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CONFORMAL DENSITY OF VISIBILITY MANIFOLD

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ABSTRACT. In this paper, we prove the existence and uniqueness of a $\delta(\Gamma)$-conformal density on the limit set of Γ acting on visibility manifold H for a Fuchsian group Γ.

1. Introduction

Let H be an n-dimensional complete simply connected Riemannian manifold without conjugate points and Γ is a discrete group of isometries on H, which acts on H freely and properly discontinuously. It is well known that H is diffeomorphic to an open disc D^n and the boundary of H at infinity is homeomorphic to a sphere S^n.

When we look at an orbit Γx of Γ in H and think of each orbit point and H as a star and a sky respectively, we can see a something like a galaxy at infinity of the sky, which is called a limit set of Γ and is denoted by $L(\Gamma)$. It is an interesting topic for geometers to get the information about the geometry of the quotient manifold $M = H/\Gamma$ by looking at this set $L(\Gamma)$. To do this, it has been an important work to present a class of measures on $L(\Gamma)$, which has been developed by Patterson, Sullivan and others.

In [9], Patterson constructs a class of measures on $L(\Gamma)$ on 2-dimensional manifold with a constant curvature -1. And Patterson showed the uniqueness of the measures using ergodic theory. His construction was extended by Sullivan to the case that the sectional curvature of H is constant -1 in all dimensions in [11]. The measure constructed by Patterson and Sullivan is called **Patterson-Sullivan measure**. Moreover, Sullivan introduced the conformal density that generalized the Patterson-Sullivan measure and showed many results related to Hausdorff dimension of the Limit set and

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estimates on the orbital counting function. In [12], Yue performed the same construction on the manifold with strictly negative curvature. Using the general notation on a Riemannian manifold, Yue showed that the Patterson Sullivan *measure on his manifold was a conformal density related to the* Busemann function. And Yue extend the Sullivan's results to the manifold with strictly negative curvature.

In this paper, the author extend some of Sullivan's results to the more general class of hyperbolic manifolds. One of the most important properties of the Hyperbolic Manifold with the curvature -1 is that any two points x and y in ∂H is joined by a geodesic line y. Eberlein defined a manifold with this property, called **visibility manifold**, and he proved many results in [4] and [5]. Of course, a manifold with strictly negative curvature is a visibility manifold. We follow the notations in [4] and [5].

It is said that H satisfies the **visibility axiom** if H has the following property;

for every point $p \in H$ *and every real number* $\epsilon > 0$, *there is* $R = R(p, \epsilon) > 0$ *such that if* $\gamma : [a, b] \to H$ *is a geodesic segment with* $d(p, \gamma) \geq R$, *then* $\angle_p(\gamma) \leq \epsilon$ *where* $d(p, \gamma) = \inf\{ d(p, \gamma(t)) \,|\, t \in \mathbb{R} \}$ *and* $\angle_p(\gamma) = \sup\{ \angle_p(\gamma(t), \gamma(s)) \,|\, t, s \in \mathbb{R} \}$.

Roughly speaking, visibility axiom means that geodesic far away look small. The visibility axiom is equivalent that for any two points x and y in ∂H there exists a geodesic line between x and y [1]. This definition means that the geodesic lines between x and y may be more one. If there are one more geodesic lines between x and y, two geodesic lines between x and y bounds a flat strip. H may be allowed some parts with the curvature 0. The **uniform visibility axiom** on H is to choose a constant $R = R(\epsilon)$ independent of $p \in H$ in the definition of visibility axiom. Then H can get only some flat strip not a flat half plane.

DEFINITION 1.1. Suppose M is a complete Riemannian manifold without any conjugate points. If the universal cover H of M satisfies the uniform visibility axiom, we call M a visibility manifold.

We want to note that a visibility manifold M has no assumption about a curvature.

Suppose H satisfies the uniform visibility axiom. For any point x in H, consider the orbit Γx of x and its closure $\overline{\Gamma x}$. The limit set of Γ is defined by $L(\Gamma) = \overline{\Gamma x} \cap \partial H$. We can consider that Γ is a set of a isometry on H. Then we extend the Γ-action on H to $\bar{H} = H \cup \partial H$ and we can get a Γ-action on ∂H.

