Bull. Korean Math. Soc. **38** (2001), No. 1, pp. 121–128

ON JACOBSON MODULES

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ABSTRACT. In this paper, we define Jacobson modules which are the generalization of Jacobson rings. We give criteria of Jacobson modules and useful properties of Jacobson modules.

0. Introduction

Throughout this paper, every ring is commutative and has an identity. Many authors have studied about Jacobson (or Hilbert) rings. By making use of these rings, we can easily prove the famous Hilbert's Nullstellensatz (cf. [3; Theorem 32]). In [1,2,3], several interesting facts about Jacobson rings are given.

In this paper, we define a Jacobson module M which is the generalization of the Jacobson ring and provide useful properties of Jacobson rings.

In the first chapter, we define Jacobson modules (Definition 1.1) and give its examples (Example 1.3). Equivalence conditions between Artinian (or Noetherian) modules and Jacobson modules (Proposition 1.6) are also given. In the next chapter, we provide criteria of Jacobson modules (Theorem 2.2) and, for given two rings R and S, when S is an integral extension of R, if an R-module M is Jacobson then the S-module $M \otimes_R S$ is Jacobson (Theorem 2.5). On the other hand, if every prime ideal of $\mathrm{Supp}_R(M)$ is the contraction of a prime ideal of $\mathrm{Supp}_S(M \otimes_R S)$, then the converse holds.

1. Preliminaries

Let R be a commutative ring with an identity, $\mathfrak{p}$ a prime ideal of R and M an R-module.

Let $\mathrm{Spec}(R)$ be the set of all prime ideals of R, $\mathrm{Max}(R)$ the set of all maximal ideals of R, $\mathrm{Ann}(M)$ the set $\{r \in R : rM = 0\}$ and $\mathrm{Supp}(M)$ the set $\{\mathfrak{p} \in \mathrm{Spec}(R) : M_{\mathfrak{p}} \neq 0\}$.

Received September 9, 1999.

2000 Mathematics Subject Classification: Primary 13C13.

Key words and phrases: Jacobson ring, Jacobson module, integral.

For $\mathfrak{p} \in \text{Supp}(M)$, the M-height of $\mathfrak{p}$, denoted by $ht_M\mathfrak{p}$, is defined to be $dim_{R_\mathfrak{p}} M_\mathfrak{p}$.

We denote by MaxSupp(M) the set of all maximal members in Supp(M) and by MinSupp(M) the set of all minimal members in Supp(M). Then

MaxSupp(M) = Max$(R) \cap$ Supp(M)

MinSupp$(M) = \{\mathfrak{p} \in \text{Supp}(M) : ht_M\mathfrak{p} = 0\}$.

Let R be an integral domain with the quotient field K. Then recall that R is called a *G-domain* if as a ring, K is generated over R by one element. Obviously, every field is a G-domain. Recall further that a prime ideal $\mathfrak{p}$ of a commutative ring R is a *G-ideal* if $R/\mathfrak{p}$ is a G-domain.

Let R be a ring and M an R-module. The set of all elements r in R such that $r^nM = 0$ for some n forms an ideal of R called the *nilradical* of M and denoted by Nil(M). Then

$$\text{Nil}(M) = \sqrt{\text{Ann}(M)}.$$

The intersection of all maximal ideals in Supp(M) is called a *Jacobson radical* of M and denoted by

$$J(M) = \bigcap_{\mathfrak{m} \in \text{MaxSupp}(M)} \mathfrak{m}.$$

Recall that a ring R is called a *Jacobson ring* if every prime ideal of R is the intersection of a family of maximal ideals. Now, we define a *Jacobson R-module* which provides us with the natural generalization of the Jacobson ring.

DEFINITION 1.1. Let R be a ring. An R-module M is called a *Jacobson R-module* if every member in Supp(M) is the intersection of a family of members in MaxSupp(M).

It is clear that every module over a Jacobson ring is Jacobson. Conversely, let R be a ring and assume that every R-module is Jacobson. Then, in particular, R itself is a Jacobson R-module. If $\mathfrak{p}$ is a prime ideal of R, then $(R\backslash\mathfrak{p}) \cap 0 = \emptyset$; hence $R_\mathfrak{p} \neq 0$. This shows that Spec(R) = Supp(R). Therefore, R is a Jacobson ring. Consequently, we have the following result:

PROPOSITION 1.2. *Let R be a ring. Then R is a Jacobson ring if and only if every R-module is a Jacobson R-module.* $\square$

There exists a Jacobson module over a non-Jacobson ring. The example of this is given below.

