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HOMOTOPY FIXED POINT SET FOR p-COMPACT TORAL GROUP

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ABSTRACT. First, we show the finiteness property of the homotopy fixed point set of p-discrete toral group. Let G_∞ be a p-discrete toral group and X be a finite complex with an action of G_∞ such that X^K is nilpotent for each finite p-subgroup K of G_∞. Assume X is $\mathbb{F}_p$-complete. Then X^{hG_∞} is $\mathbb{F}_p$-finite. Using this result, we give the condition so that X^{hG} is $\mathbb{F}_p$-finite for p-compact toral group G.

1. Introduction

Let G be a group acting on a space X. Then the fixed point set X^G is the G-equivariant mapping space from a point into X, denoted by $X^G = map^G(*, X)$. The homotopy fixed point set X^{hG} is defined to be the G-equivariant mapping space $map^G(EG, X)$ where EG is a universal contractible G-space. A G-map $f : X \to Y$ induces a map $f^{hG} : X^{hG} \to Y^{hG}$; if f is an ordinary(non-equivariant) homotopy equivalence, then f^{hG} is a homotopy equivalence. If G acts trivially on X, then X^{hG} is $map(BG, X)$. A proxy action of G on X is a space Y homotopy equivalent to X together with an action of G on Y. Standard homotopy theoretic constructions often give proxy actions of this type. If there is a proxy action of G on X under consideration we usually write X^{hG} for the associated homotopy fixed point set instead of introducing a symbol for the proxy space Y and writing Y^{hG}. Let $X_{hG} = EG \times_G X$ be a Borel construction, which is also called the homotopy orbit space of the action of G on X. Then the homotopy fixed point set is equivalent to the space of sections $\Gamma_s(X_{hG} \to BG)$ of the fibration $X_{hG} \to BG$.

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A p-discrete torus T_∞ of rank r is a discrete group which is isomorphic to $(\mathbf{Z}/p^\infty)^r$. A *p-discrete toral group* G_∞ is a discrete group which is an extension of a p-discrete torus by a finite p-group.

A *loop space* is a triple $\mathcal{X} = (\mathcal{X}, B\mathcal{X}, e)$, where $\mathcal{X}$ is a topological space, $B\mathcal{X}$ is a connected pointed classifying space of $\mathcal{X}$ and $e : \mathcal{X} \to \Omega B\mathcal{X}$ is a homotopy equivalence from $\mathcal{X}$ to the space $\Omega B\mathcal{X}$ of based loops in BX. Such a loop space is called *p-compact group* if $\mathcal{X}$ is $\mathbb{F}_p$-finite and $B\mathcal{X}$ is $\mathbb{F}_p$-complete. Here the second condition is equivalent to that $\mathcal{X}$ is $\mathbb{F}_p$-complete and $\pi_0(\mathcal{X})$ is a finite p-group. Main examples of p-compact groups are the p-completion of compact Lie groups G, $(C_{\mathbb{F}_p}(G), C_{\mathbb{F}_p}(BG), e)$, where $\pi_0(G)$ is a finite p-group and $e : \Omega C_{\mathbb{F}_p}(BG) \simeq C_{\mathbb{F}_p}(G)$. A p-compact torus T of rank r is a p-compact group such that BT is an Eilenberg-Mac Lane space of type $K((\mathbf{Z}_p)^r, 2)$. A p-compact toral group is a p-compact group which is an extension of a p-compact torus by a finite p-group.

For a loop space G, the *G-space* X is defined to be the fibration $EG \times_G X \to BG$ with X as the fibre. With such an action of G on X, the homotopy fixed point set X^{hG} is defined to be the space of sections of $X_{hG} \to BG$.

Dwyer and Wilkerson defined p-compact groups and proved a lot of their properties in ([4]), which are based on homotopy theoretic generalizations of compact Lie groups.

