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# CONFORMAL CHANGE OF THE TENSOR $S_{\omega\mu}{}^{\nu}$ IN 7-DIMENSIONAL $g$-UFT

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ABSTRACT. We investigate change of the torsion tensor $S_{\omega\mu}{}^{\nu}$ induced by the conformal change in 7-dimensional $g$-unified field theory. These topics will be studied for the second class with the first category in 7-dimensional case.

## 1. Introduction

The conformal change in a generalized 4-dimensional Riemannian space connected by an Einstein's connection was primarily studied by HLAVATÝ [10]. CHUNG [8] also investigated the same topic in 4-dimensional $^*g$-unified field theory.

The Einstein's connection induced by the conformal change for all classes in 3-dimensional case, for the second and third classes in 5-dimensional case, and for the first class in 5-dimensional $^*g$-UFT, and for the second class in 6-dimensional $g$-UFT were investigated by CHO [1-4].

In the present paper, we investigate change of the torsion tensor $S_{\omega\mu}{}^{\nu}$ induced by the conformal change in 7-dimensional $g$-unified field theory. These topics will be studied for the second class with the first category in 7-dimensional case.

## 2. Preliminaries

This chapter is a brief collection of basic concepts, notations, theorems, and results needed in our further considerations. They may be reffered to CHUNG [5-7], CHO [1-4].

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### 2.1. $n$-dimensional $g$-unified field theory

The $n$-dimensional $g$-unified field theory ($n$-$g$-UFT hereafter) was originally suggested by HLAVATÝ [10] and systematically introduced by CHUNG [9].

Let $X_n$[1] be an $n$-dimensional generalized Riemannian manifold, reffered to a real coordinate system $x^\nu$ obeying coordinate transformations $x^\nu \to x^{\nu'}$, for which

$$\text{Det}\left(\left(\frac{\partial x}{\partial x'}\right)\right) \neq 0. \tag{2.1}$$

In the usual Einstein's $n$-dimensional unified field theory, the manifold $X_n$ is endowed with a general real nonsymmetric tensor $g_{\lambda_\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$[2] :

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{2.2}$$

where

$$\text{Det}((g_{\lambda\mu})) \neq 0, \qquad \text{Det}((h_{\lambda\mu})) \neq 0. \tag{2.3}$$

Therefore we may define a unique tensor $h^{\lambda\nu} = h^{\nu\lambda}$ by

$$h_{\lambda\mu} h^{\lambda\nu} = \delta^\nu_\mu. \tag{2.4}$$

In our $n$-$g$-UFT, the tensors $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and/or lowering indices of the tensors in $X_n$ in the usual manner.

The manifold $X_n$ is connected by a general real connection $\Gamma^\nu_{\omega\mu}$ with the following transformation rule :

$$\Gamma^{\nu'}_{\omega'\mu'} = \frac{\partial x^{\nu'}}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial x^{\omega'}} \cdot \frac{\partial x^\gamma}{\partial x^{\mu'}}\Gamma^\alpha_{\beta\gamma} + \frac{\partial^2 x^\alpha}{\partial x^{\omega'}\partial x^{\mu'}}\right) \tag{2.5}$$

and satisfies the system of Einstein's equations

$$D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^\alpha g_{\lambda\alpha} \tag{2.6}$$

where $D_\omega$ denotes the covariant derivative with respect to $\Gamma^\nu_{\omega\mu}$ and

$$S_{\omega\mu}{}^\nu = \Gamma^\nu_{[\omega\mu]} \tag{2.7}$$

[1]Throughout the present paper, we assumed that $n \geq 2$.

[2]Throughout this paper, Greek indices are used for holonomic components of tensors. In $X_n$ all indices take the values $1, \cdots, n$ and follow the summation convention.

is the *torsion tensor* of $\Gamma^{\nu}_{\omega\mu}$. The connection $\Gamma^{\nu}_{\omega\mu}$ satisfying (2.6) is called the *Einstein's connection.*

In our further considerations, the following scalars, tensors, abbreviations, and notations for $p = 0, 1, 2, \cdots$ are frequently used :

(2.8)$a$
$$\mathfrak{g} = \mathrm{Det}((g_{\lambda\mu})) \neq 0, \quad \mathfrak{h} = \mathrm{Det}((h_{\lambda\mu})) \neq 0, \quad \mathfrak{k} = \mathrm{Det}((k_{\lambda\mu})),$$

(2.8)$b$
$$g = \frac{\mathfrak{g}}{\mathfrak{h}}, \qquad k = \frac{\mathfrak{k}}{\mathfrak{h}},$$

(2.8)$c$
$$K_p = k_{[\alpha_1}{}^{\alpha^1} \cdots k_{\alpha_p]}{}^{\alpha_p}, \quad (p = 0, 1, 2, \cdots)$$

