

FIXED POINT THEOREMS OF QUASICOMPACT MULTIVALUED MAPPINGS IN GENERAL TOPOLOGICAL VECTOR SPACES

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ABSTRACT. Using the notions of admissibility, local convexity and measure of noncompactness, we give new fixed point theorems for quasicompact or condensing multivalued mappings with domains that are not necessarily convex subsets of an arbitrary topological vector space.

1. Introduction

In the past forty years, the fixed point theory in not necessarily locally convex topological vector spaces is intensively developed by many authors, see [1-5,7,10-15].

O. Hadžić [4] gave some properties of measures of noncompactness in paranormed spaces and fixed point theorems for condensing multivalued mappings in general topological vector spaces. S. Hahn [11] extended Schauder's fixed point theorem to noncompact multivalued mappings with nonconvex domains in a topological vector space. See also [9].

The purpose in this paper is to obtain new fixed point theorems for quasicompact or condensing multivalued mappings with domains that are not necessarily convex subsets of an arbitrary topological vector space, where the concepts of admissible and locally convex sets play a fundamental role.

The proof of the main fixed point theorem for quasicompact mappings is based on the related result for compact mappings due to T. Jerofsky [13]. Moreover, we show that the fixed point theorem for quasicompact mappings is applied to obtain the related result for condensing mappings

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by introducing the notion of φ -measure of noncompactness in a topological vector space, see [4].

In Section 2, we prove the Leray-Schauder fixed point theorem for quasi-compact multivalued mappings with closed convex values in general topological vector spaces. In Section 3, the Schauder fixed point theorem for noncompact multivalued mappings with not necessarily convex domains in a topological vector space is presented.

In this paper all topological vector spaces are assumed to be real and Hausdorff. Let K be a subset of a topological vector space E . The closure, the convex hull, and the closed convex hull of K in E are denoted by \bar{K} , $\text{co}K$, and $\bar{\text{co}}K$, respectively. If U is a subset of K , then the boundary of U with respect to the relative topology on K is denoted by $\partial_K U$.

Further, we denote the collection of all nonempty, closed and convex subsets of K by $c(K)$, and the collection of all finite unions of closed and convex subsets of E by $uc(E)$.

Let E be a topological vector space, U and K subsets of E . A (multi-valued) mapping $F : U \rightarrow c(K)$ is said to be *upper semicontinuous* on U if for any open subset V of K , the set $\{x \in U : F(x) \subset V\}$ is open in U . F is said to be *compact* if its range $F(U)$ is relatively compact.

A mapping $F : U \rightarrow c(K)$ is said to be *quasicompact* if there exists a closed convex subset S of E with the property that $U \cap S \neq \emptyset$, $F(U \cap S) \subset S$ and $F(U \cap S)$ is relatively compact. Such a set S is called a *characteristic set* for F . See [9,11].

Let E be a topological vector space, K a nonempty subset of E , and Ψ a collection of subsets of $\bar{\text{co}}K$ with the property that for any $M \in \Psi$, the sets \bar{M} , $\text{co}M$, $M \cup \{u\}$ ($u \in K$) and every subset of M belong to Ψ .

Let A be a partially ordered set with the partial ordering \leq , and φ a function from A into itself. A function $\gamma : \Psi \rightarrow A$ is said to be a φ -*measure of noncompactness* on K if the following conditions are satisfied for any $M \in \Psi$:

- (1) $\gamma(\bar{M}) = \gamma(M)$;
- (2) if $u \in K$, then $\gamma(M \cup \{u\}) = \gamma(M)$;
- (3) if $N \subset M$, then $\gamma(N) \leq \gamma(M)$ (monotone);
- (4) $\gamma(\text{co}M) \leq \varphi(\gamma(M))$.

If the following condition is required instead of (4)

$$\gamma(\text{co}M) \leq \gamma(M)$$

then γ is called a *measure of noncompactness* on K . See [4].

Let U and K be subsets of E such that $U \subset K$, and γ a φ -measure of noncompactness on K . A mapping $F : U \rightarrow c(K)$ is said to be (γ, φ) -condensing if for every $N \subset U$, the inequality $\gamma(N) \leq \varphi(\gamma(F(N)))$ implies that $F(N)$ is relatively compact. In particular, if φ is the identity map, then F is called γ -condensing. See [8].