EXAMPLE 1.3. Let R be the set of all formal power series in $\mathbb{Q}[[x]]$ whose constant terms are integers. That is, $R = \mathbb{Z} + x\mathbb{Q}[[x]]$. Then R is a G-domain and it is not Jacobson.

In fact, every non-zero prime ideal of R contains a non-zero element in R, for example, x and so by [3; Theorem 19], R is a G-domain. Notice that

$$\mathrm{Max}(R) = \{p\mathbb{Z} + x\mathbb{Q}[[x]] : p \text{ is a prime number}\}.$$

Then since

$$\cap_{\mathfrak{m} \in \mathrm{Max}(R)} \mathfrak{m} = x\mathbb{Q}[[x]] \neq 0,$$

we can prove that the zero ideal of R which is prime is not the intersection of any family of maximal ideals of R. Hence, R is not a Jacobson ring.

Let $M = R/x\mathbb{Q}[[x]]$. Then M is obviously an R-module. (Of course, it follows from the Second Isomorphism Theorem for modules that M is R-isomorphic to $\mathbb{Z}$.) Further,

$$\begin{aligned}\mathrm{Supp}(M) &= \{p\mathbb{Z} + x\mathbb{Q}[[x]] : p \text{ is a prime number or } 0\}\\ &= \mathrm{Max}(R) \cup \{x\mathbb{Q}[[x]]\}.\end{aligned}$$

Hence, by definition, M is a Jacobson R-module. □

Every ring R admits a Jacobson R-module. In fact, there exists a maximal ideal in R and for every maximal ideal $\mathfrak{m}$ of R, $R/\mathfrak{m}$ is a Jacobson R-module.

PROPOSITION 1.4. *Let M', M and M'' be R-modules. Consider the following exact sequence.*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

Then M' and M'' are Jacobson modules if and only if so is M.

Proof. ("if" part) Since $\mathrm{Supp}(M) = \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$, it is sufficient to show that if $\mathfrak{p} \in \mathrm{Supp}(M')$ (or respectively $\mathrm{Supp}(M'')$) and $\mathfrak{p} \subset \mathfrak{m}$ for some $\mathfrak{m} \in \mathrm{Supp}(M)$ then $\mathfrak{m} \in \mathrm{Supp}(M')$ (or respectively $\mathrm{Supp}(M'')$). But this is clear.

("only if" part) It is clear. □

PROPOSITION 1.5. *Let $M_1, M_2, \cdots, M_n$ be R-modules. Then $M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is Jacobson if and only if each M_i is Jacobson.*

Proof. Use the mathematical induction on n and Proposition 1.4 to prove the result. □

PROPOSITION 1.6. *Let M be an R-module. Then we have the following.*

(1) *Suppose M has only a finite number of maximal ideals in* $\mathrm{Supp}(M)$. *Then M is Jacobson if and only if it is Artinian.*
(2) *A Noetherian module M is Jacobson if and only if, for every prime ideal $\mathfrak{p} \in \mathrm{Supp}(M)$ such that $ht_M\mathfrak{p} = dim(M) - 1$, there exist infinitely many maximal ideals containing $\mathfrak{p}$ in* $\mathrm{Supp}(M)$.

Proof. (1) Assume that M is Jacobson. Then for all $\mathfrak{p} \in \mathrm{Supp}(M)$ we have $\mathfrak{p} = \bigcap_{\mathfrak{m} \in \mathrm{MaxSupp}(M)} \mathfrak{m}$. Therefore $\mathfrak{p} \in \mathrm{MaxSupp}(M)$ and hence it is Artinian. The other part is clear.

(2) Since M is Noetherian, we have $\mathrm{Supp}(M) = \mathrm{Supp}(R/\mathrm{Ann}(M))$ and $R/\mathrm{Ann}(M)$ is Noetherian by [4; Theorem 3.5]. Hence we may assume that $M = R$ is a Noetherian ring. It follows from [3; Theorem 147]. □

2. Criteria of Jacobson modules and changes of rings

In this chapter, we give criteria of Jacobson modules and provide useful properties. For a given ring homomorphism, we investigate changes of rings about Jacobson modules.