In this paper we are interested in the homotopy fixed point set X^{hG} for p-compact toral group G. We find the condition of X so that X^{hG} is $\mathbb{F}_p$-finite. By Dwyer and Wilkerson ([4]) it is known that any p-compact toral group G has a discrete approximation $f : G_\infty \to G$ and if X is a $\mathbb{F}_p$-complete space with an action of G, then f induces a homotopy equivalence $X^{hG_\infty} \to X^{hG}$. Using this theory and the fact that X^{hG_∞} is $\mathbb{F}_p$ -finite under some condition, we show that X^{hG} is $\mathbb{F}_p$-finite. The following is the finiteness property of the homotopy fixed point set of p-discrete toral group G_∞ which we will show first.

THEOREM 1.1. *Let G_∞ be a p-discrete toral group and X be an $\mathbb{F}_p$-complete, finite complex with an action of G_∞ such that X^K is nilpotent for each finite p-subgroup K of G_∞. Then X^{hG_∞} is $\mathbb{F}_p$-finite.*

Hence we conclude the following.

COROLLARY 1.2. *Let $f : G_\infty \to G$ be a discrete approximation of the p-compact toral group G, and let X be an $\mathbb{F}_p$-complete space with an action of G such that X^K is nilpotent for each finite p-subgroup K of G_∞. Then X^{hG} is $\mathbb{F}_p$-finite.*

This paper is organized as follows. In section 2, we give some definitions and properties as a background for understanding our main result. Section 3 gives the proof of our main result with some auxiliary properties.

Notations and terminology : Let p be a fixed prime number, $\mathbb{F}_p$ the field with p-elememts, $\mathbf{Z}_p$ the ring of p-adic integers. All unspecified homology and cohomology are assumed with coefficients in $\mathbb{F}_p$. A graded vector space H^* over a field $\mathbb{F}_p$ is of *finite type* if each H^i is finite dimensional over $\mathbb{F}_p$ and is *finite dimensional* if in addition $H^i = 0$ for all but a finite number of i. A space X is $\mathbb{F}_p$-*finite* if H^*X is finite dimensional over a field $\mathbb{F}_p$. A map is an $\mathbb{F}_p$-*equivalence* if it induces an isomorphism on $H^*(\_, \mathbb{F}_p)$.

2. Preliminaries

In this section we summarize some basic definitions and properties as a background for the section 3.

Bousfield and Kan ([1]) constructed a functor $C_{\mathbb{F}_p}(\_)$ on the category of spaces, called $\mathbb{F}_p$-*completion functor*, together with a natural map $\epsilon_X : X \to C_{\mathbb{F}_p}(X)$ for any X. If $f : X \to Y$ induces an isomorphism $H_*(X) \cong H_*(Y)$ then $C_{\mathbb{F}_p}(f)$ is a homotopy equivalence. A space X is $\mathbb{F}_p$-*good* if $H_*\epsilon_X$ is an isomorphism and $\mathbb{F}_p$-*complete* if ϵ_X is a homotopy equivalence. A space X is $\mathbb{F}_p$-*local* if any $\mathbb{F}_p$-equivalence $A \to B$ induces a homotopy equivalence $Map(B, X) \to Map(A, X)$. If $f : E \to B$ is a fibration with $\mathbb{F}_p$-local fibres, then the space of sections of f is $\mathbb{F}_p$-local ; if G is a discrete group acting on a $\mathbb{F}_p$-local space X, then X^{hG} is $\mathbb{F}_p$-local. If X is any space, then $C_{\mathbb{F}_p}(X)$ is $\mathbb{F}_p$-local since $C_{\mathbb{F}_p}(X)$ is constructed as a homotopy inverse limit. A space X is $\mathbb{F}_p$-complete if and only if X is both $\mathbb{F}_p$-local and $\mathbb{F}_p$-good.

REMARK 2.1. A space X is $\mathbb{F}_p$-good if and only if $C_{\mathbb{F}_p}(X)$ is $\mathbb{F}_p$-complete, or if and only if $C_{\mathbb{F}_p}(X)$ is $\mathbb{F}_p$-good [1].