REMARK 1.2. ([5])

(1) (the topological trichotomy) One of the following possibilities must occur ;

(i) $L(\Gamma)$ is a singleton, (ii) $L(\Gamma)$ consists of two points (iii) $L(\Gamma)$ is infinite.

(2) If $L(\Gamma)$ is infinite, a set of fixed points of hyperbolic elements in Γ is dense in $L(\Gamma)$

(3) If $L(\Gamma)$ is infinite, the Γ-orbit of is dense in $L(\Gamma)$.

(4) If $L(\Gamma)$ is infinite, $L(\Gamma)$ is a perfect subset of ∂H. Either $L(\Gamma) = \partial H$ or $L(\Gamma)$ is nowhere dense in ∂H.

Γ is called **Fuchsian** if its limit set satisfies (iii) of (1), that is, $L(\Gamma)$ is infinite. From now on we always assume Γ to be a Fuchsian group through our paper. We define a set in $L(\Gamma)$, that is extremely useful when studying the properties of measure on $L(\Gamma)$. The radial limit set $L^r(\Gamma)$ is the set of all $\eta \in L(\Gamma)$ such that any geodesic ray joining $x \in H$ and η intersects some ϵ-neighborhood of Γx infinitely many times. Obviously $L^r(\Gamma)$ is non empty and hence it is dense in $L(\Gamma)$ by Remark 1.2.

Let $x, y \in H$ and $\psi \in \partial H$. Busemann function $\rho_{x,\psi} : H \to \mathbb{R}$ is defined by

$$\rho_{x,\psi}(y) = \lim_{t\to\infty}(c(t) - d(y, c(t))),$$

where c is a geodesic from x to ψ and $d(.,.)$ is the Riemannian metric.

DEFINITION 1.3. Let $\alpha > 0$. A family of finite Borel measures $\{\sigma_x\}_{x\in H}$ on ∂H is called an α-conformal density if $\{\sigma_x\}_{x\in H}$ satisfies the followings;

(1) σ_x is supported on the limit set of Γ

(2) for $x, x' \in H$, $\sigma_x, \sigma_{x'}$ are absolutely continuous with respect to each other and the Radon-Nikodym derivative satisfies

$$\left[\frac{d\sigma_x}{d\sigma_{x'}}\right](\psi) = \exp(-\alpha\rho_{x,\psi}(x'))$$

(3) $\gamma^*\sigma_x = \sigma_{\gamma(x)}$, for $\gamma \in \Gamma$.

In Section 2 we construct a conformal density on our visibility manifold and showed when a conformal density is unique. The construction is the same as that of Patterson's. When Sullivan and Yue proved the properties including the uniqueness of the conformal density, they used the property of the curvature in their manifold strongly. So the author modified the method without using any curvature property.

2. Conformal density

In order to get a conformal density on H we generalize Patterson's construction. He has constructed a conformal density for any Fuchsian group Γ acting on a negatively curved surface. For positive real number s and two fixed points x, y in H, we consider the Poincare series

$$g_s(x,y) = \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma y)},$$

where $d(x,\gamma y)$ is the hyperbolic distance in H [9], [11], [12]. Then there is a positive number $\delta(\Gamma)$ such that $g_s(x,y)$ diverges for $s < \delta(\Gamma)$ and $g_s(x,y)$ converges for $s > \delta(\Gamma)$. Using the triangle inequality, it is easy to see that $e^{-sd(x,y)}g_s(y,y) \leq g_s(x,y) \leq e^{sd(x,y)}g_s(y,y)$, which implies that $\delta(\Gamma)$ is independent of the choice of $x, y \in H$.

