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A STUDY ON ADDITIVE ENDOMORPHISMS OF RINGS

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Abstract. In this paper, we initiate the investigation of rings in which all the additive endomorphisms are generated by ring endomorphisms(AGE-rings). This study was motivated by the work on the Sullivan's Research Problem [11]: Characterize those rings in which every additive endomorphism is a ring endomorphism(AE-rings). The purpose of this paper is to obtain a certain characterization of AGE-rings, and investigate some relations between AGE and LSD-generated rings.

1. Introduction

Throughout this paper, R denotes an associative ring not necessarily with identity, $End(R,+)$ the ring of additive endomorphisms of R, and $End(R,+,\cdot)$ the monoid of ring endomorphisms of R. For $X \subseteq R$, we use $gp < X >$ for the subgroup of $(R,+)$ generated by X.

We will consider that property $(*)$: Every additive mapping from R into itself is multiplicative, that is, every additive endomorphism of R is a ring endomorphism.

In 1977, R. P. Sullivan suggested the problem: Characterize all rings with the property $(*)$ in his "Research Problem 23" [11]. Since then, many ring theorists researched this problem. In 1981, K. H. Kim and F. W. Roush [10] classified finite rings, also in 1987, S. Dhompongsa and J. Sanwong [5] classified reduced case, and in 1988, S. Feigelstock [7] characterized torsion case with the property $(*)$.

In recent years, G. F. Birkenmeier and H. E. Heatherly [1], Y. Hirano [9] and M. Dugas, J. Hausen and J. A. Johnson [6] developed separate but equivalent formulations for AE-rings. This formulation included Feigelstock's solution of the torsion case from Birkenmeier and Heatherly [The-

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orem 4] as a Corollary, they characterized all non cube zero rings with the property $(*)$ from [1, Corollary 6].

S. Feigelstock defined a ring R with the property $(*)$, that is, in case

$$End(R,+) = End(R,+,\cdot),$$

R is called an *AE-ring*. Sometimes, we will use the notations $End(R,+)$ as $End_{\mathbb{Z}}(R)$ and $End(R,+,\cdot)$ as $End(R)$. We will generalize these AE-rings and then are going to characterize these general concepts.

2. Some results on AGE-rings

We begin by defining a general concept of AE-rings which will come up in this paper and will give their examples. First of all, before we can get down to the discussion of these rings, we will introduce the following notation and lemma:

$$GE(R) := gp < End(R,+,\cdot) >= gp < End(R) > .$$

LEMMA 2.1. *$(GE(R),+,\circ)$ is a subring of $End(R,+)$, where $\circ$ is a composition of mappings.*

Thus, we have the following new definition and examples.

DEFINITION 2.2. In case $End_{\mathbb{Z}}(R) = GE(R)$, R is called an AGE-ring.

Clearly, we see that every AE-ring is AGE, but not conversely from the following examples.

EXAMPLES 2.3.

(1) $\mathbb{Z}$ and $\mathbb{Z}_n$ ($n \geq 1$ in $\mathbb{Z}$) are AGE-rings but they are not AE-rings. For, $\mathbb{Z}$ and $\mathbb{Z}_n$ are additively generated by 1, and $End_{\mathbb{Z}}(\mathbb{Z}) \cong \mathbb{Z}$, $End_{\mathbb{Z}}(\mathbb{Z}_n) \cong \mathbb{Z}_n$, we see that $\mathbb{Z}$ and $\mathbb{Z}_n$ are both AGE-rings. However, $\mathbb{Z}$ and $\mathbb{Z}_n$ are all not AE-rings except the cases $\mathbb{Z}_1$ and $\mathbb{Z}_2$, because any nontrivial on $\mathbb{Z}$ or $\mathbb{Z}_n$ is additive endomorphism but which is not ring endomorphism.

(2) $\mathbb{Z} \oplus \mathbb{Z}$(or $\mathbb{Z}_n \oplus \mathbb{Z}_n$) is an AGE-ring. Indeed, from L. Fuch's Book [8, p182], we see the following:

$$End_{\mathbb{Z}}(\mathbb{Z} \oplus \mathbb{Z}) \cong M_2(End_{\mathbb{Z}}(\mathbb{Z}) \cong M_2(\mathbb{Z}).$$

Let $f \in End(\mathbb{Z} \oplus \mathbb{Z},+)$. Then we can regard f as $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ in $M_2(\mathbb{Z})$. Putting f_{ij} is 2×2-matrix with entries 1 for ij-th place and 0 otherwise. It is a straightforward verification that the f_{ij} are ring endomorphisms for $i = 1, 2$ and $j = 1, 2$, and that

$$f = a_{11}f_{11} + a_{12}f_{12} + a_{21}f_{21} + a_{22}f_{22},$$

in other words, additive endomorphism f is generated by ring endomorphisms. Hence $\mathbb{Z} \oplus \mathbb{Z}$ is an AGE-ring, but it is not an AE-ring, because above f is not a ring endomorphism.