A space X is called *nilpotent* if the action of $\pi_1 X$ on each $\pi_i X$ is nilpotent. Any nilpotent space is $\mathbb{F}_p$-good ([1]).

PROPOSITION 2.2.([1]) (Fibre Lemma) *Let $F \to E \to B$ be a fibration over the connected pointed space B. Assume that the monodromy action of $\pi_1 B$ on $H_i F$ is nilpotent for each $i \geq 0$. Then the induced sequence $C_{\mathbb{F}_p}(F) \to C_{\mathbb{F}_p}(E) \to C_{\mathbb{F}_p}(B)$ is also a fibration sequence.*

A *homomorphism* $f : H \to G$ of p-compact groups or loop spaces is a pointed map $Bf : BH \to BG$. A homomorphism f is an *equivalence* if Bf

is a homotopy equivalence and *trivial* if Bf is null homotopic. A homomorphism f is said to be a *monomorphism* if homotopy fiber G/H is $\mathbb{F}_p$-finite, and an *epimorphism* if $\Omega G/H$ is a p-compact group. A homomorphism of p-compact groups which is both a monomorphism and an epimorphism is an equivalence.

Suppose that $f : G_\infty \to G$ is a (loop space) homomorphism, where G_∞ is a p-discrete toral group and G is a p-compact toral group. If Bf is an $\mathbb{F}_p$-equivalence, then G_∞ is said to be a *discrete approximation* to G and G is said to be a closure of G_∞.

PROPOSITION 2.3.([4, 6.7]) *Let $f : G_\infty \to G$ be a discrete approximation of the p-compact toral group G, and let X be an $\mathbb{F}_p$-complete space with an action of G. Then f induces a homotopy equivalence $X^{hG} \to X^{hG_\infty}$.*

3. Homotopy fixed point set of the p-discrete toral group

In this section, first we will give properties of homotopy fixed point sets associated to actions of p-discrete toral groups on finite CW complexes. In this situation, the properties are contingent upon the spaces involved being $\mathbb{F}_p$-complete and nilpotent. Finally we show that X^{hG} is $\mathbb{F}_p$-finite for p-compact toral group G under some condition.

PROPOSITION 3.1.([4, 6.19]) *If G_∞ is a p-discrete toral group, then there exists an increasing chain $G_n \subset G_{n+1} \subset \cdots\cdots$ of finite subgroups of G_∞ such that $G_\infty = \cup_{m \geq n} G_m$.*

The following is the generalized Sullivan's conjecture proved independently by G. Carlsson, J. Lannes and H. Miller.

THEOREM 3.2.([2, 5, 6]) *Let A be a finite p-group, X be a finite A-complex. Then $C_{\mathbb{F}_p}(X^A) \to (C_{\mathbb{F}_p}(X))^{hA}$ is a homotopy equivalence.*

LEMMA 3.3. *Let G_∞ be a p-discrete toral group and G_∞ act on a finite complex X. Assume X is $\mathbb{F}_p$-complete. Then there exists N such that X^{hG_N} is homotopy equivalent to X^{hG_i} for $i \geq N$ where $\{G_m \mid m \geq 1\}$ is an increasing chain of finite subgroups of G_∞.*

Proof. Since X is a finite G_m-complex for each m, there is a decreasing chain $X = X^{\{1\}} \supset X^{G_1} \supset X^{G_2} \supset \cdots \supset X^{G_m} \supset \cdots$ for the increasing chain of finite subgroups of G_∞ as in 3.1. The fixed point set X^{G_m} consists of a finite number of equivariant cells since X^{G_m} is a finite complex. Thus the sequence must stabilize, and hence $X^{G_N} = X^{G_i}$ for each $i \geq N$. Taking $\mathbb{F}_p$-completion, $C_{\mathbb{F}_p}(X^{G_N}) = C_{\mathbb{F}_p}(X^{G_i})$ for $i \geq N$. For each m, $C_{\mathbb{F}_p}(X^{G_m}) \to (C_{\mathbb{F}_p}(X))^{hG_m}$ is a homotopy equivalence by 3.2. This implies $(C_{\mathbb{F}_p}(X))^{hG_N}$ is homotopy equivalent to $(C_{\mathbb{F}_p}(X))^{hG_i}$ for $i \geq N$. Therefore X^{hG_N} is homotopy equivalent to X^{hG_i} for $i \geq N$ since X is $\mathbb{F}_p$-complete. □

