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ALMOST REGULAR OPERATORS ARE REGULAR

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ABSTRACT. We give a characterization of regular operators that allows us to prove that a bounded operator acting between Banach spaces is almost regular if and only if it is regular, solving an open problem in [5]. As an application, we show that some operators in the closure of the set of all regular operators are regular.

Recently, Lee and Choi [5] introduced a concept of almost regular operators, following a suggestion in [3, Preface]. They proved that if X and Y are Hilbert spaces, then $T \in L(X,Y)$ is almost regular if and only if T is regular. However, for X and Y incomplete normed spaces, they gave an example of an almost regular operator which is not regular. In the case that X and Y are Banach spaces they propose as an open problem whether almost regular operators and regular operators coincide.

Here we give a positive answer to this problem.

Along the paper, X and Y denote real or complex Banach spaces and $L(X,Y)$ the set of all (bounded linear) operators acting from X into Y. For every $T \in L(X,Y)$ we denote by $R(T)$ and $N(T)$ the range and the kernel of T, respectively.

DEFINITION 1. An operator $T \in L(X,Y)$ is called *almost regular* if there exists a bounded sequence $\{A_n\} \subset L(Y,X)$ such that

$$\|TA_nT - T\| \to 0 \text{ as } n \to \infty.$$

The operator $T \in L(X,Y)$ is called *regular* if there exists $A \in L(Y,X)$ such that $TAT = T$.

It is clear that regular operators are almost regular and that regular operators have close range.

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Recall that a subspace M of a Banach space X is said to be *complemented* if there exists a projection $P \in L(X, X)$ such that $R(P) = M$. In particular, complemented subspaces are closed.

REMARK 1. An operator $T \in L(X, Y)$ is regular if and only if $N(T)$ and $R(T)$ are complemented.

Indeed, if there is $A \in L(Y, X)$ such that $TAT = T$, then it is easy to check that TA is a projection with range $R(TA) = R(T)$ and AT is a projection with kernel $N(AT) = N(T)$.

On the other hand, if $R(T)$ and $N(T)$ are complemented, then T has a continuous inverse $T|_M^{-1}$ on any closed complement M of $N(T)$. We can define $A \in L(Y, X)$ equal to $T|_M^{-1}$ on $R(T)$ and equal to 0 in a fixed closed complement of $R(T)$, and we obtain $TAT = T$.

We recall some concepts about ultrapowers of Banach spaces and operators. See [4] for more information.

We fix a non-trivial ultrafilter $\mathcal{U}$ on the set $\mathbb{N}$ of all positive integers. For every Banach space X, we consider the Banach space $\ell_\infty(X)$ of all bounded sequences (x_i) in X, endowed with the norm $\|(x_i)\|_\infty := \sup\{\|x_i\| : i \in \mathbb{N}\}$. Let $N_\mathcal{U}(X)$ be the closed subspace of all sequences $(x_i) \in \ell_\infty(X)$ which converge to 0 following $\mathcal{U}$. The *ultrapower of X following $\mathcal{U}$* is defined as the quotient

$$X_\mathcal{U} := \frac{\ell_\infty(X)}{N_\mathcal{U}(X)}.$$

The element of $X_\mathcal{U}$ including the sequence $(x_i) \in \ell_\infty(X)$ as a representative is denoted by $[x_i]$. Its norm in $X_\mathcal{U}$ is given by

$$\|[x_i]\| = \lim_\mathcal{U} \|x_i\|.$$

The constant sequences generate a subspace of $X_\mathcal{U}$ isometric to X. So we can consider the space X embedded in $X_\mathcal{U}$. Moreover, every operator $T \in L(X, Y)$ admits an extension $T_\mathcal{U} \in L(X_\mathcal{U}, Y_\mathcal{U})$, defined by

$$T_\mathcal{U}([x_i]) := [Tx_i], \ \ [x_i] \in X_\mathcal{U}.$$

PROPOSITION 1. *Let $T \in L(X, Y)$ be an almost regular operator. Then $R(T)$ is closed.*

