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A NOTE ON LATTICE IMPLICATION ALGEBRAS

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Abstract. In this paper, a simple axiom system of lattice implication algebras is presented, it is convenient for verifying whether an algebra of type (2,2,2,1,0,0) becomes a lattice implication algebra.

1. Introduction

In order to do research on the logical system whose propositional value is given in a lattice, Y. Xu [6] proposed the concept of lattice implication algebras. Some of their fundamental properties were obtained in [1, 2, 4, 6, 7]. In this paper we will give an equivalent axiom system of lattice implication algebras, which simplifies Xu's axioms in [6]. This is convenient for verifying whether an algebra of type (2,2,2,1,0,0) becomes a lattice implication algebra.

2. Preliminaries

According to the notion of lattice implication algebras, originally given by Y. Xu [6], we can also describe it as follows:

Definition. An algebra $(X, \vee, \wedge, \to, ', 0, 1)$ of type (2,2,2,1,0,0) is called

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a *lattice implication algebra* if it satisfies the following axioms:

(L1) $x \wedge x = x$, (L$'$1) $x \vee x = x$,

(L2) $x \wedge y = y \wedge x$, (L$'$2) $x \vee y = y \vee x$,

(L3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, (L$'$3) $(x \vee y) \vee z = x \vee (y \vee z)$,

(L4) $x \wedge (x \vee y) = x$, (L$'$4) $x \vee (x \wedge y) = x$,

(B1) $x \wedge 0 = 0$, (B$'$1) $x \vee 1 = 1$,

(L) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$,(L$'$) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,

(C1) $x \wedge y = x \Rightarrow x' \wedge y' = y'$,

(C2) $(x')' = x$,

(I1) $x \to (y \to z) = y \to (x \to z)$,

(I2) $x \to x = 1$,

(I3) $x \to y = y' \to x'$,

(I4) $x \to y = 1 = y \to x \Rightarrow x = y$,

(I5) $(x \to y) \to y = (y \to x) \to x$,

for all $x, y, z \in X$.

LEMMA 1 ([6]). *In a lattice implication algebra* $(X, \vee, \wedge, \to, ', 0, 1)$*, the following hold (for all* $x, y, z \in X$*):*

(1) $0 \to x = 1$,
(2) $x \to 0 = x'$,
(3) $x \vee y = (x \to y) \to y = (y \to x) \to x$,
(4) $x \wedge y = ((y \to x) \to y')' = ((x \to y) \to x')'$.

LEMMA 2 ([5]). *An algebra* $(X; *, 0)$ *of type* $(2, 0)$ *is a commutative* BCK*-algebra if and only if the following conditions are satisfied (for all* $x, y, z \in X$*):*

(1) $0 * x = 0$,
(2) $(x * y) * (x * z) = (z * y) * (z * x)$,
(3) $x * 0 = x$.

LEMMA 3 ([3]). *Suppose* $(X; *, 0)$ *is a bounded commutative* BCK*-algebra with unit 1. For any* x, y, z *in* X*, let* $Nx = 1 * x$*,* $x \wedge y = y * (y * x)$*,* $x \vee y = N(Nx \wedge Ny)$*. Then* $(X, \vee, \wedge)$ *is a distributive lattice with a least element 0 and a greatest element 1.*

3. Result

The following theorem gives a very simpler axiom system of lattice implication algebras.

THEOREM 1. *An algebra* $(X, \vee, \wedge, \rightarrow, ', 0, 1)$ *of type* $(2,2,2,1,0,0)$ *is a lattice implication algebra if and only if it satisfies the following conditions (for all* $x, y, z \in X$*):*

(Z1) $0 \rightarrow x = 1$,

(Z2) $(x \rightarrow z) \rightarrow (x \rightarrow y) = (z \rightarrow x) \rightarrow (z \rightarrow y)$,

(Z3) $x \rightarrow y = y' \rightarrow x'$,

(Z4) $(x')' = x$,

(Z5) $x' = x \rightarrow 0$,

(Z6) $x \vee y = (x \rightarrow y) \rightarrow y$,

(Z7) $(x \wedge y)' = x' \vee y'$.