PROPOSITION 3.4. *Let G_∞ be a p-discrete toral group and G_∞ act on a finite complex X. Assume X is $\mathbb{F}_p$-complete. Then there is a finite subgroup A of G_∞ such that X^{hG_∞} is homotopy equivalent to X^{hA}.*

Proof. Now $G_\infty = \cup_{m \geq n} G_m$ as in 3.1. Then the space X^{hG_∞} is equivalent to the homotopy inverse limit of the tower $\{X^{hG_m} \mid m \geq n\}$. Therefore by the elementary property of homotopy inverse limit and 3.3, X^{hG_∞} is homotopy equivalent to X^{hA} for some finite subgroup A of G_∞. □

PROPOSITION 3.5. [1] *The class of $\mathbb{F}_p$-complete space is closed under the process of taking homotopy inverse limits.*

PROPOSITION 3.6. *Let G_∞ be a p-discrete toral group and X be an $\mathbb{F}_p$-complete, finite complex with an action of G_∞ such that X^K is nilpotent for each finite p-subgroup K of G_∞. Then X^{hK} is also $\mathbb{F}_p$-complete for each K.*

Proof. By using 2.1. and 3.2,

$$\begin{aligned} C_{\mathbb{F}_p}(X^{hK}) &\simeq C_{\mathbb{F}_p}((C_{\mathbb{F}_p}(X))^{hK}) \\ &\simeq C_{\mathbb{F}_p}(C_{\mathbb{F}_p}(X^K)) \\ &\simeq C_{\mathbb{F}_p}(X^K) \\ &\simeq (C_{\mathbb{F}_p}(X))^{hK} \\ &\simeq X^{hK}. \end{aligned}$$

Therefore X^{hK} is $\mathbb{F}_p$-complete. □

THEOREM 3.7.([4, 4.6]) *Let X be a space with an action of the finite p-group A. Assume that X is $\mathbb{F}_p$-finite and that for each subgroup $K \subset A$, X^{hK} is $\mathbb{F}_p$-complete. Then X^{hA} is $\mathbb{F}_p$-finite.*

THEOREM 3.8. *Let G_∞ be a p-discrete toral group and X be an $\mathbb{F}_p$-complete, finite complex with an action of G_∞ such that X^K is nilpotent for each finite p-subgroup K of G_∞. Then the homotopy fixed point set X^{hG_∞} is $\mathbb{F}_p$-finite.*

Proof. There exists a finite subgroup A of G_∞ such that X^{hG_∞} is homotopy equivalent to X^{hA} by 3.4. Since X^{hK} is $\mathbb{F}_p$-complete for each finite subgroup K of A by 3.6, X^{hA} is $\mathbb{F}_p$-finite by 3.7. Therefore X^{hG_∞} is $\mathbb{F}_p$-finite. □

COROLLARY 3.9. *Let $f : G_\infty \to G$ be a discrete approximation of the p-compact toral group G, and let X be an $\mathbb{F}_p$-complete finite complex with an action of G such that X^K is nilpotent for each finite p-subgroup K of G_∞. Then X^{hG} is $\mathbb{F}_p$-finite.*

Proof. The action of G on X induces the action of G_∞ by the proof of 2.3. Hence X^{hG_∞} is $\mathbb{F}_p$-finite by 3.8. Now X^{hG_∞} is homotopy eqivalent to X^{hG} by 2.3. Therefore X^{hG} is $\mathbb{F}_p$-finite. □

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