Consider the family of measures

$$\mu_x^s = \frac{1}{g_s(y,y)} \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma y)} \delta_{\gamma y}, \qquad s > \delta(\Gamma),$$

where $\delta_{\gamma y}$ is the Dirac mass at γy. Thus $\{\mu_x^s\}_{s>\delta(\Gamma)}$ is a family of finite measures with uniformly bounded total mass. Let $\mu_x = \lim_{s\to\delta(\Gamma)+} \mu_x^s$ be a weak limit in the space of uniformly bounded measures on $H \cup \partial H$. Note that μ_x^s is concentrated on $L(\Gamma)$ for $s > \delta(\Gamma)$. When at $s = \delta(\Gamma)$ μ_x^s diverges, Γ is of divergence type. Otherwise, Γ is of convergence type. We assume that Γ is of divergence type. Then μ_x is concentrated on $L(\Gamma)$. Without this restriction, we can construct the conformal density by adding a slowly increasing weight as in [9]. From the construction, it is easy to see that for any other point $x' \in H$ the limit $\lim_{s\to\delta(\Gamma)+} \mu_{x'}^s = \mu_{x'}$ also exists and moreover the Radon-Nikodym derivative at $\xi \in L(\Gamma)$ satisfies

$$\frac{d\mu_{x'}}{d\mu_x}(\psi) = e^{-\delta(\Gamma)\rho_{x,\xi}(x')}.$$

The proof of the following theorem is similar to that in [12]. But we describe the proof in detail.

THEOREM 2.1. *For any Fuchsian group Γ, $\delta(\Gamma) > 0$, and there exists a $\delta(\Gamma)$-conformal density.*

Proof. First, we show the existence of $\delta(\Gamma)$-conformal density. Since we showed that $\{\mu_x\}$ satisfies (1) and (2) in the Definition 1.3, it is sufficient only to show that

$$V^*\mu_x = \mu_{V(x)}, \tag{2.1}$$

for all $V \in \Gamma$.

Let E be a Borel measurable subset in H and $s > \delta(\Gamma)$.

$$V^*\mu_x^s(E) = \mu_x^s(V(E)) = \frac{1}{g_s(y,y)} \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma y)} 1_{V(E)}(\gamma y),$$

where 1_E is the characteristic function on E. Set $\eta = V^{-1}\gamma$ and note that $\gamma(y) \in V(E)$ if and only if $V^{-1}\gamma(y) \in E$. As γ runs over Γ, η also runs over Γ and we have

$$\mu_x^s(V(E)) = \frac{1}{g_s(y,y)} \sum_{\eta\in\Gamma} e^{-sd(V^{-1}x,\eta y)} 1_{V(E)}(\eta y) = \mu_{V^{-1}x,s}(E).$$

Then $\{\mu_x\}_{x\in H}$ is a $\delta(\Gamma)$-conformal density.

Next, we show that $\delta(\Gamma) > 0$. If the critical exponent $\delta(\Gamma) = 0$ then for all $x, x' \in H$, $\mu_x = \mu_{x'}$. The above (2.1) implies that for every Borel set $E \subset \partial H$ and every $\gamma \in \Gamma$, $\mu_x(E) = \mu_x(\gamma(E))$. So μ_x is a finite Γ-invariant measure on the limit set. Let E be a Borel measurable set in $H(\infty)$ with positive measure. Since Γ is Fuchsian, we can suppose that E has two distinct points, say $\eta, \xi \in L(\Gamma)$, which are fixed by hyperbolic isometries in Γ. Let γ_1, γ_2 be hyperbolic isometries in Γ fixing η, ξ, respectively. Then we can choose a integer $n > 0$ such that $\gamma_1^n(E)$ and $\gamma_2^n(E)$ are disjoint subset of E [4]. Then we have

$$\mu_x(E) \geq \mu_x(\gamma_1^n(E)) + \mu_x(\gamma_2^n(E)) = 2\mu_x(E) > 0,$$

which is a contradiction. □

The weak limit in Patterson's construction is by no means unique. The following statement means that uniqueness is closely related to ergodic theory. The proof is in [12].

THEOREM 2.2. *Let $\{\mu_x\}_{x\in H}$ be any α-conformal density of Γ. Then any other α-conformal density $\{\nu_x\}_{x\in H}$ coincides with μ up to a scalar multiplication if and only if the Γ action on ∂H is ergodic with respect to the measure class defined by μ.*

We consider when the conformal density is unique. Using the theorem 2.2, it is sufficient when the Γ action on ∂H is ergodic with respect to the measure class $\{\mu_x\}$.