Similarly, $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, \cdots\cdots$ are all AGE-rings.

(3) $\mathbb{Z}_2 \oplus \mathbb{Z}$ is an AGE-ring.

For finite rings case, we get the following examples:

(4) For each positive integer n, $\mathbb{Z}_n \oplus \mathbb{Z}_n, \mathbb{Z}_n \oplus \mathbb{Z}_n \oplus \mathbb{Z}_n, \cdots\cdots$ are all AGE-rings.

(5) For each two positive integers m and n with g.c.d. of m and n equal to 1, $\mathbb{Z}_m \oplus \mathbb{Z}_n$ is an AGE-ring.

From now onward, we investigate some properties of AGE-rings and relations with LSD-generated rings, after that, we will obtain another examples, and then characterize AGE-rings.

We can now extend the above results of (2) and (4) in Examples 2.3, as following:

PROPOSITION 2.4. *For every AGE-ring R, and for any positive integer n, we get that $\oplus_{i=1}^{n} R_i$ is an AGE-ring, where $R_i \cong R$, for all $i = 1, 2, \ldots, n$.*

Proof. We prove the case for $n = 2$, that is, $R \oplus R$. Similarly, we can prove for the case $n > 2$. We must show that

$$End_{\mathbb{Z}}(R \oplus R) = GE(R \oplus R).$$

Since $End_{\mathbb{Z}}(R \oplus R) \cong Mat_2(End_{\mathbb{Z}}(R))$, we obtain that

$$End_{\mathbb{Z}}(R \oplus R) \cong \begin{bmatrix} End_{\mathbb{Z}}(R) & End_{\mathbb{Z}}(R) \\ End_{\mathbb{Z}}(R) & End_{\mathbb{Z}}(R) \end{bmatrix} = \begin{bmatrix} GE(R) & GE(R) \\ GE(R) & GE(R) \end{bmatrix}.$$

Let $f \in End_{\mathbb{Z}}(R \oplus R)$ such that

$$f = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}, \ f_{ij} \in GE(R).$$

Then

$$f_{11} = \sum_i \lambda_i h_i, \ f_{12} = \sum_j \lambda_j h_j, \ f_{21} = \sum_k \lambda_k h_k, \ f_{22} = \sum_t \lambda_t h_t,$$

where, $\lambda's \in \mathbb{Z}$ and $h's \in End(R)$. Thus f is expressed of the form

$$f = \sum_i \lambda_i \begin{bmatrix} h_i & 0 \\ 0 & 0 \end{bmatrix} + \sum_j \lambda_j \begin{bmatrix} 0 & h_j \\ 0 & 0 \end{bmatrix} + \sum_k \lambda_k \begin{bmatrix} 0 & 0 \\ h_k & 0 \end{bmatrix} + \sum_t \lambda_t \begin{bmatrix} 0 & 0 \\ 0 & h_t \end{bmatrix}.$$

Since all $\begin{bmatrix} h_i & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & h_j \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ h_k & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & h_t \end{bmatrix}$ are ring endomorphisms of $R \oplus R$. Hence $R \oplus R$ is an AGE-ring. □

Obviously, we obtain the following lemmas.

LEMMA 2.5. *R is an AGE-ring if and only if there exists a subset S of $End(R)$ such that $End(R, +) = gp < S >$.*

The following notations and definitions are followed from G. F. Birkenmeier and H. E. Heatherly [2], [4].

$$L(R) = \{x \in R | xab = xaxb, \quad \text{for all} \quad a, b \in R\},$$

$$R(R) = \{x \in R | abx = axbx, \quad \text{for each} \quad a, b \in R\},$$

and

$$D(R) = L(R) \cap R(R).$$

In case $R = L(R)$, R is called an *LSD-ring*, $R = R(R)$, R is an *RSD-ring*, and $R = D(R)$, R is called a *SD-ring*.

Furthermore, if

$$R = gp < L(R) >, \ R = gp < R(R) > \quad \text{and} \quad R = gp < D(R) >,$$

then R is said to be *LSD-generated, RSD-generated and SD-generated* respectively.