A nonempty subset X of a topological vector space E is said to be *admissible* in the sense of Klee [14] provided that, for every compact subset K of X and every neighborhood V of the origin 0 in E , there exists a continuous function $h : K \rightarrow X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E . See [12].

It is known that every nonempty convex subset of a locally convex topological vector space is admissible.

A subset K of a topological vector space E is said to be *locally convex* if for every $x \in K$ there exists a base of neighborhoods $U(x)$ of x in K such that $U(x) = W(x) \cap K$ and $W(x)$ is a convex subset of E . See [10,15].

Every subset of a locally convex topological vector space is a locally convex set. Every subset of a locally convex set in a topological vector space is locally convex.

2. The Leray-Schauder fixed point theorem for quasicompact mappings

Using the concepts of admissibility and local convexity, we give new fixed point theorems for quasicompact multivalued mappings with the Leray-Schauder boundary condition in general topological vector spaces. As applications, the existence of fixed point for condensing mappings is guaranteed.

We begin with the following result which can be found in [13, Folgerung 4.3.5].

LEMMA 2.1. *Let E be a topological vector space, and K an admissible subset of E with $K \in uc(E)$ such that K is starshaped with respect to $u \in K$. Let $Y \subset K$ be an in K closed neighborhood of u . Let $F : Y \rightarrow c(K)$ be a compact upper semicontinuous mapping. If $x \notin tF(x) + (1-t)u$ for every $x \in \partial_K Y$ and $t \in (0, 1)$, then F has a fixed point.*

THEOREM 2.2. *Let E be a topological vector space, K starshaped with respect to $u \in K$ with $K \in uc(E)$, and $U \subset K$ an in K closed neighborhood of u . Let $F : U \rightarrow c(K)$ be a quasicompact upper semicontinuous mapping with a characteristic set S containing u such that $K \cap S$ is admissible and*

$x \notin tF(x) + (1-t)u$ for every $x \in \partial_K U$ and $t \in (0, 1)$. Then there exists a point $x \in U$ such that $x \in F(x)$.

Proof. By hypotheses, S is a closed and convex subset of E , $u \in U \cap S$, $F(U \cap S) \subset S$ and $F(U \cap S)$ is relatively compact. Let $Y := U \cap S$ and $K_0 := K \cap S$. Then Y is an in K_0 closed neighborhood of u , and K_0 is admissible, starshaped with respect to u and $K_0 \in uc(E)$ because S is a closed and convex subset of E containing u and $K \in uc(E)$.

Let $F_0 := F|_Y : Y \rightarrow c(K_0)$ be the restriction of F to Y . Then F_0 has nonempty, closed and convex values in K_0 since $F_0(y) = F(y) \cap S$ and $F(y) \in c(K)$ for each $y \in Y$. Moreover, it is clear that F_0 is a compact upper semicontinuous mapping. Since $\partial_{K_0} Y \subset \partial_K U$, we have $x \notin tF_0(x) + (1-t)u$ for every $x \in \partial_{K_0} Y$ and $t \in (0, 1)$. By Lemma 2.1, there exists a point $x \in Y (\subset U)$ such that $x \in F_0(x) = F(x)$. This completes the proof. \square

The following result is an immediate consequence of Theorem 2.2.

THEOREM 2.3. *Let E be a topological vector space, K a closed and convex subset of E , and $U \subset K$ an in K closed neighborhood of u . Let $F : U \rightarrow c(K)$ be a quasicompact upper semicontinuous mapping with a characteristic set S containing u such that $K \cap S$ is admissible and $x \notin tF(x) + (1-t)u$ for every $x \in \partial_K U$ and $t \in (0, 1)$. Then F has a fixed point.*

COROLLARY 2.4 [6, SATZ 2]. *Let E be a locally convex topological vector space, K a closed and convex subset of E with $0 \in K$, and W a closed neighborhood of 0 in E . Let $F : W \cap K \rightarrow c(E)$ be a compact upper semicontinuous mapping with $F(W \cap K) \subset K$ such that $\beta x \notin F(x)$ for every $x \in \partial_E W \cap K$ and $\beta > 1$. Then F has a fixed point.*

Proof. This result follows from Theorem 2.3 because every nonempty convex subset of a locally convex topological vector space is admissible. \square

The following fact which gives the relation between admissible and local convex sets is useful for the fixed point theory in topological vector spaces. See [13, Satz 1.5.3].