LEMMA 2.1. *Let M be an R-module. Then we have the following.*

(1) *If I is an ideal of R such that $I \supset \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Supp}(M)$, then the radical of I is the intersection of all G-ideals containing I in* $\mathrm{Supp}(M)$.
(2) *If M is finitely generated, then the nilradical* $\mathrm{Nil}(M)$ *of M is the intersection of all G-ideals in* $\mathrm{Supp}(M)$.

Proof. (1) If $\mathfrak{q} \in \mathrm{Supp}(R/I)$ then $\mathfrak{q} \in \mathrm{Supp}(M)$. Thus the part (1) follows immediately from [3; Theorem 26].

(2) Since $\mathrm{Supp}(R/\mathrm{Nil}(M)) = \mathrm{Supp}(R/\mathrm{Ann}(M)) = \mathrm{Supp}(M)$ and $R/\mathrm{Nil}(M)$ is nilpotent, the result follows from [3; Theorem 26]. □

THEOREM 2.2. *Let M be an R-module. Then the following statements are equivalent.*

(1) *M is a Jacobson module.*
(2) *Every radical ideal I such that $I \supset \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Supp}(M)$ is the intersection of maximal ideals in* $\mathrm{Supp}(M)$.
(3) *Every G-ideal in* $\mathrm{Supp}(M)$ *is the intersection of maximal ideals in* $\mathrm{Supp}(M)$.
(4) *Every G-ideal in* $\mathrm{Supp}(M)$ *is a maximal ideal of R.*
(5) *Every non-maximal prime ideal in* $\mathrm{Supp}(M)$ *is the intersection of a set of prime ideals which strictly contain it.*
(6) *$J(R/\mathfrak{p}) = 0$ for each $\mathfrak{p} \in \mathrm{Supp}(M)$.*
(7) *$R/\mathfrak{p}$ is a Jacobson ring for each $\mathfrak{p} \in \mathrm{Supp}(M)$.*
(8) *$R/\mathfrak{p}$ is a Jacobson ring for each $\mathfrak{p} \in \mathrm{MinSupp}(M)$.*
(9) *R/I is a Jacobson ring for each ideal I of R such that $I \supset \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{MinSupp}(M)$.*
(10) *$M[x_1, \ldots, x_n]$ is a Jacobson module.*

Proof. (1) $\Rightarrow$ (2) Let $B = \bigcap_{I \subset \mathfrak{m} \in \mathrm{MaxSupp}(M)} \mathfrak{m}$. We show $B = \sqrt{I}$. We have

$$B = \bigcap_{\sqrt{I} \subset \mathfrak{m} \in \mathrm{MaxSupp}(M)} \mathfrak{m}$$

since if $I \subset \mathfrak{m}$ then $\sqrt{I} \subset \mathfrak{m}$. Hence $\sqrt{I} \subset B$.

On the other hand, let $\mathfrak{q} \in \mathrm{MinSupp}(R/I)$. Then we have $\mathfrak{q} \in \mathrm{Supp}(M)$. Hence by the assumption

$$\begin{aligned} B = \bigcap_{I \subset \mathfrak{m} \in \mathrm{MaxSupp}(M)} \mathfrak{m} &\subset \bigcap_{\mathfrak{q} \in \mathrm{MinSupp}(R/I)} \left(\bigcap_{\mathfrak{q} \subset \mathfrak{m} \in \mathrm{MaxSupp}(R/I)} \mathfrak{m}\right) \\ &= \bigcap_{\mathfrak{q} \in \mathrm{MinSupp}(R/I)} \mathfrak{q} = \sqrt{I}. \end{aligned}$$

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (4) Suppose that a G-ideal $\mathfrak{p}$ is not maximal.
Then $\mathfrak{p} = \bigcap_{\mathfrak{p} \subset \mathfrak{m} \in \mathrm{MaxSupp}(M)} \mathfrak{m}$. Hence by [3; Theorem 19] there is an element $r \in R$ such that, for all maximal ideal $\mathfrak{m}$ with $\mathfrak{p} \subset \mathfrak{m}$, $r \in \mathfrak{m} \setminus \mathfrak{p}$. That is, we have the following contradiction.