(2.8)$d$
$$^{(0)}k_{\lambda}{}^{\nu} = \delta^{\nu}_{\lambda}, \qquad ^{(1)}k_{\lambda}{}^{\nu} = k_{\lambda}{}^{\nu}, \qquad ^{(p)}k_{\lambda}{}^{\nu} = {}^{(p-1)}k_{\lambda}{}^{\alpha} k_{\alpha}{}^{\nu},$$

(2.8)$e$
$$K_{\omega\mu\nu} = \nabla_{\nu} k_{\omega\mu} + \nabla_{\omega} k_{\nu\mu} + \nabla_{\mu} k_{\omega\nu},$$

(2.8)$f$
$$\sigma = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

where $\nabla_{\omega}$ is the symbolic vector of the convariant derivative with respect to the Christoffel symbols $\left\{ {\nu \atop \lambda\mu} \right\}$ defined by $h_{\lambda\mu}$. The scalars and vectors introduced in (2.8) satisfy

(2.9)$a$
$$K_0 = 1; K_n = k \text{ if } n \text{ is even}; \qquad K_p = 0 \text{ if } p \text{ is odd},$$

(2.9)$b$
$$g = 1 + K_2 + \cdots + K_{n-\sigma},$$

(2.9)$c$
$$^{(p)}k_{\lambda\mu} = (-1)^{p(p)} k_{\mu\lambda}, \qquad ^{(p)}k^{\lambda\nu} = (-1)^{p(p)} k^{\nu\lambda}.$$

Furthermore, we also use the following useful abbrevations, denoting an arbitrary tensor $T_{\omega\mu\nu}$, skew-symmetric in the first two indices, by $T$ :

(2.10)$a$
$$\overset{pqr}{T} = \overset{pqr}{T}{}_{\omega\mu\nu} = T_{\alpha\beta\gamma} {}^{(p)}k_{\omega}{}^{\alpha(q)} k_{\mu}{}^{\beta(r)} k_{\nu}{}^{\gamma},$$

(2.10)$b$
$$T = T_{\omega\mu\nu} = \overset{000}{T},$$

(2.10)$c$
$$2\overset{pqr}{T}{}_{\omega[\lambda\mu]} = \overset{pqr}{T}{}_{\omega\lambda\mu} - \overset{pqr}{T}{}_{\omega\mu\lambda},$$

(2.10)$d$
$$2\overset{(pq)r}{T}{}_{\omega\lambda\mu} = \overset{pqr}{T}{}_{\omega\lambda\mu} + \overset{qpr}{T}{}_{\omega\lambda\mu}.$$

We then have

(2.11)
$$\overset{pqr}{T}{}_{\omega\lambda\mu} = -\overset{qpr}{T}{}_{\lambda\omega\mu}.$$

If the system (2.6) admits $\Gamma^{\nu}_{\lambda\mu}$, using the above abbreviations it was shown that the connection is of the form

(2.12)
$$\Gamma^{\nu}_{\omega\mu} = \left\{ {}^{\ \nu}_{\omega\mu} \right\} + S_{\omega\mu}{}^{\nu} + U^{\nu}{}_{\omega\mu}$$

where

(2.13)
$$U_{\nu\omega\mu} = 2\overset{001}{S}{}_{\nu(\omega\mu)}.$$

The above two relations show that *our problem of determining* $\Gamma^{\nu}_{\omega\mu}$ *in terms of* $g_{\lambda\mu}$ *is reduced to that of studying the tensor* $S_{\omega\mu}{}^{\nu}$. On the other hand, it has also been shown that the tensor $S_{\omega\mu}{}^{\nu}$ satisfies

(2.14)
$$S = B - 3\overset{(110)}{S}$$

where

(2.15)
$$2B_{\omega\mu\nu} = K_{\omega\mu\nu} + 3K_{\alpha[\mu\beta}k_{\omega]}{}^{\alpha}k_{\nu}{}^{\beta}.$$

### 2.2. Some results for the second class in 7-$g$-UFT

In this section, we introduce some results of 7-$g$-UFT without proof, which are needed in our subsequent considerations.

They may be reffered to CHUNG [5-7].

DEFINITION 2.1. In 7-$g$-UFT, the tensor $g_{\lambda\mu}(k_{\lambda\mu})$ is said to be the second class with the first category, if $K_2 \neq 0$, $K_4 = K_6 = 0$.

