilon^2 \setminus H_\epsilon$. Then for all $\alpha < s$, there exists a constant $C_2 > 0$ satisfying*

$$\int_{[\tau|_n, w|_m]} \mid \Pi(\tau) - \Pi(w) \mid^{-\alpha} d\mu_2 \leq C_2 \epsilon^{2s-\alpha}.$$

PROOF. The hypothesis $[\ \tau|_n,\ w|_m\ ] \in \mathcal{C}_\epsilon^2 \setminus H_\epsilon$ means that $[\tau|_n]$ and $[w|_m]$ are ϵ-cylinders in Σ such that $\Lambda_{\tau_1,\tau_2,\cdots,\tau_n} \cap \Lambda_{w_1,w_2,\cdots,w_m} = \emptyset$. Since $\theta(K) > 1$, by the definition of thickness, $\theta(\Lambda_{\tau_1,\tau_2,\cdots,\tau_n}) > 1$ and $\theta(\Lambda_{w_1,w_2,\cdots,w_m}) > 1$. Using the Lemma 3.2, one of the sets $\Lambda_{\tau_1,\tau_2,\cdots,\tau_n}$ and $\Lambda_{w_1,w_2,\cdots,w_m}$ lies in a connected component of the complement of the other one.
Write $\bar{1} \equiv 111\cdots$ and $\bar{l} \equiv lll\cdots$. We have

$$|\Pi(\tau) - \Pi(w)| \geq \min\{|\Pi(\tau) - \Pi(\tau|_n\bar{1})|,\ |\Pi(\tau) - \Pi(\tau|_n\bar{l})|\} \quad --(*)$$

for $\tau \in [\tau|_n],\ w \in [w|_m]$.
Let $A_k \equiv [\ \tau|_n\ 1^k\ ] \setminus [\ \tau|_n\ 1^{k+1}\ ]$, $\quad B_k \equiv [\ \tau|_n\ l^k\ ] \setminus [\ \tau|_n\ l^{k+1}\ ]$ for $k \geq 1$, and $A_0 \equiv [\ \tau|_n\ ] \setminus ([\ \tau|_n 1\ ] \cup [\ \tau|_n l\ ])$, $\quad B_0 \equiv \emptyset$. Then

$$[\ \tau|_n\ ] = \bigcup_{k=0}^{\infty} (\ A_k \cup B_k\ ).$$

Using the condition (*iii*) in the Section 2 and the Remark 2.2, we have,

$$\mid \Pi(\tau) - \Pi(\tau|_n\bar{1}) \mid \geq c\ r^{\tau|_n}\ (r_1)^k \ \geq c r_0\ \epsilon r_1^k \quad \text{for } \tau \in A_k$$

and $\quad | \Pi(\tau) - \Pi(\tau|_n \bar{l}) | \geq (1-c)\ r^{\tau|_n}(r_l)^k\ \geq (1-c)r_0\epsilon(r_l)^k$ for $\tau \in B_k$.
Take $d = \min\{cr_0,\ (1-c)r_0\}$. Then, using the inequality $(*)$, we get $|\Pi(\tau) - \Pi(w)| \geq d\epsilon\ r_1^k$ for $\tau \in A_k$ or $|\Pi(\tau) - \Pi(w)| \geq d\epsilon r_l^k$ for $\tau \in B_k$.
Clearly, for $k \geq 0$,

$$\mu(A_k) \leq \mu(\ [\tau|_n 1^k]\ ) \leq \epsilon^s\ r_1^{ks} \text{ and } \mu(B_k) \leq \epsilon^s\ (r_l)^{ks}.$$

Hence, for $\alpha < s$,

$$\begin{aligned}
&\int_{[\tau|_n, w|_m]} |\ \Pi(\tau) - \Pi(w)\ |^{-\alpha}\ d\mu_2 \\
&= \int_{[w|_m]} \sum_{k=0}^{\infty} \int_{A_k \cup B_k} |\ \Pi(\tau) - \Pi(w)\ |^{-\alpha}\ d\mu(\tau)\ d\mu(w) \\
&\leq d^{-\alpha} \int_{[w|_m]} \sum_{k=0}^{\infty} \left(\epsilon^{-\alpha} r_1^{-\alpha k}\ \epsilon^s\ r_1^{ks}\ +\ \epsilon^{-\alpha}\ r_l^{-\alpha k}\ \epsilon^s\ r_l^{ks} \right) d\mu(w) \\
&= d^{-\alpha}\ \epsilon^{s-\alpha} \int_{[w|_m]} \sum_{k=0}^{\infty} \left(r_1^{(s-\alpha)k}\ +\ r_l^{(s-\alpha)k} \right) d\mu(w) \\
&< C_2\ \epsilon^{s-\alpha} \int_{[w|_m]} d\mu(w) \\
&\leq C_2\ \epsilon^{2s-\alpha}
\end{aligned}$$

