Bull. Korean Math. Soc. **38** (2001), No. 1, pp. 137–142

A STUDY ON ANNIHILATOR CONDITIONS OF POLYNOMIALS

Yong Uk Cho

Abstract. In this paper, we initiate the study of some annihilator conditions on polynomials which were used by Kaplansky [4] to abstract algebras of bounded linear operators on a Hilbert spaces with Baer condition. On the other hand, p.p. rings were introduced by A. Hattori [3] to study the torsion theory. The purpose of this paper is to introduce near-rings with Baer condition and near-rings with p.p. condition which are somewhat different from the ring case, and to extend a results of Armendariz [1] to polynomial near-rings with Baer condition in somewhat different way of Birkenmeier and Huang [2].

1. Introduction

Throughout this paper, all rings are associative with identity and all near-rings are right near-rings. We denote a ring by R and a near-ring by N respectively. In 1965, Kaplansky [4] introduced a ring with Baer condition as following: R is a Baer ring in case the (left) annihilator of every nonempty subset of R is generated, as a left ideal, by an idempotent. This is equivalent to the condition that for every nonempty subset S of R, the (left) annihilator of S is the annihilator of some idempotent e of R. In 1974, Armendarz obtained the following result ([1], Theorem B). Let R be a reduced ring. Then R is a Baer ring if and only if the polynomial ring $R[x]$ is Baer. It is natural to define a near-ring with Baer condition and to investigate the polynomial near-ring with Baer condition when R is a Baer ring, where the multiplication of polynomial near-ring is the "substitution $\circ$".

Let G be an additively written (but not necessarily abelian) group with zero 0 and $M_0(G) = \{f : G \to G \mid f(0) = 0\}$ the near-ring of all zero

Received August 12, 2000.
2000 Mathematics Subject Classification: 16Y30.
Key words and phrases: Baer rings, p.p. rings, Baer near-rings, p.p. near-rings.

respecting mappings on G. We shall show that $M_0(G)$ is a Baer near-ring. From this result, we show that every zero-symmetric near-ring can be embedded into a Baer near-ring. For a reduced ring R, it is known that R is a Baer (resp. p.p.) ring if and only if the polynomial ring $R[x]$ is a Baer (resp. p.p.) ring (See Armendariz [1], Hattori [3]). Let R be a commutative ring with identity. Corresponding to these results, we will prove that the zero-symmetric part of $R[x]$ is a Baer (resp. p.p.) near-ring if and only if R is a Baer (resp. p.p.) ring. We also study the structure of the near-ring $R \oplus M$, where R is an associative ring with identity and M is a unital left R-module. Then $R \oplus M$ is a p.p. near-ring if and only if R is a p.p. ring.

2. Baer near-rings and p.p. near-rings

A (right) near-ring is a set N with two binary operations $+$ and $\cdot$ such that $(N, +)$ is a not necessarily abelian group with identity 0, $(N, \cdot)$ is a semigroup and $(x+y)z = xz+yz$ for all $x, y, z \in N$. Some basic definitions and concepts in near-ring theory can be found in Meldrum [5] and Pilz [6].

For a subset S of a near-ring N, the set $\{n \in N \mid nS = 0\}$ is called the *annihilator of* S in N and is denoted by $\text{Ann}_N(S) = \text{Ann}(S)$.

A near-ring is called a *Baer near-ring* if, for any nonempty subset S of N, $\text{Ann}(S) = \text{Ann}(e)$ for some idempotent $e \in N$. The following propositions is obvious.

PROPOSITION 1. *Let N be a near-ring with identity, then N is Baer if and only if for any nonempty subset S of N, $Ann(S) = Ne$, as a left N-subgroup, for some central idempotent $e \in N$.*

PROPOSITION 2. *Let N_i $(i \in I)$ be a family of near-rings. Then the direct product $\prod_{i\in I} N_i$ is a Baer near-ring if and only if N_i is a Baer near-ring for each $i \in I$.*

A near-ring N is said to be *integral* if N has no nonzero divisors of zero [6, 1.14, p.11].