LEMMA 2.6. *Let R be a ring and let $h \in End(R)$. If h is an onto mapping, then $L(R)$, $R(R)$ and $D(R)$ are all fully invariant under h.*

PROPOSITION 2.7. *Let R be a ring with identity. If R is an AGE-ring with $S \subseteq End(R)$ such that $End_{\mathbb{Z}}(R) = gp < S >$, and each element of S is onto, then R is an LSD-generated, moreover SD-generated.*

Proof. Let $x \in R$. Consider a left translation mapping $\phi_x : R \longrightarrow R$ by $\phi_x(a) = xa$ for each $a \in R$, which is a group endomorphism. Since R is an AGE ring,

$$\phi_x = \sum_i^n \lambda_i h_i,$$

where $\lambda_i \in \mathbb{Z}$ and $h_i \in End(R)$ such that h_i is onto, $i = 1, 2, \cdots, n$. Since $1 \in R$, $\phi_x(1) = \sum_i^n \lambda_i h_i(1)$, that is, $x = \sum_i^n \lambda_i h_i(1)$ and since $1 \in L(R) \cap R(R)$ by Lemma 2.6, $h_i(1) \in L(R) \cap R(R)$. Hence R is $LSD-$generated and $RSD-$generated, so $SD-$generated. □

EXAMPLES 2.8. Rings additively generated by central idempotents and one sided unities are LSD-generated and RSD-generated, so that SD-generated. In particular, we see that $\mathbb{Z}$ and $\mathbb{Z}_n$ are both LSD-generated and RSD-generated rings, further SD-generated rings. On the other hand, $x \in L(R)$ implies $x^3 = x^n$ for $n > 3$, then $L(S) = \{0\}$ for any nonzero proper subring S of $\mathbb{Z}$. Hence any nonzero proper subring of $\mathbb{Z}$ is an AGE-ring which is not LSD-generated and SD-generated.

From Examples 2.3, 2.8 and Proposition 2.4, there exist numerously many examples of AGE-rings and LSD-generated rings.

In Proposition 2.7, R is an AGE-ring by Lemma 2.5, we say that this kind of AGE-ring is an $AGOE$*-ring*.

The following is an extension of Lemma 2.6.

PROPOSITION 2.9. *If R is an $AGOE$-ring, then $gp < L(R) >$, $gp < R(R) >$ and $gp < D(R) >$ are all fully invariant subgroups of $(R, +)$.*

EXAMPLE 2.10[3]. Let S be an LSD-semigroup (i.e, $xab = xaxb$, for all $x, a, b \in S$). Then the semigroup ring $K[S]$, where K is $\mathbb{Z}$ or $\mathbb{Z}_n$, is an LSD-generated ring. In particular, let S be a nonempty set and define multiplication on S by $st = t$, for each $s, t \in S$. Then $\mathbb{Z}[S]$ and $\mathbb{Z}_n[S]$ are LSD-generated rings. Furthermore if $|S| = 2$, then $\mathbb{Z}_2[S]$ is an LSD-ring which is not an AGE-ring.

LEMMA 2.11. *If m and n are positive integers with $(m, n) = 1$, then*

$$Hom_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n) = 0 = Hom_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}_m).$$

From this Lemma, we obtain the following statement.

PROPOSITION 2.12. *Let m and n be positive integers. If m and n are relatively prime, then $\mathbb{Z}_m \oplus \mathbb{Z}_n$ is an AGE-ring.*

Proof Sketch.

$$End_{\mathbb{Z}}(\mathbb{Z}_m \oplus \mathbb{Z}_n) = \begin{bmatrix} End_{\mathbb{Z}}(\mathbb{Z}_m) & Hom_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}_m) \\ Hom_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n) & End_{\mathbb{Z}}(\mathbb{Z}_n) \end{bmatrix}$$

$$= \begin{bmatrix} GE(\mathbb{Z}_m) & 0 \\ 0 & GE(\mathbb{Z}_n) \end{bmatrix}. \qquad \square$$

Finally, we can improve above results and obtain a characterization of AGE-rings.

PROPOSITION 2.13. *Let $R = \oplus_{i=1}^n A_i$, where A_i is a ring for each i. Then R is an AGE-ring if and only if for each pair (i,j), $f_{\mathrm{ij}} \in Hom_Z(A_j, A_i)$ is of the form $f_{\mathrm{ij}} = \sum_\alpha \lambda_\alpha h_\alpha$, where $\lambda_\alpha \in \mathbb{Z}$ and h_α is a ring homomorphism from A_j into A_i.*

Proof. For convenience, we prove the case for $n = 2$. The case for $n > 2$ is similar. We see that $End_{\mathbb{Z}}(R) = End_{\mathbb{Z}}(A_1 \oplus A_2) \simeq M$, where

$$M = \begin{bmatrix} End_{\mathbb{Z}}(A_1) & Hom_{\mathbb{Z}}(A_2, A_1) \\ Hom_{\mathbb{Z}}(A_1, A_2) & End_{\mathbb{Z}}(A_2) \end{bmatrix}.$$

So we can represent $f \in End_{\mathbb{Z}}(R)$ by the matrix

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix},$$

where $f_{jk} \in Hom_{\mathbb{Z}}(A_k, A_j)$.