Fix a point $x_0 \in H$. For any x in H and $d > 0$, consider the shadow of the ball $B(x,d)$ from x_0 to ∂H defined by

$$O_{x_0}(x,d) = \{\eta \in \partial H \mid c_{x_0,\eta} \cap B(x,d) \neq \emptyset\},$$

where $B(x_0,d)$ is a geodesic ball with a center x_0 and a radius d and $c_{x_0,\eta}$ is the geodesic ray from x_0 to η. In [11], Sullivan proved the following Theorem 2.3, which is called the Sullivan's shadow lemma. This theorem has given much useful information on the local structure of a conformal density. We prove Theorem 2.3 by using only the uniform visibility axiom.

THEOREM 2.3. *Let $\{\mu_x\}_{x\in M}$ be a α-conformal density of Γ and $x_0 \in H$. Suppose μ_{x_0} does not consist of a single atom. Then there is $C_1 \geq 1$ and $b_0 \geq 0$ such that for all $b \geq b_0$ and*

$$C_1^{-1} e^{-\alpha d(x_0,\gamma^{-1}x_0)} \leq \mu_{x_0}(O_{x_0}(\gamma^{-1}x_0, b)) \leq C_1 e^{-\alpha d(x_0,\gamma^{-1}x_0)+2b\alpha}.$$

Proof. Let m_0 be the largest mass of atoms with respect to μ_{x_0}. Let $A := \mu_{x_0}(\partial H)$. Since μ_{x_0} does not consist of a single atom, we can choose $m > 0$ such that $m_0 \leq m < A$. By the definition of m, there is $\delta > 0$ such that for all set $D \subset \partial H$ with a angle at x_0 less than δ, $\mu_{x_0}(D) < m$.

Let $\gamma \in \Gamma$ be a isometry in H. Choose two points $\xi, \eta \in \gamma(\partial\tilde{M} - O_{x_0}(\gamma^{-1}x_0, b)) = \partial\tilde{M} - \gamma(O_{x_0}(\gamma^{-1}x_0, b)$. Then we have $\gamma^{-1}\xi, \gamma^{-1}\eta \in \partial\tilde{M} - O_{x_0}(\gamma^{-1}x_0, b)$. Let g_1 and g_2 be geodesic rays from γx_0 to ξ and η, respectively. Then $\gamma^{-1}g_1, \gamma^{-1}g_2$ are geodesic rays from x_0 to $\gamma^{-1}\xi, \gamma^{-1}\eta$, respectively. Therefore $d(\gamma^{-1}g_1, \gamma^{-1}x_0) \geq b$ and $d(\gamma^{-1}g_2, \gamma^{-1}x_0) \geq b$.

By the definition of visibility axiom, there exists $b_0 > 0$ such that for all geodesic $c : \mathbb{R} \to H$ with $d(x_0, c) \geq b_0$, then $\angle_{x_0}(c) < \frac{\delta}{2}$. Let $b \geq b_0$ be chosen. Since $d(\gamma^{-1}g_1, \gamma^{-1}x_0) \geq b$ and $d(\gamma^{-1}g_2, \gamma^{-1}x_0) \geq b$, we have $\angle_{x_0}(\gamma x_0, \xi) < \frac{\delta}{2}$ and $\angle_{x_0}(\gamma x_0, \eta) < \frac{\delta}{2}$. Then we get for all $\xi, \eta \in \partial\tilde{M} - \gamma(O_{x_0}(\gamma^{-1}x_0, b)$,

$$\angle_{x_0}(\xi,\eta) \leq [\,\angle_{x_0}(\gamma x_0, \xi) + \angle_{x_0}(\gamma x_0, \eta)\,] < \delta.$$