EXAMPLE 1. (1) Every integral near-ring with identity is a Baer near-ring.

(2) Every constant near-ring is a Baer near-ring.

(3) A direct product of integral near-rings with identity is a Baer near-ring.

Let G be an additively written (but not necessarily abelian) group with zero 0 and $M_0(G) = \{f : G \to G \mid f(0) = 0\}$ the near-ring of all zero respecting mappings on G.

LEMMA 3. *The near-rings $M_0(G)$ and $M(G)$ are Baer.*

Proof. Let S be a subset of $M_0(G)$ and let $H = \{s(g) \mid s \in S,\ g \in G\}$. Let e be a mapping on G such that if $x \in H$, then $e(x) = x$ and for any $y \in G - H$ $e(y) = 0$. Then e is an idempotent of $M_0(G)$ and $\text{Ann}(S) = \text{Ann}(e)$. This implies that $M_0(G)$ is a Baer near-ring. Similarly, $M(G)$ is a Baer near-ring. □

THEOREM 4. *Every zero-symmetric near-ring can be embedded into a Baer near-ring.*

Proof. By [6, 1.102], every zero-symmetric near-ring can be embedded into a zero-symmetric near-ring with identity. Let N be a zero-symmetric near-ring with identity. By Lemma 3, $M_0(N)$ is a Baer near-ring. Let $r \in N$. Then the mapping $f_r : t \in N \to rt \in N$ is an element of $M_0(N)$. Since N contains an identity it follows that the mapping $f : N \to M_0(N)$ is a near-ring monomorphism. □

An associative ring R called a *left p.p. ring* if every principal left ideal of R is projective. This is equivalent to the condition that, for any $a \in R$, $\text{Ann}(a) = \text{Ann}(e)$ for some idempotent $e \in R$. This condition is a "Rickart type annihilator" in ring with involution. Similarly we can define for near-rings. N is called a *p.p. near-ring* if for any $a \in N$, $\text{Ann}(a) = \text{Ann}(e)$ for some idempotent $e \in N$.

EXAMPLE 2. Recall a near-ring N is called regular if, for any $x \in N$, there exists $y \in N$ such that $xyx = x$. Then xy is an idempotent and $\text{Ann}(x) = \text{Ann}(xy)$. Hence every regular near-ring is a p.p. near-ring.

Let R be a commutative ring with identity and let $R[x]$ denote the set of all polynomials in one indeterminate over R. Under usual addition $+$ and substitution $\circ$ of polynomials, $(R[x], +, \circ)$ becomes a near-ring. Following Pilz [6], $R_0[x]$ denote the zero symmetric part of $R[x]$, that is $R_0[x] = \{\sum_{i=1}^{n} a_i x^i \mid a_i \in R, n \geq 1\}$.

A ring (or near-ring) without non-zero nilpotent element is called *reduced*.

THEOREM 5. *Let R be a commutative ring with identity. Then the following conditions are equivalent:*

1) $R_0[x]$ is a p.p. near-ring.

2) R is a p.p. ring.

Proof. 1) $\Rightarrow$ 2). First we claim that R is reduced. Suppose that $a \in R$ with $a^2 = 0$. By hypothesis, there exists an idempotent $f \in R_0[x]$ such that $\mathrm{Ann}(ax) = \mathrm{Ann}(f)$. Let $f = a_1x + a_2x^2 + \cdots + a_nx^n$ with $a_i \in R$. Since f is an idempotent, we have $a_1^2 = a_1$. Since $ax \in \mathrm{Ann}(ax)$, $ax \circ f = af = 0$. In particular, $aa_1 = 0$. Since $x - f \in \mathrm{Ann}(f)$, $0 = (x-f) \circ ax = ax - f(ax)$. Hence $ax = a_1ax = 0$, that is $a = 0$. This proves that R is reduced.

