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STABILITY IN VARIATION FOR NONLINEAR VOLTERRA DIFFERENCE SYSTEMS

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ABSTRACT. We investigate the property of h-stability, which is an important extension of the notions of exponential stability and uniform Lipschitz stability in variation for nonlinear Volterra difference systems.

1. Introduction

Consider the nonlinear Volterra difference system

$$x(n+1) = f(n, x(n)) + \sum_{s=n_0}^{n} g(n, s, x(s)) \tag{1}$$

where $f : \mathbb{N}(n_0) \times \mathbb{R}^m \to \mathbb{R}^m$, $g : \mathbb{N}(n_0) \times \mathbb{N}(n_0) \times \mathbb{R}^m \to \mathbb{R}^m$, $\mathbb{N}(n_0) = \{n_0, n_0+1, \cdots, n_0+k, \cdots\}$ (n_0 a nonnegative integer), $\mathbb{R}^m$ is the $m-$dimensional real euclidean space. We assume that $f_x = \frac{\partial f}{\partial x}$ and $g_x = \frac{\partial g}{\partial x}$ exist and are continuously invertible on $\mathbb{N}(n_0)\times\mathbb{R}^m$, $\mathbb{N}(n_0)\times\mathbb{N}(n_0)\times\mathbb{R}^m$, respectively. Also, we assume that $f(n, 0) = 0$ and $g(n, s, 0) = 0$. Let $x(n) = x(n, n_0, x_0)$ be the solution of system (1) with $x(n_0, n_0, x_0) = x_0$.

Also, we consider the associated Volterra variational systems

$$v(n+1) = f_x(n, 0)v(n) + \sum_{s=n_0}^{n} g_x(n, s, 0)v(s) \tag{2}$$

and

$$z(n+1) = f_x(n, x(n, n_0, x_0))z(n) + \sum_{s=n_0}^{n} g_x(n, s, x(s))z(s) \tag{3}$$

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of (1). The fundamental matrix solution $\Phi(n, n_0, 0)$ of (2) is given by

$$\Phi(n, n_0, 0) = \frac{\partial x(n, n_0, 0)}{\partial x_0}$$

and the fundamental matrix solution $\Phi(n, n_0, x_0)$ of (3) is given by

$$\Phi(n, n_0, x_0) = \frac{\partial x(n, n_0, x_0)}{\partial x_0}.$$

See [1] or [6].

If we assume that $x(n, n_0, x_0)$ and $x(n, n_0, y_0)$ are the solutions of (1) through (n_0, x_0) and (n_0, y_0) respectively, which exist for $n \geq n_0$, and such that x_0 and y_0 belong a convex subset of $\mathbb{R}^m$, then for $n \geq n_0$, we have

$$(4) \quad x(n, n_0, y_0) - x(n, n_0, x_0) = \int_0^1 \Phi(n, n_0, x_0 + s(y_0 - x_0))ds \cdot (y_0 - x_0).$$

See [6, Lemma 2.1]. Furthermore, from the formula (4), we obtain

$$(5) \qquad x(n, n_0, x_0) = \int_0^1 \frac{\partial x(n, n_0, sx_0)}{\partial x_0} ds \cdot x_0.$$

The purpose of this paper is to study the h-stability in variation for the system (1). In stability theory for difference systems the notion of h-stability introduced by Medina and Pinto [8, 9] is an important extension of the notions of exponential asymptotic stability and uniform Lipschitz stability [5]. It is very useful because, when we study the asymptotic stability it is not easy to work with non-exponential types of stability. The systematic approach of h-stability for nonlinear difference system $x(n+1) = f(n, x(n))$ can be found in [8] and [9]. Also, we studied the h-stability in variation for nonlinear difference systems in [4].

We recall some definitions of stability in [8]. The symbol $|\cdot|$ will be used to denote any convenient vector norm on $\mathbb{R}^m$.

DEFINITION 1. System (1) is called an *h-system* around the null solution, or more briefly an *h-system*, if there exist a positive function $h : \mathbb{N}(n_0) \to \mathbb{R}$ and a constant $c \geq 1$, such that

$$|x(n, n_0, x_0)| \leq c|x_0|h(n)h^{-1}(n_0), \quad n \geq n_0$$

for $|x_0|$ small enough (here $h^{-1}(n) = \frac{1}{h(n)}$).