And $\angle_{x_0}(\partial\tilde{M}-\gamma(O_{x_0}(\gamma^{-1}x_0,b))\leq\delta$ and $\mu_{x_0}(\partial\tilde{M}-\gamma(O_{x_0}(\gamma^{-1}x_0,b))<m$. Therefore we get

$$A-m\leq\mu_{x_0}(\gamma(O_{x_0}(\gamma^{-1}x_0,b))\leq A. \tag{2.2}$$

On the other hand, The definition (3)and (2) of α-conformal density of Γ says that

$$\mu_{x_0}(\gamma(O_{x_0}(\gamma^{-1}x_0,b))=\mu_{\gamma^{-1}x_0}(O_{x_0}(\gamma^{-1}x_0,b)),$$

and

$$\frac{d\mu_{x_0}}{d\mu_{\gamma^{-1}x_0}}(\eta)=e^{\rho_{\gamma^{-1}x_0,\eta}(x_0)}.$$

By the definition of Busemann function $\rho_{x,\eta(y)}$ and the uniform visibility axiom of H, we get

$$d(x_0,\gamma^{-1}x_0)-2b\leq\rho_{\gamma^{-1}x_0,\eta(x_0)}\leq d(x_0,\gamma^{-1}x_0).$$

Therefore we get

$$e^{-\alpha d(x_0,\gamma^{-1}x_0)}\leq\frac{\mu_{x_0}(O_{x_0}(\gamma^{-1}x_0,b))}{\mu_{\gamma^{-1}x_0}(O_{x_0}(\gamma^{-1}x_0,b))}\leq e^{-\alpha d(x_0,\gamma^{-1}x_0)+2b\alpha}$$

This inequality and (2.2) implies our theorem. □

We can explain the radial limit set in ∂H in the relation with the shadow of $O_{x_0}(x,d)$ of a ball $B(x,d)$ from x_0. $\zeta\in\partial H$ is a radial limit point if and only if for some $c>0$ and $x\in H$, ζ belongs to infinitely many shadows $O_x(\gamma x,c)$, for $\gamma\in\Gamma$. We denote $L^r(\Gamma)$ the radial limit set. As we show the below properties of conformal density, the measurement of the radial limit set is important for them.

THEOREM 2.4. *Let μ be a α-conformal density of Γ and let x be any point in H. If $\sum_{\gamma\in\Gamma}e^{-\alpha d(x,\gamma^{-1}x)}<\infty$, then $\mu_x(L^r(\Gamma))=0$.*

Proof. Since Γ is a Fuchsian group, we can let $\Gamma=\{\gamma_n\}_{n\in\mathbb{N}}$. For given $\epsilon>0$, there exists $N>0$ such that $\sum_{n\geq N}e^{-\alpha d(x,\gamma_n^{-1}x)}\leq\epsilon$. By Theorem 2.3, we have a constant $b_0>0$ and $C_2>0$ such that for $b\geq b_0$

$$\sum_{n\geq N}\mu_x(O_x(\gamma_n^{-1}x,b))\leq C_1e^{-\alpha d(x_0,\gamma^{-1}x_0)+2b\alpha}\leq C_2e^{2b\alpha}\epsilon.$$

But for $0 < b < b_0$ we also have

$$\sum_{n\geq N} \mu_x(O_x(\gamma_n^{-1}x, b)) \leq \sum_{n\geq N} \mu_x(O_x(\gamma_n^{-1}x, b_0)) \leq C_1 e^{2b_0\alpha}\epsilon.$$

By Borel-Cantelli Lemma, we have $\mu_x[\cap_{N\geq 1} \cup_{n>N} O_x(\gamma_n^{-1}x, b)] = 0$, for all $b > 0$. Since $L^r(\Gamma) = \cup_{b>0}[\cap_{N\geq 1} \cup_{n>N} O_x(\gamma_n^{-1}x, b)]$, we prove

$$\mu_x(L^r(\Gamma)) = \mu_x\left(\cup_{b>0}[\cap_{N\geq 1} \cup_{n>N} O_x(\gamma_n^{-1}x, b)]\right) = 0. \qquad \square$$

Now, we show that a radial limit point can never be a atom with point mass.