Since R is reduced, the set of idempotents of $R_0[x]$ is just $\{ex \mid e^2 = e \in R\}$. Now let r be an arbitrary element of R. By hypothesis, there exists an idempotent $e \in R$ such that $\mathrm{Ann}(rx) = \mathrm{Ann}(ex)$. Clearly this implies that $\{s \in R \mid sr = 0\} = R(1-e)$. Hence R is a p.p. ring.

2) $\Rightarrow$ 1). Let $f = a_1x + \cdots + a_nx^n \in R_0[x]$ and $g = b_1x + \cdots + b_mx^m \in R_0[x]$. First we claim that $f \circ g = 0$ if and only if $a_ib_j = 0$ for all i, j. It suffices to prove the 'only if' part. Let P be an arbitrary prime ideal of R and let $\bar{f}$ and $\bar{g}$ denote the image of f and g in $(R/P)[x]$, respectively. Since R/P is an integral domain and since $\bar{f} \circ \bar{g} = 0$, we can easily see that either $\bar{f} = 0$ or $\bar{g} = 0$ holds. Hence $a_ib_j \in P$ for all i, j. Since a prime ideal P is arbitrary, this implies that $a_ib_j \in \mathrm{Rad}(R)$, where $\mathrm{Rad}(R)$ denote the prime radical of R. Since R is a p.p. ring, R is reduced and hence $\mathrm{Rad}(R) = 0$. This proves our claim. Therefore $a_1, \ldots, a_n \in \mathrm{Ann}_R(b_1, \ldots, b_m)$. Since R is a p.p. ring, for each i, there exists an idempotent $e_i \in R$ such that $\mathrm{Ann}(b_i) = \mathrm{Ann}(e_i)$. If $n = 2$, then $f = e_1 + e_2 - e_1e_2$ is an idempotent and $\mathrm{Ann}_R(b_1, b_2) = \mathrm{Ann}(f)$. Using induction on n, we can find an idempotent e of R such that $\mathrm{Ann}_R(b_1, \ldots, b_m) = \mathrm{Ann}(e)$. Then ex is an idempotent of $R_0[x]$ and $\mathrm{Ann}(g) = \mathrm{Ann}(ex)$. Therefore $R_0[x]$ is a p.p. near-ring. $\square$

From this Theorem 5, we get the following statement as a corollary.

COROLLARY 6. *Let R be a commutative p.p. ring with identity. Then R is reduced. Moreover, the following conditions are equivalent:*

1) R is reduced.

2) $(R[x], +, \cdot)$ is reduced.

3) $(R_0[x], +, \circ)$ is reduced.

THEOREM 7. *Let R be a commutative ring with identity. Then the following conditions are equivalent:*

1) $R_0[x]$ is a Baer near-ring.

2) R is a Baer ring.

Proof. 1) $\Rightarrow$ 2). Let T be a subset of R and consider the subset $S = \{tx \mid t \in T\}$ of $R_0[x]$. As saw in the proof of 1) $\Rightarrow$ 2) in Theorem 5, the set of idempotents of $R_0[x]$ is just $\{ex \mid e^2 = e \in R\}$. Since $R_0[x]$ is Baer, $\mathrm{Ann}(S) = \mathrm{Ann}(ex)$ for some idempotent $e \in R$. Then we can easily see that $\mathrm{Ann}_R(T) = \mathrm{Ann}_R(e)$. Hence R is a Baer ring.