THEOREM 2.2. *(Main recurrence relations) For the second class with the first category in 7-g-UFT, the following recurrence relation hold*

$$ {}^{(p+3)}k_{\lambda}{}^{\nu} = -K_2 {}^{(p+1)}k_{\lambda}{}^{\nu}, \qquad (p = 0, 1, 2, \cdots). \tag{2.16} $$

THEOREM 2.3. *(For the second class with the first category in 7-g-UFT). A necessary and sufficient condition for the existence and uniqueness of the solution of (2.5) is*

$$ 1 - (K_2)^2 \neq 0. \tag{2.17} $$

*If the condition (2.17) is satisfied, the unique solution of (2.14) is given by*

$$ (1 - K_2{}^2)(S - B) = 1 - K_2{}^2 - (1 - K_2{}^2)2\overset{110}{B} - 2\overset{(10)1}{B} + \overset{112}{B} - K_2\overset{002}{B} \tag{2.18} $$

## 3. Conformal change of the 7-dimensional torsion tensor for the second class

In this final chapter we investigate the change $S_{\omega\mu}{}^{\nu} \to \overline{S}_{\omega\mu}{}^{\nu}$ of the torsion tensor induced by the conformal change of the tensor $g_{\lambda\mu}$, using the recurrence relations and theorems introduced in the preceding chapter.

We say that $X_n$ and $\overline{X}_n$ are conformal if and only if

$$ \overline{g}_{\lambda\mu}(x) = e^{\Omega} g_{\lambda\mu}(x) \tag{3.1} $$

where $\Omega = \Omega(x)$ is an at least twice differentiable function. This conformal change enforces a change of the torsion tensor $S_{\omega\mu}{}^{\nu}$. An explicit representation of the change of 7-dimensional torsion tensor $S_{\omega\mu}{}^{\nu}$ for the second class with the first category will be exhibited in this chapter.

AGREEMENT 3.1. Throughout this section, we agree that, if $T$ is a function of $g_{\lambda\mu}$, then we denote $\overline{T}$ the same function of $\overline{g}_{\lambda\mu}$. In particular, if $T$ is a tensor, so is $\overline{T}$. Furthermore, the indices of $T$ $(\overline{T})$ will be raised and/or lowered by means of $h^{\lambda\nu}(\overline{h}^{\lambda\nu})$ and/or $h_{\lambda\mu}(\overline{h}_{\lambda\mu})$.

The results in the following theorems are needed in our further considerations. They may be refferred to CHUNG [8-9], CHO [1-3].

THEOREM 3.2. *In n-g-UFT, the conformal change* (3.1) *induces the following changes :*

$$(3.2)a \qquad {}^{(p)}\overline{k}_{\lambda\mu} = e^{\Omega(p)}\,{}^{(p)}k_{\lambda\mu}, \qquad {}^{(p)}\overline{k}_{\lambda}{}^{\nu} = {}^{(p)}k_{\lambda}{}^{\nu}, \qquad {}^{(p)}\overline{k}^{\lambda\nu} = e^{-\Omega(p)}\,{}^{(p)}k^{\lambda\nu}$$

$$(3.2)b \qquad \overline{g} = g, \qquad \overline{K_p} = K_p \qquad (p = 1, 2, \cdots).$$

THEOREM 3.3. *(For all classes in 7-g-UFT). The change of the tensor* $B_{\omega\mu\nu}$ *induced by the conformal change* (3.1) *may be given by*

$$(3.3) \qquad \begin{aligned} \overline{B}_{\omega\mu\nu} =& e^{\Omega}(B_{\omega\mu\nu} + k_{\nu[\omega}\Omega_{\mu]} - k_{\omega\mu}\Omega_{\nu} \\ & - h_{\nu[\omega}k_{\mu]}{}^{\delta}\Omega_{\delta} + 2\,{}^{(2)}k_{\nu[\omega}k_{\mu]}{}^{\delta}\Omega_{\delta} + k_{\omega\mu}\,{}^{(2)}k_{\nu}{}^{\delta}\Omega_{\delta}). \end{aligned}$$

Now, we are ready to derive representations of the changes $S_{\omega\mu}{}^{\nu} \to \overline{S}_{\omega\mu}{}^{\nu}$ in 7-$g$-UFT for the second class with the first category induced by the conformal change (3.1).

THEOREM 3.4. *The conformal change* (3.1) *induces the following change :*

$$(3.4) \qquad \begin{aligned} 2\overline{\overset{(10)1}{B}}_{\omega\mu\nu} =& e^{\Omega}[2\overset{(10)1}{B}_{\omega\mu\nu} + (-2\,{}^{(4)}k_{\nu[\omega}k_{\mu]}{}^{\delta} \\ & + 2\,{}^{(2)}k_{\nu[\omega}k_{\mu]}{}^{\delta} - k_{\nu[\omega}\,{}^{(2)}k_{\mu]}{}^{\delta})\Omega_{\delta} - {}^{(3)}k_{\nu[\omega}\Omega_{\mu]}]. \end{aligned}$$