for some $C_2 > 0$. □

THEOREM 3.4. *Let $\theta(\ \Lambda\ ) > 1$. Then*

$$D_2(\Lambda) = 2s - \overline{\dim}_B Z.$$

PROOF. Owing to the Proposition 3.1, it is sufficient only to show that $D_2(\Lambda) \geq 2s - \overline{\dim}_B Z$, *i.e.* $I_\alpha(\nu) < \infty$ for any $\alpha < 2s - \overline{\dim}_B Z$.

Note that Z contains $\{(\tau, \tau) : \tau \in \Sigma\}$, hence $\overline{\dim}_B Z \geq \dim_B \Sigma = s$ and so $\alpha < s$. Let A_ϵ be the union of ϵ-cylinders in Σ^2 intersecting Z.
For $0 < \delta < 1$, consider a sequence $\{\epsilon_n\}$ such that $\epsilon_n = \delta^n$ for $n = 0, 1, 2, \cdots$. Clearly, $Z = \cap_{n=1}^{\infty} A_{\epsilon_n}$. Since $A_1 = \Sigma^2$, we have

$$\Sigma^2 = \bigcup_{n=0}^{\infty} \left(A_{\epsilon_n} \setminus A_{\epsilon_{n+1}} \right)\ \cup Z.$$

By the Lemma 3.3,

$$\begin{aligned}
I_\alpha(\nu) &= \int_{\Sigma^2} \mid \Pi(\tau) - \Pi(w) \mid^{-\alpha} d\mu_2 \\
&= \sum_{n=0}^{\infty} \int_{A_{\epsilon_n} \setminus A_{\epsilon_{n+1}}} \mid \Pi(\tau) - \Pi(w) \mid^{-\alpha} d\mu_2 + \int_Z \mid \Pi(\tau) - \Pi(w) \mid^{-\alpha} d\mu_2 \\
&\leq C_2 \sum_{n=0}^{\infty} N_{\epsilon_{n+1}} \, \epsilon_{n+1}^{2s-\alpha} + \int_Z \mid \Pi(\tau) - \Pi(w) \mid^{-\alpha} d\mu_2.
\end{aligned}$$

We may assume that $\overline{\dim}_B Z < 2s$. Then $\mu_2(Z) = 0$ by Proposition 2.4. Since $\alpha < 2s - \overline{\dim}_B Z$ *i.e.* $\limsup_{n\to\infty} N_{\delta^n}/-n\log\delta < 2s - \alpha$, we have for $\beta > 0$, $N_{\delta^n} \leq \delta^{n(\alpha+\beta-2s)}$ for all n.
Therefore, for all $\alpha < 2s - \overline{\dim}_B Z$,

$$\begin{aligned}
I_\alpha(\nu) &\leq C_2 \sum_{n=0}^{\infty} N_{\epsilon_{n+1}} \epsilon_{n+1}^{2s} \\
&\leq C_2 \sum_{n=0}^{\infty} \delta^{(n+1)(\alpha+\beta-2s)} \delta^{(n+1)(2s-\alpha)} \\
&= C_2 \sum_{n=0}^{\infty} \delta^{\beta(n+1)} \; < \; \infty.
\end{aligned}$$

This completes the proof of Theorem. □

REMARK 3.5. As we apply Theorem 3.4 to three cases in the Remark 2.1, we obtain the following results.
(1) In case of a self-similar set or loosely self-similar set Λ in $\mathbb{R}^1$, by the disjoint property of basic sets of Λ, we obtain $\overline{\dim}_B Z = s$. Hence Theorem 3.4 comes to the same conclusion in [5] for the push-down measure ν on Λ, that is,

$$D_2(\Lambda) = s = \dim_H \Lambda$$

where $\dim_H \Lambda$ is denoted the Hausdorff dimension of Λ.
(2) In case of a self-similar Cantor set with overlaps([7]), Theorem 3.4 is also true.

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