2) $\Rightarrow$ 1). Let S be a subset of $R_0[x]$ and consider the set T of all coefficients of $g(x) \in S$. Let $f = a_1x + \cdots + a_nx^n \in \mathrm{Ann}(S)$. As saw in the proof of 2) $\Rightarrow$ 1) in Theorem 5, $a_i \in \mathrm{Ann}_R(T)$ for all i. Since R is a Baer ring, there exists an idempotent e such that $\mathrm{Ann}_R(T) = \mathrm{Ann}_R(e)$. Now we can easily see that $\mathrm{Ann}(S) = \mathrm{Ann}(ex)$. This proves that $R_0[x]$ is a Baer near-ring. □

Let R be an associative ring with identity and let M be a unital left R-module. If we define a multiplication on the additive group $R \oplus M$ by $(a, b) \circ (c, d) = (ac, ad + b)$ for any (a, b), $(c, d) \in R \oplus M$, then $R \oplus M$ becomes a near-ring with identity $(1, 0)$.

THEOREM 8. *Let R be an associative ring with identity and let M be a unital left R-module. Then the following conditions are equivalent:*

1) $R \oplus M$ is a p.p. near-ring.

2) R is a p.p. ring.

Proof. 2) $\Rightarrow$ 1). We can easily see that, for $(c, d) \in R \oplus M$, $\mathrm{Ann}(c, d) = \{(a, -ad) \mid a \in \mathrm{Ann}(a)\}$. Since R is a left p.p. ring, there is an idempotent $e \in R$ such that $\mathrm{Ann}_R(c) = \mathrm{Ann}(e)$. Then $(e, (1-e)d)$ is an idempotent of $R \oplus M$ and $\mathrm{Ann}(c, d) = \mathrm{Ann}(e, (1-e)d)$. Thus $R \oplus M$ is a p.p. near-ring.
1) $\Rightarrow$ 2). We first note that the set of idempotents of $R \oplus M$ is equal to $\{(e, (1-e)x) \mid e = e^2 \in R,\ x \in M\}$. Hence, for any $c \in R$, there exists idempotent $e \in R$ and $x \in M$ such that $\mathrm{Ann}(c, 0) = \mathrm{Ann}(e, (1-e)x)$. By the way, $\mathrm{Ann}(c, 0) = \{(a, 0) \mid a \in \mathrm{Ann}(c)\}$. On the other hand, $(1-e, -(1-e)x) \in \mathrm{Ann}\ (e, (1-e)x)$. Hence $(1-e)x = 0$, and so $\mathrm{Ann}\ (c, 0) = \mathrm{Ann}\ (e, 0)$. This implies $\mathrm{Ann}\ (c) = \mathrm{Ann}(e)$. Therefore R is a p.p. ring. □

We can extend the result [1, Theorem B] as following statement without reducibility.

COROLLARY 9. *Let R be a commutative ring with identity. Then the following conditions are equivalent:*

1) R is a Baer ring.

2) $(R[x], +, \cdot)$ is a Baer ring.

3) $(R_0[x], +, \circ)$ is a Baer near-ring.

ACKNOWLEDGEMENTS. The author gratefully acknowledges the kind hospitality he enjoyed at Okayama University.

References

[1] E. P. Armendariz, *A note on extensions of Baer and p.p. rings*, Australian Math. Soc. **18** (1974), 470–473.
[2] G. F. Birkenmeier and F. K. Huang, *Annihilator conditions on polynomials*, Comm. in Algebra, to appear.
[3] A. Hattori, *A foundation of the torsion theory over general rings*, Nagoya Math. J. **17** (1960), 147–158.
[4] I. Kaplansky, *Rings of Operators*, Math. Lecture Notes Series, Benjamin, New York, 1965.
[5] J. D. P. Meldrum, *Near-Rings and Their Links with Groups,* Pitman Advanced Publishing Program, Boston-London-Melbourne, 1985.
[6] G. Pilz, *Near-Rings,* North-Holland Publishing Company, Amsterdam-New York-Oxford, 1983.
[7] D. Ramakotaiah and V. Sambasivarao, *Reduced near-ring*, in Near-rings and Near-fields, G. Betsch (editor), North-Holland (1987), 233–243.

Department of Mathematics, Silla University, Pusan 617-736, Korea
E-mail: yucho@silla.ac.kr