The function h as well as the constant c depends only on f. If h is a bounded function, then an h-system permits the following types of stability :

DEFINITION 2. The zero solution of system (1), or more briefly system (1), is said to be

(hS) *h-stable* if $c \geq 1$, δ exist as well as a positive bounded function $h : \mathbb{N}(n_0) \to \mathbb{R}$ such that

$$|x(n, n_0, x_0)| \leq c|x_0|h(n)h^{-1}(n_0), \quad n \geq n_0$$

for $|x_0| \leq \delta$,

(GhS) *globally h-stable* if system (1) is hS for every $x_0 \in D$, where $D \subset \mathbb{R}^m$ is a region which includes the origin,

(hSV) *h-stable in variation* if the zero solution of system (3) is hS,

(GhSV)*globally h-stable in variation* if the zero solution of system (3) is GhS.

The various notions about h-stability given by Definition 2 include several types of known stability properties as uniform stability, uniform Lipschitz stability and exponential asymptotic stability. See [2, 3, 7, 10]. Also, some examples about hS for difference systems are presented in [8].

2. Main results

For the linear difference system $x(n+1) = A(n)x(n)$, where $A(n)$ is an $m \times m$ matrix, Medina and Pinto [8] showed that the following diagram holds :

$$\text{GhSV} \iff \text{GhS} \iff \text{hS} \iff \text{hSV}.$$

Also the associated variational Volterra difference system inherits the property of hS from the original nonlinear difference system with $g(n, s, x) = 0$ in system (1). i.e., the zero solution $v = 0$ of (2) is hS when the zero solution $x = 0$ of (1) is hS [8, Theorem 2].

Now, we will show that the above diagram holds for nonlinear Volterra difference system (1) by assuming that the variational Volterra difference system (2) is h-stable. To do this, we need the following lemmas.

LEMMA 1. *Assume that $f : \mathbb{N}(n_0) \times \mathbb{R}^m \to \mathbb{R}^m$ and $g : \mathbb{N}(n_0) \times \mathbb{N}(n_0) \times \mathbb{R}^m \to \mathbb{R}^m$ have partial derivatives. Let $x(n, n_0, x_0)$ be the solution of* (1), *which exists for $n \geq n_0$, and let*

$$H(n, n_0, x_0) = \frac{\partial f(n, x(n, n_0, x_0))}{\partial x}$$

and

$$G(n, s, x_0) = \frac{\partial g(n, s, x(s, n_0, x_0))}{\partial x(s)}.$$

Then $\Phi(n, n_0, x_0) = \frac{\partial x(n, n_0, x_0)}{\partial x_0}$ exists and is the solution of

(6)
$$\Phi(n+1, n_0, x_0) = H(n, n_0, x_0)\Phi(n, n_0, x_0) + \sum_{s=n_0}^{n} G(n, s, x_0)\Phi(s, n_0, x_0)$$

Proof. By differentiating (1) with respect to x_0, we have

$$\frac{\partial x(n+1)}{\partial x_0} = \frac{\partial f(n, x(n))}{\partial x(n)} \frac{\partial x(n)}{\partial x_0} + \sum_{s=n_0}^{n} \frac{\partial g(n, s, x(s))}{\partial x(s)} \frac{\partial x(s)}{\partial x_0}.$$

Then the result of Lemma follows from the definition $\Phi(n)$. □

The following lemma states a result corresponding to Fubini's theorem, which can be proved by induction.

LEMMA 2 [11, Lemma 2.1]. *Let $L(n, s)$ and $K(n, s)$ be $m \times m$ matrices defined for $s, n \geq n_0$ such that L and K are zero matrices for $s, n \leq n_0$. Then the relation*

$$\sum_{s=n_0}^{n-1} L(n, s+1) \sum_{\sigma=n_0}^{s-1} K(s, \sigma)x(\sigma) = \sum_{s=n_0}^{n-1} \sum_{\sigma=s+1}^{n-1} L(n, \sigma+1)K(\sigma, s)x(s)$$

holds, where $x : \mathbb{N}(n_0) \to \mathbb{R}^m$ is a vector function defined on $\mathbb{N}(n_0)$.