LEMMA 2.5. *A radial limit point $\zeta \in L^r(\Gamma)$ cannot be the atom of any α-conformal density μ of Γ.*

Proof. Assume that $\eta \in \partial H$ is a radial limit point and a atom. There exists a strictly increasing sequence $\{\gamma_i\}$ in Γ and $x \in H$ so that $\lim_{i\to\infty} e^{-\rho_{x,\eta_i}(\gamma x)} = \infty$, Consider the stabilizer $\Gamma_\eta = \{\gamma \in G \,|\, \gamma(\eta) = \eta\}$ of η. Then an argument that is similar to that in the proof in Theorem 2.1 shows that Γ_η has no hyperbolic element.

First, suppose that Γ_η has no parabolic element. Then η is not parabolic. Then Γ is composed of entirely elliptic elements, thus has finite order. Since Γ is a torsion free, Γ_η has only the identity element. Then we have

$$\sum_{\gamma\in\Gamma} e^{-\alpha\rho_{x,\eta(\gamma x)}} = \sum_{\gamma\in\Gamma} \frac{\mu_x(\gamma^{-1}\eta)}{\mu_x(\eta)} \leq \frac{\mu_x(\partial H)}{\mu_x(\eta)} < \infty.$$

This is a contradiction.

Next, suppose that η is a parabolic, then the stabilizer Γ_η of η preserves all horospheres centered at η. Then the sequence $\{\gamma_i\}$ contains no two elements from the same coset of Γ/Γ_η.

$$\sum_{\gamma\in\Gamma} e^{-\alpha\rho_{x,\eta(\gamma x)}} = \sum_{\gamma\in\Gamma} \frac{\mu_x(\gamma^{-1}\eta)}{\mu_x(\eta)} \leq \frac{\mu_x(\partial H)}{\mu_x(\eta)} < \infty,$$

which is a contradiction. $\square$

THEOREM 2.6. *Let μ be an α-conformal density of Γ. If A is a Γ-invariant subset of $L^r(\Gamma)$. Then either $\mu_x(A) = 0$ or $\mu_x(A) = \mu_x(\partial H)$.*

Proof. Suppose $\mu_x(A) > 0$. Then μ-almost all $\xi \in A$ is a density point. There exist $b_0 > 0$ and $\{\gamma_n^{-1}x\}$ converging to ξ radially such that for all $b > b_0$

$$\lim_{n\to\infty} \frac{\mu_x(O_x(\gamma_n^{-1}x, b) \cap A)}{\mu_x(O_x(\gamma_n^{-1}x, b))} = 1.$$

Let m_0 be the largest point mass of μ_x. Using the similar argument to the proof in Theorem 2.3, for given $\epsilon > 0$ and for sufficiently large n and sufficiently large $b > 0$, we have

$$\mu_x(\partial H - O_{\gamma_n x}(x, b)) = \mu_x(\partial H) - \mu_x(O_{\gamma_n x}(x, b)) \leq m_0 + \epsilon.$$

Since $\mu_x(O_{\gamma_n x}(x, b)) = \mu_{\gamma_n x}(O_x(\gamma_n x, b))$, we have

$$\mu_{\gamma_n x}(O_x(\gamma_n x, b)) \geq \mu_x(\partial H) - m_0 - \epsilon.$$

Furthermore we get

$$\begin{aligned}
&\frac{\mu_{\gamma_n^{-1}x}(O_x(\gamma_n^{-1}x, b) \cap A)}{\mu_{\gamma_n^{-1}x}(O_x(\gamma_n^{-1}x, b))} \\
=&1 - \frac{\int_{O_x(\gamma_n^{-1}x, b) - A} 1 d\mu_{\gamma_n^{-1}x}}{\int_{O_x(\gamma_n^{-1}x, b)} 1 d\mu_{\gamma_n^{-1}x}} \\
=&1 - \frac{\int_{O_x(\gamma_n^{-1}x, b) - A} e^{-\alpha\rho_{x,\psi}(\gamma_n^{-1}x)} d\mu_x}{\int_{O_x(\gamma_n^{-1}x, b)} e^{-\alpha\rho_{x,\psi}(\gamma_n^{-1}x)} d\mu_x}.
\end{aligned}$$