PROPOSITION 2.5. *Let E be a topological vector space and K a locally convex subset of E with $K \in uc(E)$. Then K is admissible.*

THEOREM 2.6. *Let K be a locally convex subset of a topological vector space E such that K is starshaped with respect to $u \in K$ and $K \in uc(E)$. Let $U \subset K$ be an in K closed neighborhood of u , and γ a φ -measure of*

noncompactness on E . Let $F : U \rightarrow c(K)$ be a (γ, φ) -condensing upper semicontinuous mapping such that $x \notin tF(x) + (1-t)u$ for every $x \in \partial_K U$ and $t \in (0, 1)$. Then F has a fixed point.

Proof. Let $\Sigma = \{S \subset E : S = \overline{\text{co}}S, u \in S, F(U \cap S) \subset S\}$. Then Σ is nonempty since $\overline{\text{co}}(F(U) \cup \{u\}) \in \Sigma$. Let $S_0 = \bigcap_{S \in \Sigma} S$ and $S_1 = \overline{\text{co}}(F(U \cap S_0) \cup \{u\})$. From $S_0 \in \Sigma$ it follows that $S_1 \subset S_0$ and so $S_1 \in \Sigma$ (since $F(U \cap S_1) \subset F(U \cap S_0) \subset S_1$) and hence $S_0 \subset S_1$. Thus, $S_0 = \overline{\text{co}}(F(U \cap S_0) \cup \{u\})$. Since $\gamma(U \cap S_0) \leq \gamma(\text{co}(F(U \cap S_0) \cup \{u\})) \leq \varphi(\gamma(F(U \cap S_0)))$, the set $F(U \cap S_0)$ is relatively compact. We conclude that F is quasicompact with characteristic set S_0 . As K is locally convex, the set $K \cap S_0$ is also locally convex. Because of $K \cap S_0 \in \text{uc}(E)$, by Proposition 2.5, $K \cap S_0$ is admissible. By Theorem 2.2, F has a fixed point. This completes the proof. \square

COROLLARY 2.7 [8, SATZ 5]. Let E be a locally convex topological vector space, K a closed and convex subset of E with $0 \in K$, and W an open neighborhood of 0 in E . Let γ be a measure of noncompactness on E and $F : \overline{W \cap K} \rightarrow c(E)$ a γ -condensing upper semicontinuous mapping with $F(\overline{W \cap K}) \subset K$ such that $\beta x \notin F(x)$ for every $x \in \partial_K(W \cap K)$ and $\beta > 1$. Then F has a fixed point.

Proof. Since K as a subset of locally convex topological vector space E is locally convex, the conclusion follows by applying Theorem 2.6. \square

REMARK. If γ is a c -measure of noncompactness on K , then we refer to [10, Theorem].

Now we show that Theorem 2.3 implies a result of O. Hadžić, see [4, Theorem 1].

THEOREM 2.8. Let E be a topological vector space, K a closed and convex subset of E with the property that every closed and convex subset of K is admissible, and $U \subset K$ an in K closed neighborhood of u . Let γ be a φ -measure of noncompactness on K and $F : U \rightarrow c(K)$ a (γ, φ) -condensing upper semicontinuous mapping such that $x \notin tF(x) + (1-t)u$ for every $x \in \partial_K U$ and $t \in (0, 1)$. Then F has a fixed point.

Proof. Let $\Sigma = \{S \subset E : S = \overline{\text{co}}S, u \in S, F(U \cap S) \subset S\}$. Then Σ is nonempty with $S = K$. Let $S_0 = \bigcap_{S \in \Sigma} S$ and $S_1 = \overline{\text{co}}(F(U \cap S_0) \cup \{u\})$. As in the proof of Theorem 2.6, it follows that $S_0 = \overline{\text{co}}(F(U \cap S_0) \cup \{u\}) \subset K$, and the set $F(U \cap S_0)$ is relatively compact. Since the set S_0 is a closed and convex subset of K and so admissible, by Theorem 2.3, F has a fixed point. This completes the proof. \square

3. The Schauder fixed point theorem for quasicompact mappings

In this section, generalized versions of the Schauder fixed point theorem for quasicompact or condensing multivalued mappings with not necessarily convex domains in a topological vector space are obtained.