$$r \in \bigcap_{\mathfrak{p} \subset \mathfrak{m} \in \mathrm{MaxSupp}(M)} \mathfrak{m} = \mathfrak{p}.$$

(4) $\Rightarrow$ (1) Since every maximal ideal is a G-ideal, by Lemma 2.1(1) and the hypothesis, for each $\mathfrak{p} \in \mathrm{Supp}(M)$, we have

$$\mathfrak{p} = \bigcap_{\mathfrak{p} \subset \mathfrak{m} \in \mathrm{MaxSupp}(M)} \mathfrak{m}.$$

(1) $\Rightarrow$ (5) Suppose that M is a Jacobson module. Then, if $\mathfrak{p} \in \mathrm{Supp}(M)$ such that $\mathfrak{p}$ is not maximal, then we have

$$\mathfrak{p} = \bigcap_{\mathfrak{p} \subset \mathfrak{m} \in \mathrm{MaxSupp}(M)} \mathfrak{m}.$$

Hence if $\mathfrak{p}$ is strictly contained in $\mathfrak{q}$ for some $\mathfrak{q} \in \mathrm{MinSupp}(M)$, then there is a maximal ideal $\mathfrak{m} \in \mathrm{Supp}(M)$ such that $\mathfrak{q} \subset \mathfrak{m}$. Therefore we get

$$\mathfrak{p} \subset \bigcap_{\mathfrak{p} \subset \mathfrak{q} \text{ and } \mathfrak{p} \neq \mathfrak{q}} \mathfrak{q} \subset \bigcap_{\mathfrak{p} \subset \mathfrak{m} \in \mathrm{MaxSupp}(M)} \mathfrak{m} = \mathfrak{p}.$$

(5) $\Rightarrow$ (1) We use the descending induction on the M-height. Let $\mathfrak{p} \in \mathrm{Supp}(M)$ such that $ht_M \mathfrak{p} = dim(M) - 1$. Then from the hypothesis

$$\mathfrak{p} = \bigcap_{\mathfrak{p} \subset \mathfrak{m} \in \mathrm{MaxSupp}(M)} \mathfrak{m}.$$

If $\mathfrak{p} \in \mathrm{Supp}(M)$ such that $ht_M\mathfrak{p} = dim(M) - n$ $(n > 1)$, then from (5) and the descending induction hypothesis

$$\mathfrak{p} = \bigcap_{\mathfrak{p}\subset\mathfrak{q} \text{ and } \mathfrak{p}\neq\mathfrak{q}} \mathfrak{q} = \bigcap_{\mathfrak{p}\subset\mathfrak{m}\in\mathrm{MaxSupp}(M)} \mathfrak{m}.$$

(1) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) $\Leftrightarrow$ (9) These follow from the definition of Jacobson module.

(1) $\Leftrightarrow$ (10) Since $M[x_1, \ldots, x_n] = M \otimes_R R[x_1, \ldots, x_n]$, we have

$$\mathrm{Supp}(M[x_1, \ldots, x_n]) \subset \mathrm{Supp}(M).$$

On the other hand, since $R[x_1, \ldots, x_n]$ is a free R-module, we may assume that M is a submodule of $M[x_1, \ldots, x_n]$. Thus we get $\mathrm{Supp}(M) = \mathrm{Supp}(M[x_1, \ldots, x_n])$ and then the assertion follows from this. □

PROPOSITION 2.3. *Let M and N be R-modules. Then we have the following.*

(1) *If $R/\mathrm{Ann}(M)$ is a Jacobson ring, then M is a Jacobson module.*
(2) *If M or N is Jacobson, then so is $M \otimes_R N$.*
(3) *If M or N is Jacobson, where M is finitely presented, then so is $\mathrm{Hom}_R(M, N)$.*
(4) *If S is a multiplicatively closed subset of R and M is a Jacobson R-module, then $S^{-1}M$ is Jacobson as an R-module.*

Proof. (1) The assertion holds from $\mathrm{Supp}(M) \subset \mathrm{Supp}(R/\mathrm{Ann}(M))$.

(2) We have the result from $\mathrm{Supp}(M \otimes_R N) \subset \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$.