Note that for all $\eta, \psi \in O_x(\gamma_n^{-1}x, b)$, we can get some constant $C > 0$ such that

$$-C \leq \rho_{x,\eta(\gamma_n^{-1})} - \rho_{x,\psi(\gamma_n^{-1})} \leq C.$$

Therefore we have

$$\begin{aligned}
&\frac{\mu_{\gamma_n^{-1}x}(O_x(\gamma_n^{-1}x, b) \cap A)}{\mu_{\gamma_n^{-1}x}(O_x(\gamma_n^{-1}x, b))} \\
\leq&1 - e^{2C} \times \frac{\mu_x(O_x(\gamma_n^{-1}x, b) - A)}{\mu_x(O_x(\gamma_n^{-1}x, b))} \\
\leq&1 - \epsilon,
\end{aligned}$$

for sufficiently large n.

We get for all $\epsilon > 0$,

$$\begin{aligned}\mu_x(A) &\geq \mu_x(O_{\gamma_n x}(x,b) \cap A) \\ &\leq (1-\epsilon)\mu_{\gamma_n^{-1}x}(O_x(\gamma_n^{-1}x,b)) \\ &= (1-\epsilon)[(\mu_x(\partial H)) - m_0 - \epsilon]. \end{aligned} \tag{2.3}$$

If μ_x has any atom, it has at least two since Γ is non elementary, and (2.3) implies A must have a atom, which is a contradiction to Lemma 2.5, that is, $A \subset L^r(\Gamma)$ has no atom. □

By Theorem 2.6, we can say that Γ is **ergodic** on ∂H with respect to the conformal measure class. Note that Γ is ergodic on ∂H with respect to the conformal measure μ if any invariant Borel set $A \subset \partial H$ under Γ then for all $x \in H$ either $\mu_x(A) = 0$ or $\mu_x(A^c) = 0$ for all $x \in H$.

If $x, y \in H$ and $r > 0$, the orbital counting function $N(r,x,y)$ can be defined by

$$N(r,x,y) = \#\{\gamma \in \Gamma \,|\, d(x,\gamma y) < r\}.$$

Using the Lemma 2.3, we can control the orbital counting function. And Lemma 2.7 play a important role to prove the uniqueness of conformal density, that is, we have only $\delta(\Gamma)$-conformal density.

LEMMA 2.7. *For any $x \in H$ and $\gamma \in \Gamma$, there exists a constant $C = C(x) > 0$ and $r_0 > 0$ so that $N(r,x,\gamma x) \leq Ce^{\delta(\Gamma)r}$, for all $r \geq r_0$.*

Proof. Let $\Gamma_k = \{\gamma \in \Gamma | k-1 < d(x,\gamma^{-1}x) \leq k\}$ and $S_k = \#\Gamma_k$ for all integers $k > 0$. Then we have that for any $n > 0$, $N(n,x) \leq S_1 + \cdots + S_n$. First, we have to show that there exists $C_2 = C_2(x) > 0$ such that for all $\eta \in \partial \tilde{M}$,

$$\#\{\gamma \in \Gamma_k \,|\, \eta \in O_x(\gamma^{-1}x, d_0)\} \leq C_2,$$

where d_0 is the constant in Theorem 2.3. That is why this means that $\{O_x(\gamma^{-1}x,d_0) \,|\, \gamma \in \Gamma_k\}$ covers $\cup_{\gamma \in \Gamma_k} O_x(\gamma^{-1}x, d_0)$ at most C_2-times.

For all $\eta \in \partial H$ we choose $\gamma_1, \gamma_2 \in \Gamma_k$ so that $\eta \in O_x(\gamma_1^{-1}x,d_0) \cap O_x(\gamma_2^{-1}x,d_0)$. Choose a geodesic ray c from x to η. By the definition of $O_x(\gamma^{-1}x,d_0)$, there are two real numbers $t_1, t_2 > 0$ such that $d(c(t_1),\gamma_1^{-1}x) < d_0$ and $d(c(t_2),\gamma_2^{-1}x) < d_0$. Then for $i = 1,2$, we have

$$d(\gamma_i^{-1}x,x) - d(c(t_i),\gamma_i^{-1}x) \leq d(c(t_i),x) \leq d(\gamma_i^{-1}x,x) + d(c(t_i),\gamma_i^{-1}x).$$