Proof. Suppose $(X, \vee, \wedge, \rightarrow, ', 0, 1)$ is a lattice implication algebra. By the definition of lattice implication algebras and Lemma1, (Z1) and (Z3)~(Z6) hold. For (Z2), from (I1) and (I3)~ (I5) we obtain

$$\begin{aligned}(x \rightarrow z) \rightarrow (x \rightarrow y) &= x \rightarrow ((x \rightarrow z) \rightarrow y) = x \rightarrow (y' \rightarrow (x \rightarrow z)') \\ &= y' \rightarrow (x \rightarrow (x \rightarrow z)') = y' \rightarrow ((z' \rightarrow x') \rightarrow x') \\ &= y' \rightarrow ((x' \rightarrow z') \rightarrow z') = (x' \rightarrow z') \rightarrow (y' \rightarrow z') \\ &= (z \rightarrow x) \rightarrow (z \rightarrow y).\end{aligned}$$

For (Z7), by (Z3), (Z4), (Z6) and Lemma 1 (4), we have

$$x' \vee y' = (x' \rightarrow y') \rightarrow y' = (y \rightarrow x) \rightarrow y' = (x \wedge y)'.$$

So far, the necessity is proved.

Conversely, suppose $(X, \vee, \wedge, \rightarrow, ', 0, 1)$ satisfies the conditions (Z1)~(Z7). For all $x, y, z \in X$, let

$$x * y = (x \rightarrow y)'.$$

We first prove $(X; *, 0)$ is a bounded commutative BCK-algebra. In fact, by (Z1) and (Z5) we have $0' = 1$, and so $1' = 0$ by (Z4). Therefore, by (Z1) and (Z3)$\sim$ (Z5) we have

$$\begin{aligned}0*x &= (0\to x)' = 1' = 0,\\ x*0 &= (x\to 0)' = (x')' = x,\\ x*1 &= (x\to 1)' = (0\to x')' = 1' = 0.\end{aligned}$$

Moreover, by (Z2)$\sim$(Z5) we have

$$\begin{aligned}(x*y)*(x*z) &= ((x\to y)'\to(x\to z)')' = ((x\to z)\to(x\to y))'\\ &= ((z\to x)\to(z\to y))' = ((z\to y)'\to(z\to x)')'\\ &= (z*y)*(z*x).\end{aligned}$$

Thus, by Lemma 2 it is proved that $(X;*,0)$ is a bounded commutative BCK-algebra with unit 1.

Next, by (Z3)$\sim$(Z5) we have

$$Nx = 1*x = (1\to x)' = (x'\to 0)' = (x'')' = x'.$$

By (Z3), (Z4),(Z6) and (Z7) we obtain

$$\begin{aligned}x\wedge y &= (x'\vee y')' = ((x'\to y')\to y')'\\ &= (y\to(y\to x)')' = y*(y*x),\\ x\vee y &= (x\to y)\to y = y'\to(y'\to x')'\\ &= (y'*(y'*x'))' = N(Nx\wedge Ny).\end{aligned}$$

Summarizing the above discussions, by Lemma 3 it is proved that $(X,\vee,\wedge)$ is a distributive lattice with a least element 0 and a greatest element 1. Therefore (L1)$\sim$(L4), (L$'$ 1)$\sim$(L$'$ 4), (B1) and (B$'$ 1) hold, and so (I5) holds by (L$'$ 2) and (Z6). Since $x\wedge y = x$ implies $y = x\vee y$, by (Z4) and (Z7) we have $y' = (x\vee y)' = x'\wedge y'$, i.e., (C1) holds.

Finally, from the fundamental properties [3] of bounded commutative

BCK-algebras we obtain

$$\begin{aligned}
&\text{(I1)}\quad x \to (y \to z) = N(x * N(y * z)) = N((y * z) * Nx) \\
&\qquad = N((y * Nx) * z) = N((x * Ny) * z) \\
&\qquad = N((x * z) * Ny) = N(y * N(x * z)) \\
&\qquad = y \to (x \to z). \\
&\text{(I2)}\quad x \to x = N(x * x) = N0 = 1. \\
&\text{(I4)}\quad x \to y = 1 = y \to x \Leftrightarrow N(x \to y) = N1 = N(y \to x) \\
&\qquad \Leftrightarrow x * y = 0 = y * x \Leftrightarrow x = y. \\
&\text{(L)}\quad (x \wedge y) \to z = N((x \wedge y) * z) = N((x * z) \wedge (y * z)) \\
&\qquad = N(x * z) \vee N(y * z) = (x \to z) \vee (y \to z). \\
&\text{(L}'\text{)}\quad (x \vee y) \to z = N((x \vee y) * z) = N((x * z) \vee (y * z)) \\
&\qquad = N(x * z) \wedge N(y * z) = (x \to z) \wedge (y \to z).
\end{aligned}$$

Noticing that (I3) and (C2) hold in the case of the sufficiency, we obtain all the axioms of the definition are satisfied. So $(X, \vee, \wedge, \to, ', 0, 1)$ is a lattice implication algebra. The proof is complete. □

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