Proof. We take a bounded sequence $\{A_n\} \subset L(Y,X)$ such that $\|TA_nT - T\| \to 0$ as $n \to \infty$, and define $\mathbf{A} \in L(Y_{\mathcal{U}}, X_{\mathcal{U}})$ by

$$\mathbf{A}([y_i]) := [A_i y_i], \ \ [y_i] \in Y_{\mathcal{U}}.$$

Clearly $\mathbf{A}$ is well-defined and satisfies $T_{\mathcal{U}}\mathbf{A}T_{\mathcal{U}} = T_{\mathcal{U}}$. Indeed,

$$\|T_{\mathcal{U}}\mathbf{A}T_{\mathcal{U}} - T_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|TA_nT - T\| = 0.$$

Therefore $T_{\mathcal{U}}$ is regular; in particular $R(T_{\mathcal{U}})$ is closed; hence $R(T)$ is closed [2, Proposition 16]. □

REMARK 2. (1) If X and Y are Hilbert spaces, then Proposition 1 implies that almost regular operators are regular.

Since every closed subspace of a Hilbert space is complemented, in this case the regular operators are precisely the operators with closed range.

(2) If X is reflexive, then we can give a direct proof of the fact that every almost regular $T \in L(X,Y)$ is regular, using ultrafilter techniques.

Indeed, take a bounded sequence $\{A_n\} \subset L(Y,X)$ such that $\|TA_nT - T\| \to 0$ as $n \to \infty$. Since every bounded sequence in X is relatively weakly compact, for every $y \in Y$ the sequence $\{A_i y\}$ is weakly convergent following $\mathcal{U}$ [4]. Hence, we can define $A \in L(Y,X)$ by $Ay :=$ weak-$\lim_{\mathcal{U}} A_i y$, and it is not difficult to check that $TAT = T$; hence T is regular.

The following characterization of regular operators was known from [3, Theorem 3.8.2], however we give a different proof.

THEOREM 1. *An operator $T \in L(X,Y)$ is regular if and only if there exists $A \in L(Y,X)$ so that $R(TAT) = R(T)$ and $N(TAT) = N(T)$. In this case,*

$$X = N(T) \oplus R(AT) \text{ and } Y = N(TA) \oplus R(T).$$

Proof. The direct implication is clear. So we only have to prove the converse one. Moreover, by [3, Theorem 4.8.2] and the closed graph theorem, it is enough to prove that $X = N(T) \oplus R(AT)$ and $Y = N(TA) \oplus R(T)$ algebraically.

Suppose that $x \in N(T) \cap R(AT)$. Then $x = ATz$ for some $z \in X$; hence $0 = Tx = TATz$. Thus $z \in N(TAT) = N(T)$ and we conclude that $x = ATz = 0$.

Moreover, for every $x \in X$ we can find $z \in X$ so that $Tx = TATz$. So we can write $x = (x - ATz) + ATz$, and we have proved that $X = N(T) \oplus R(AT)$.

For the remaining equality, suppose that $y \in N(TA) \cap R(T)$. Then $y = Tx$ for some $x \in X$; hence $0 = TAy = TATx$. Thus $x \in N(TAT) = N(T)$ and we conclude that $y = Tx = 0$.

Moreover, observe that $R(TA) = R(TAT)$. Therefore, for every $y \in Y$ we can find $z \in X$ so that $TAy = TATz$. So we can write $y = (y - Tz) + Tz$, and we have proved that $Y = N(TA) \oplus R(T)$. □

For $T \in L(X,Y)$, we consider the *minimum modulus* $\gamma(T)$, defined by

$$\gamma(T) := \inf\{\|Tx\| : x \in X, \operatorname{dist}(x, N(T)) = 1\}.$$

It is well-known that $R(T)$ is closed if and only if $\gamma(T) > 0$ [1, Theorem IV.1.6].