Since γ_1 and γ_2 are in Γ_k, we can get $k-1 \leq d(x,\gamma_i^{-1}x) \leq k$, for $i=1,2$. And $k - 1 - d_0 \leq d(x,c(t_i)) \leq k + d_0$. Since $c(t_i)$ lies on the ray between x and η, we have

$$d(c(t_1),c(t_2)) \leq |d(c(t_1),x) - d(c(t_2),x)| \leq 2d_0 + 1. \tag{2.4}$$

Therefore (2.4) implies

$d(\gamma_1^{-1}x, \gamma_2^{-1}x) \leq [\, d(\gamma_1^{-1}x, c(t_1)) + d(c(t_1), c(t_2)) + d(c(t_2), \gamma_2^{-1}x)\,] \leq 4d_0 + 1.$

Since Γ acts on H properly discontinuously, we can get a constant $C_2(x) = C_2 > 0$ independent of γ such that $\{\gamma_i \in \Gamma_k | \gamma_i x \in B(\gamma^{-1}x, 4d_0 + 1)\} \leq C_2$. Therefore we have

$$\#\{\gamma \in \Gamma_k | \eta \in O_x(\gamma^{-1}x, d_0)\} \leq C_2.$$

Note that for any integer $n > 0$, $N(n, x) \leq S_1 + \cdots + S_n$. For all $1 \leq k \leq n$, we get

$$\begin{aligned} S_k C_1^{-1} e^{-\delta(\Gamma) d(x, \gamma^{-1}x)} &\leq \sum_{\gamma \in \Gamma_k} \mu_x(O_x(\gamma^{-1}x, d_0)) \\ &\leq C_2 \mu_x(\cup_{\gamma \in \Gamma_k} O_x(\gamma^{-1}x, d_0)) \\ &\leq C_2 \mu_x(\partial H), \end{aligned} \tag{2.5}$$

where $C_1 > 0$ is the constant in Theorem 2.3. And we can get $S_k \leq C_1 C_2 \mu_x(\partial H) e^{\delta(\Gamma) k}$, by (2.5), which complete our theorem. □

Lemma 2.7 and the proof modified a theorem in [2]. Coornaert proved a theorem like Lemma 2.7 on a tree.

COROLLARY 2.8. *Suppose that there exists α-conformal density of Fuchsian group Γ with $\mu_x(L^r(\Gamma)) > 0$. Then*

(1) $\mu_x(L^r(\Gamma)) = \mu_x(\partial H)$

(2) $\alpha = \delta(\Gamma)$

(3) *μ is the unique $\delta(\Gamma)$-conformal density of Γ and Γ is ergodic on H with respect to μ.*

(4) *Γ is a divergent type.*

Proof. (1) It is clear by Theorem 2.6.

(2) By Theorem 2.4 and $\mu_x(L^r(\Gamma)) > 0$, $\sum_{\gamma \in \Gamma} e^{-\alpha(x, \gamma x)} = \infty$, we know that $\alpha \leq \delta(\Gamma)$. On the other hand, Lemma 2.7 implies there exists r_0 such that for all $r \geq r_0$, $N(r, x) \leq C$, where C is depending only on Γ and r in Lemma 2.7. Then for all $s \geq \alpha$,

$$\begin{aligned} \sum_{\gamma \in \Gamma} e^{d(x, \gamma x)} &= \lim_{R \to \infty} \int_0^R e^{-st} dN(t, x) \\ &= \lim_{R \to \infty} [N(R, x) e^{-sR} + s \int_0^R e^{-st} N(t, x) dt] \\ &\leq \lim_{R \to \infty} [C_3 e^{R(\alpha - s)} + s \int_0^R e^{t(\alpha - s)} dt] \\ &< \infty. \end{aligned}$$

Therefore, we can get $s \geq \delta(\Gamma)$.

(3) That follows from Theorem 2.2 and Theorem 2.6.

(4) That follows from Theorem 2.5. □

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