THEOREM 3.5. *The conformal change* (3.1) *induces the following change :*

$$(3.5) \qquad \begin{aligned} \overline{\overset{ppq}{B}}_{\omega\mu\nu} =& e^{\Omega}[\overset{ppq}{B}_{\omega\mu\nu} + (-1)^{p}\{2\,{}^{(p+q+2)}k_{\nu[\omega}\,{}^{(p+1)}k_{\mu]}{}^{\delta} \\ & + {}^{(2p+1)}k_{\omega\mu}\,{}^{(2+q)}k_{\nu}{}^{\delta} - {}^{(2p+1)}k_{\omega\mu}\,{}^{(q)}k_{\nu}{}^{\delta} \\ & + {}^{(p+q+1)}k_{\nu[\omega}\,{}^{(p)}k_{\mu]}{}^{\delta} - {}^{(p+q)}k_{\nu[\omega}\,{}^{(p+1)}k_{\mu]}{}^{\delta}\}\Omega_{\delta}]. \\ & \begin{pmatrix} p = 0, 1, 2, 3, 4, \cdots \\ q = 0, 1, 2, 3, 4, \cdots \end{pmatrix} \end{aligned}$$

By the above relation (3.5), we obtain $\overset{\overline{110}}{B}$, $\overset{\overline{112}}{B}$, $\overset{\overline{002}}{B}$.

THEOREM 3.6. *The change $S_{\omega\mu}{}^{\nu} \to \overline{S}_{\omega\mu}{}^{\nu}$ induced by conformal change (3.1) may be represented by*

$$\begin{aligned}\overline{S}_{\omega\mu}{}^{\nu} = {} & S_{\omega\mu}{}^{\nu} + 1 - h^{\nu}{}_{[\omega}k_{\mu]}{}^{\delta}\Omega_{\delta} \\ & + (K_2 - 1)k_{\omega\mu}\Omega^{\nu} + (1 - K_2)k_{\omega\mu}{}^{(2)}k^{\nu\delta}\Omega_{\delta} \\ & + \frac{1}{K_2{}^2 - 1}[(-1 + K_2)k^{\nu}{}_{[\omega}\Omega_{\mu]} \\ & \quad + (-1 + 2K_2 + K_2{}^2)^{(2)}k^{\nu}{}_{[\omega}k_{\mu]}{}^{\delta}\Omega_{\delta} \\ & \quad + (K_2 + K_2{}^2 - 2K_2{}^3)k^{\nu}{}_{[\omega}{}^{(2)}k_{\mu]}{}^{\delta}\Omega_{\delta}].\end{aligned} \tag{3.6}$$

*Proof.* In virtue of (2.18) and Agreement (3.1), we have

$$(1 - \overline{K}_2{}^2)(\overline{S} - \overline{B}) = 1 - \overline{K_2{}^2} - (1 - \overline{K_2{}^2})\overset{\overline{110}}{B} - 2\overset{\overline{(10)1}}{B} + \overset{\overline{112}}{B} - K_2\overset{\overline{002}}{B}. \tag{3.7}$$

The relation (3.6) follows by substituiting (3.2), (3.3), (3.4), (3.5), (2.16), Definition (2.1), into (3.7). □

## References


[1] C. H. Cho, *Conformal change of the connection in 3- and 5-dimensional* $^*g^{\lambda\nu}$*-unified Field Theory*, BIBS, Inha Univ. **13** (1992).

[2] ———, *Conformal change of the connection for the first class in 5-dimensional* $^*g^{\lambda\nu}$*-unified Field Theory*, Kangweon-Kyungki Math. Jour. **2** (1994), 19–33.

[3] ———, *Conformal change of the tensor* $S_{\lambda\mu}{}^{\nu}$ *for the second category in 6-dimensional g-UFT,*, Kangweon-Kyungki Math. Jour. **4** (1996), 163–171.

[4] ———, *Conformal change of the torsion tensor in 6-dimensional g-unified field theory*, Kangweon-Kyungki Math. Jour. **3** (1995), no. 1, 71–79.

[5] K. T. Chung, *A Stusy on* $X_7$*. I. Basic vectors and scalars in* $X_7$ *for the first and second classes*, Yonsei Nonchong **30** (1994).

[6] ———, *A Stusy on* $X_7$*. II. Recurrence relations in* $X_7$, Yonsei Nonchong **30** (1994).

[7] ———, *A Stusy on* $X_7$*. III. 7-dimensional Einstein's connection for the first and second classes*, Yonsei Nonchong **30** (1994).

[8] ———, *Conformal change in Einstein's* $^*g^{\lambda\nu}$*-unified field theory*, Nuove Cimento **(X) 58B** (1968).

[9] ———, *Einstein's connection in terms of* $^*g^{\lambda\nu}$, Nuove Cimento **(X) 27** (1963).

[10] V. HLAVATÝ, *Geometry of Einstein's unified field theory*, P. Noordhoff Ltd (1957).



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