THEOREM 2. *Every almost regular operator $T \in L(X,Y)$ is regular.*

Proof. Let $\{A_n\} \subset L(Y,X)$ be a sequence such that $\|TA_nT - T\| \to 0$ as $n \to \infty$. Since $R(T)$ is closed by Proposition 1, we can consider T and TA_nT as operators in $L(X, R(T))$. Thus the conjugate T^* is bounded below (as an operator in $L(R(T)^*, X^*)$). Moreover, $\|T^*A_n^*T^* - T^*\| \to 0$ as $n \to \infty$. Therefore, there exists an integer n_1 so that $T^*A_n^*T^*$ is bounded below (and hence TA_nT is surjective) for $n > n_1$. In this way, $R(T) = R(TA_nT)$ for $n > n_1$. Moreover, $N(T) \neq N(TA_nT)$ implies $\|T - TA_nT\| \geq \gamma(T) > 0$. Hence there exists an integer n_2 so that $N(T) = N(TA_nT)$ for $n > n_2$. Taking $n > \max\{n_1, n_2\}$ we obtain $R(T) = R(TA_nT)$ and $N(T) = N(TA_nT)$. Thus, the result follows from Theorem 1. □

REMARK 3. (1) In the definition of almost regular operator, the condition $\{A_n\}$ bounded is not superfluous.

For instance, the operator $T : \ell_2 \longrightarrow \ell_2$ given by $T(x_n) := (x_n/n)$ is not regular. However, the operators $A_n : \ell_2 \longrightarrow \ell_2$, given by

$$A_n(x_1, x_2, \ldots) := (x_1, 2x_2, \ldots, nx_n, 0, 0, \ldots)$$

satisfy $\|TA_nT - T\| \to 0$ and $\{A_n\}$ is not bounded.

(2) An operator $T \in L(X,Y)$ is regular if and only if it has closed range and there exists a (not necessarily bounded) sequence $\{A_n\}$ in $L(Y,X)$ so that $\|TA_nT - T\| \to 0$.

It is enough to check the proof of Theorem 2.

(3) If in the definition of almost regular operator T the operators A_n can be taken to be bijective, then $\dim N(T) = \dim Y/R(T)$. This can be seen as a "zero index" condition, although sometimes $\dim N(T) = \dim Y/R(T) = \infty$.

Observe that, from the time that $N(T) = N(TA_nT)$ and $R(T) = R(TA_nT)$, the operators A_n^{-1} apply $N(T)$ onto a complement of $R(T)$, which is isomorphic to $Y/R(T)$.

Finally, we give a result for operators in the closure of the set of all regular operators.

THEOREM 3. *Let $\{T_n\} \subset L(X,Y)$ be a sequence of regular operators. Assume that $T_n \to T$ as $n \to \infty$ and there exists a bounded sequence $\{U_n\} \subset L(Y,X)$ such that $T_nU_nT_n = T_n$ for all $n \in \mathbb{N}$. Then T is regular.*

Proof. From the equality $T_nU_nT_n = T_n$, it follows that

$$\begin{aligned} \|TU_nT - T\| &= \|(T-T_n)U_nT + T_nU_n(T-T_n) + T_n - T\| \\ &\leq \|T-T_n\|\|U_nT\| + \|T_nU_n\|\|T-T_n\| + \|T_n - T\|. \end{aligned}$$

Since $\{U_n\}$ is bounded, we obtain that $\|TU_nT - T\| \to 0$ as $n \to \infty$. Hence T is regular by Theorem 2. □

REMARK 4. (1) The condition of existence of a bounded sequence $\{U_n\}$ in Theorem 3 is not necessary in order that the limit of a sequence of regular operators be regular.

Given a Banach space X, the sequence of operators $\{T_n\} \subset L(X\times X, X\times X)$ defined by $T_n(x,y) := (x, y/n)$, converges to a regular operator, but there is no bounded sequence $\{T_n\} \subset L(X\times X, X\times X)$ so that $T_nU_nT_n = T_n$ for every n.

(2) If the sequence $\{U_n\}$ in Theorem 3 is unbounded, then there exists a sequence $\{Z_n\} \subset L(Y,X)$ of norm one operators such that $TZ_nT \to 0$ as $n \to \infty$.

Without lost of generality, we assume that $\|U_n\| \to \infty$. We define $Z_n := \dfrac{U_n}{\|U_n\|}$ and, proceeding as in the proof of Theorem 3, we get $\|TZ_nT\| \to 0$ as $n \to \infty$.

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