Consider the linear Volterra difference system

$$x(n+1) = A(n)x(n) + \sum_{s=n_0}^{n} B(n, s)x(s), \quad x(n_0) = x_0, \tag{7}$$

and its perturbation

$$y(n+1) = A(n)y(n) + \sum_{s=n_0}^{n} B(n,s)y(s) + g(n), \tag{8}$$

where $A(n)$ and $B(n,s)$ are $m \times m$ matrix functions on $\mathbb{N}(n_0)$ and $\mathbb{N}(n_0) \times \mathbb{N}(n_0)$, respectively and $g(n)$ is a vector function on $\mathbb{N}(n_0)$.

We are now going to develop the variation of constant formula for system (8) using the fundamental matrix solution $\Phi(n, n_0, x_0)$ of unperturbed linear Volterra difference system (7).

LEMMA 3. *The unique solution* $y(n, n_0, y_0)$ *of* (8) *satisfying* $y(n_0, n_0, y_0) = y_0$ *is given by*

$$y(n, n_0, y_0) = \Phi(n, n_0)y_0 + \sum_{s=n_0}^{n-1} \Phi(n, s+1)g(s). \tag{9}$$

Proof. From (6) and Lemma 2, it is easily seen by calculation that the function $y(n, n_0, y_0)$ satisfies the system (8). Thus we have

$$\begin{aligned} y(n+1) &= \Phi(n+1, n_0)y_0 + \sum_{s=n_0}^{n-1} \Phi(n+1, s+1)g(s) + g(n) \\ &= A(n)[\Phi(n, n_0)y_0 + \sum_{s=n_0}^{n-1} \Phi(n, s+1)g(s)] + g(n) \\ &\quad + \sum_{s=n_0}^{n} B(n,s)\Phi(s, n_0)y_0 + \sum_{s=n_0}^{n-1} \sum_{\sigma=s+1}^{n} B(n,\sigma)\Phi(\sigma, s+1)g(s) \\ &= A(n)y(n) + g(n) + \sum_{s=n_0}^{n} B(n,s)\Phi(s, n_0)y_0 \\ &\quad + \sum_{s=n_0}^{n-1} \sum_{\sigma=n_0}^{s-1} B(n,s)\Phi(s, \sigma+1)g(\sigma) \\ &\quad + \sum_{s=n_0}^{n-1} B(n,n)\Phi(n, s+1)g(s) \\ &= A(n)y(n) + g(n) + \sum_{s=n_0}^{n} B(n,s)[\Phi(s, n_0)y_0 + \sum_{\sigma=n_0}^{s-1} \Phi(s, \sigma+1)g(\sigma)] \\ &= A(n)y(n) + \sum_{s=n_0}^{n} B(n,s)y(s) + g(n). \end{aligned}$$

This completes the proof. □

LEMMA 4 [9, Lemma 3.1]. *The linear Volterra difference system (7) is an h-system if and only if there exist a constant $c \geq 1$ and a positive function $h : \mathbb{N}(n_0) \to \mathbb{R}$ such that for every $x_0 \in \mathbb{R}^m$,*

$$|\Phi(n, n_0, x_0)| \leq ch(n)h^{-1}(n_0), \quad \text{for } n \geq n_0,$$

where Φ is the fundamental matrix solution of

$$\Phi(n+1, n_0, x_0) = A(n)\Phi(n, n_0, x_0) + \sum_{s=n_0}^{n} B(n, s)\Phi(s, n_0, x_0), \quad n \geq n_0 \tag{10}$$

with $\Phi(n_0, n_0, x_0) = I$ (the identity matrix).

For the linear Volterra difference system, since the systems (1) and (3) coincide and the fundamental matrix $\Phi(n, n_0, x_0)$ of (3) is given by $\Phi(n, n_0, x_0) = \Phi(n, n_0, 0)$, we note that the following diagram holds:

$$\text{GhSV} \Longleftrightarrow \text{GhS} \Longleftrightarrow \text{hS} \Longleftrightarrow \text{hSV}.$$

Also, this diagram holds for all system (1) when the fundamental matrix solution $\Phi(n, n_0, x_0)$ of (3) is independent of x_0.