THEOREM 3.1. *Let E be a topological vector space, G a starshaped set with respect to $u \in G$ with $G \in uc(E)$, and $F : G \rightarrow c(G)$ a quasicompact upper semicontinuous mapping with a characteristic set S containing u such that $G \cap S$ is an admissible subset of E . Then F has a fixed point.*

Proof. By assumptions, S is a closed and convex subset of E , $u \in G \cap S$, $F(G \cap S) \subset S$, and $F(G \cap S)$ is relatively compact. Let $F_0 := F|_{G_0}$ be the restriction of F to $G_0 := G \cap S$. Then G_0 is admissible, starshaped with respect to u and $G_0 \in uc(E)$, and F_0 is a compact upper semicontinuous mapping of G_0 into $c(G_0)$. By Lemma 2.1, F_0 has a fixed point and so F . This completes the proof. \square

THEOREM 3.2. *Let G be a nonempty, closed and convex subset of a topological vector space E , and $F : G \rightarrow c(G)$ a quasicompact upper semicontinuous mapping with a characteristic set S such that $G \cap S$ is an admissible subset of E . Then F has a fixed point.*

From Theorem 3.2, we deduce the following known results for quasicompact or condensing mappings, see [11, Theorem 1], [4, Theorem 3].

THEOREM 3.3. *Let E be a topological vector space, G a nonempty, locally convex, closed and convex subset of E , and $F : G \rightarrow c(G)$ a quasicompact upper semicontinuous mapping. Then F has a fixed point.*

Proof. Together with characteristic set S for F , the set $G \cap S$ is closed, convex and locally convex and so admissible by Proposition 2.5. Theorem 3.2 implies that F has a fixed point. \square

THEOREM 3.4. *Let E be a topological vector space, and G a nonempty, closed and convex subset of E such that every closed and convex subset of G is admissible. Suppose that γ is a φ -measure of noncompactness on G and $F : G \rightarrow c(G)$ is a (γ, φ) -condensing upper semicontinuous mapping. Then F has a fixed point.*

Proof. Let $z \in G$ and $\Sigma = \{S \subset G : S = \overline{\text{co}}S, z \in S, F(S) \subset S\}$. Then Σ is nonempty since $S = G$. Let $Z = \bigcap_{S \in \Sigma} S$ and $Z_1 = \overline{\text{co}}(F(Z) \cup \{z\})$. From $Z \in \Sigma$ it follows that $Z_1 \subset Z$ and so $Z_1 \in \Sigma$ and hence $Z \subset Z_1$. Consequently, $Z = \overline{\text{co}}(F(Z) \cup \{z\}) \subset G$. By properties of γ , we have $\gamma(Z) \leq$

$\varphi(\gamma(F(Z) \cup \{z\})) = \varphi(\gamma(F(Z)))$ and hence $F(Z)$ is relatively compact. We have shown that F is a quasicompact mapping with characteristic set $Z (= G \cap Z)$ which is an admissible subset of E . By Theorem 3.2, F has a fixed point. This completes the proof. \square

THEOREM 3.5. *Let E be a topological vector space, and G a nonempty, locally convex, closed and convex subset of E . Suppose that γ is a φ -measure of noncompactness on G and $F : G \rightarrow c(G)$ is a (γ, φ) -condensing upper semicontinuous mapping. Then F has a fixed point.*

Proof. Following the proof of Theorem 3.4, we conclude that F is quasicompact. By Theorem 3.3, F has a fixed point. \square

COROLLARY 3.6. *Let G be a nonempty, closed and convex subset of a locally convex topological vector space E , γ a measure of noncompactness on G , and $F : G \rightarrow c(G)$ a γ -condensing upper semicontinuous mapping. Then F has a fixed point.*

REMARK. In fact, we can see that Theorem 3.4 and Theorem 3.5 remain true without assuming the condition of monotonicity in the definition of φ -measure of noncompactness.

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