(3) By [5; Theorem 3.84], we have $\mathrm{Supp}(\mathrm{Hom}_R(M, N)) \subset \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$. The conclusion follows from this fact.

(4) Since $\mathrm{Supp}(S^{-1}M) \subset \mathrm{Supp}(M)$, by (2) $S^{-1}M = M \otimes_R S^{-1}R$ is a Jacobson R-module. □

REMARK 2.4. If S is a multiplicatively closed subset of R and M is a Jacobson R-module, then $S^{-1}M$ is Jacobson as an $S^{-1}R$-module if and only if it is Artinian by 1.6.

THEOREM 2.5. *Let $f : R \longrightarrow S$ be a ring homomorphism and $f^* : \mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R)$ the mapping associated with f. Then we have the following.*

(1) *If M is a Jacobson R-module and S is integral over R, then $M \otimes_R S$ is a Jacobson S-module.*
(2) *If M is a Jacobson R-module and S is an extension ring of R which is finitely generated as an R-module, then $M \otimes_R S$ is a Jacobson S-module.*

Moreover, suppose S is an integral extension ring of R. Then we have the following.

(3) *If M is an R-module such that* $\mathrm{Supp}_R(M) = f^*(\mathrm{Supp}_S(M \otimes_R S))$ *and* $M \otimes_R S$ *is a Jacobson S-module, then M is a Jacobson R-module.*

(4) *If there is a Jacobson S-module N which is faithfully flat over R, then R is a Jacobson ring.*

Proof. (1) By [1; p.107, Proposition 19] we have

$$f^*(\mathrm{Supp}_S(M \otimes_R S)) \subset \mathrm{Supp}_R(M).$$

Hence we have, for each $\mathfrak{p} \in \mathrm{Supp}_S(M \otimes_R S)$, there is $f^*(\mathfrak{p}) \in \mathrm{Supp}_R(M)$. Then $R/f^*(\mathfrak{p})$ is a Jacobson ring by 2.2(7) and $S/\mathfrak{p}$ is integral over $R/f^*(\mathfrak{p})$ by [1; p.305, Proposition 2]. Therefore $S/\mathfrak{p}$ is a Jacobson ring by [1; p.352, Proposition 5]. From 2.2(7), $M \otimes_R S$ is a Jacobson S-module.

(2) By [1; p.305, Proposition 1], S is integral over R, it follows from (1).

(3) For each $\mathfrak{p} \in \mathrm{Supp}_R(M)$ there is $\mathfrak{p}' \in \mathrm{Supp}_S(M \otimes_R S)$ such that $\mathfrak{p}'$ lies above $\mathfrak{p}$. Then $S/\mathfrak{p}'$ is a Jacobson ring by 2.2(7) and $S/\mathfrak{p}'$ is integral over $R/\mathfrak{p}$ by [1; p.305, Proposition 2]. Therefore by 2.2(6) we have

$$J(S/\mathfrak{p}') = \mathfrak{p}'.$$

By [1; p.328, Corollary 3] we have

$$J(R/\mathfrak{p}) = J(S/\mathfrak{p}') \cap R = \mathfrak{p}' \cap R = \mathfrak{p}.$$

Hence by 2.2(6) we have M is a Jacobson R-module.

(4) Since, for each $\mathfrak{p} \in \mathrm{Spec}(R)$, there is $\mathfrak{p}' \in \mathrm{Supp}_S(N)$ such that $\mathfrak{p}'$ lies above $\mathfrak{p}$ by [4; Theorem 7.3], a similar proof of (3) gives the conclusion. □

Corollary 2.6. *Let $f : R \longrightarrow S$ be a ring homomorphism and f^* :* $\mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R)$ *the mapping associated with f. Then we have the following.*

(1) *If R is Jacobson and S is integral over R, then S is Jacobson.*

(2) *If R is Jacobson and S is an extension ring of R which is finitely generated as an R-module, then S is Jacobson.*

Moreover, suppose S is an integral extension ring of R. Then we have the following.

(3) *If* $\mathrm{Spec}(R) = f^*(\mathrm{Spec}(S))$ *and S is Jacobson, then R is Jacobson.*

(4) *If S is a Jacobson ring which is faithfully flat over R, then R is a Jacobson ring.* □

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