($\Longrightarrow$). Assume that R is an AGE-ring and $f_{jk} \in Hom_{\mathbb{Z}}(A_k, A_j)$. Consider $j = 2$ and $k = 1$. Then $\begin{bmatrix} 0 & 0 \\ f_{21} & 0 \end{bmatrix} \in M$. So $\begin{bmatrix} 0 & 0 \\ f_{21} & 0 \end{bmatrix} = \sum_{\alpha \in \Lambda} k_\alpha h_\alpha$, where each $k_\alpha \in \mathbb{Z}$ and each $h_\alpha \in M$ is a ring endomorphism. Thus

$$h_\alpha = \begin{bmatrix} h_{\alpha 11} & h_{\alpha 12} \\ h_{\alpha 21} & h_{\alpha 22} \end{bmatrix}.$$

By definition, each $h_{\alpha jk}$ is additive. Let $x, y \in A_1$. Then

$$\begin{bmatrix} h_{\alpha 11}(xy) \\ h_{\alpha 21}(xy) \end{bmatrix} = h_\alpha(\begin{bmatrix} x \\ 0 \end{bmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix}) = h_\alpha(\begin{bmatrix} x \\ 0 \end{bmatrix}) h_\alpha(\begin{bmatrix} y \\ 0 \end{bmatrix})$$

$$= \begin{bmatrix} h_{\alpha 11} & h_{\alpha 12} \\ h_{\alpha 21} & h_{\alpha 22} \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} \begin{bmatrix} h_{\alpha 11} & h_{\alpha 12} \\ h_{\alpha 21} & h_{\alpha 22} \end{bmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} h_{\alpha 11}(x) \\ h_{\alpha 21}(x) \end{bmatrix} \begin{bmatrix} h_{\alpha 11}(y) \\ h_{\alpha 21}(y) \end{bmatrix}.$$

Thus $f_{21} = \sum_{\alpha\in\Lambda} k_\alpha h_{\alpha 21}$, where each $h_{\alpha 21} : A_1 \longrightarrow A_2$ is a ring endomorphism.

Similarly, f_{11}, f_{12} and f_{22} are shown to have the desired properties.

($\Longleftarrow$) Let $f \in End_{\mathbb{Z}}(R)$ with matrix representation $\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \in M$. Then

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} \sum_{\alpha\in\Lambda} k_{\alpha 11} h_{\alpha 11} & \sum_{\alpha\in\Lambda} k_{\alpha 12} h_{\alpha 12} \\ \sum_{\alpha\in\Lambda} k_{\alpha 21} h_{\alpha 21} & \sum_{\alpha\in\Lambda} k_{\alpha 22} h_{\alpha 22} \end{bmatrix},$$

where each $k_{\alpha jk} \in Z$ and each $h_{\alpha jk} : A_k \longrightarrow A_j$ is a ring homomorphism.

Let $x, y \in A_1$ and $\alpha \in \Lambda$. Consider

$$\begin{bmatrix} 0 & 0 \\ h_{\alpha 21} & 0 \end{bmatrix} \begin{bmatrix} xy \\ 0 \end{bmatrix} = \begin{bmatrix} h_{\alpha 21}(xy) \\ 0 \end{bmatrix} = \begin{bmatrix} h_{\alpha 21}(x) h_{\alpha 21}(y) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ h_{\alpha 21} & 0 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ h_{\alpha 21} & 0 \end{bmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix}.$$

Clearly, $\begin{bmatrix} 0 & 0 \\ h_{\alpha 21} & 0 \end{bmatrix}$ is additive. Hence $\begin{bmatrix} 0 & 0 \\ h_{\alpha 21} & 0 \end{bmatrix}$ represents a ring endomorphism on R.

Similarly, $\begin{bmatrix} h_{\alpha 11} & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & h_{\alpha 12} \\ 0 & 0 \end{bmatrix}$ *and* $\begin{bmatrix} 0 & 0 \\ 0 & h_{\alpha 22} \end{bmatrix}$ are all ring endomorphisms on R. Thus

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \sum_{\delta\in\Delta} k_\delta h_{\delta,}$$

where each $k_\delta \in \mathbb{Z}$ and each h_δ represents a ring endomorphism on R. Therefore R is an AGE-ring. □

COROLLARY 2.14. *Let $R = \oplus_{i=1}^n A_i$, each A_i is an AGE-ring. If $Hom_{\mathbb{Z}}(A_i, A_j) = 0$ for each $i \neq j$, then R is an AGE-ring.*

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