THEOREM 5. *If $x = 0$ of (1) is hS, then $v = 0$ of (2) is hS.*

Proof. By the assumption, there exist $c \geq 1$, $\delta > 0$ and a positive bounded function $h : \mathbb{N}(n_0) \to \mathbb{R}$ such that

$$|x(n, n_0, x_0)| \leq c|x_0|h(n)h^{-1}(n_0)$$

for $n \geq n_0$ and $|x_0| \leq \delta$. Let $y_{0j} = (0, 0, \cdots, x_{0j}, 0, \cdots, 0), j = 1, 2, \cdots, m$. Then $|y_{0j}| = s \leq \delta, j = 1, 2, \cdots, m$. From the fact that $x(n, n_0, 0) = 0$, it follows that for each $j = 1, 2, \cdots, m$

$$\begin{aligned}
\left|\frac{\partial x(n, n_0, 0)}{\partial x_{0j}}\right| &= \left|\lim_{s\to 0} \frac{x(n, n_0, y_{0j}) - x(n, n_0, 0)}{s}\right| \\
&\leq \lim_{s\to 0} \frac{c|y_{0j}|h(n)h^{-1}(n_0)}{s} \\
&= \lim_{s\to 0} \frac{csh(n)h^{-1}(n_0)}{s} = ch(n)h^{-1}(n_0).
\end{aligned}$$

Thus $|\Phi(n, n_0, 0)| \le ch(n)h^{-1}(n_0)$, and $v = 0$ of (2) is hS by Lemma 4. □

We note that the converse of Theorem 5 does not hold in general.

EXAMPLE. We consider the following Volterra difference equation

$$x(n+1) = f(n, x(n)) + \sum_{s=0}^{n} g(n, s, x(s)), \tag{V}$$

where $f(n,x) = x^2$ and $g(n,s,x) = 2^{n-s-1}x^2$. Then any solution $x(n, 0, x_0)$ of (V) through the initial point $|x(0, 0, x_0)| = |x_0| > 1$ is not bounded. Thus the zero solution $x = 0$ of (V) is not hS. Also, we can easily obtain the following variational Volterra difference equations of (V) :

$$\begin{aligned} v(n+1) &= f_x(n, 0)v(n) + \sum_{s=0}^{n} g_x(n, s, 0)v(s) \\ &= 0 \end{aligned} \tag{V-1}$$

and

(V-2)

$$z(n+1) = f_x(n, x(n, 0, x_0))z(n) + \sum_{s=0}^{n} g_x(n, s, x(s))z(s).$$

Then the zero solution $v = 0$ of (V-1) is hS but the zero solution $x = 0$ of (V) is not hS. Hence the converse of the Theorem 5 does not hold in general. Furthermore, if $f(n,x) = 2x$ and $g(n,s,x) = 2^{n-s}x$, then any solution $x(n, 0, x_0)$ of (V) through the initial point $x(0, 0, x_0) = x_0 \neq 0$ is given by

$$x(n) = \frac{1}{3}x(0)[1 + 2 \cdot 4^n], \quad n \in \mathbb{N}(0).$$

Thus fundamental matrix solution of variational Volterra difference equation (V-2) is given by

$$\Phi(n, 0, x_0) = \frac{\partial x(n, 0, x_0)}{\partial x_0} = \frac{1}{3}[1 + 2 \cdot 4^n].$$

Thus we obtain

$$\begin{aligned} |\Phi(n, 0, x_0)| &\le 4^n \\ &= ch(n)h^{-1}(0) \end{aligned}$$

where $c = 1$ and $h(n) = 4^n$. Therefore (V-2) is an h-system.

The following theorem says that hSV implies hS for the system (1).

THEOREM 6. *If $z = 0$ of (3) is hS, then $x = 0$ of (1) is also hS.*

Proof. Let $x(n) = x(n, n_0, x_0)$ be any solution of (1). Since $z = 0$ of (3) is hS, it follows from Definition 2 and Lemma 4 that

$$|\Phi(n, n_0, x_0)| \leq ch(n)h^{-1}(n_0), \quad n \geq n_0,$$

whenever $|x_0| \leq \delta$ for some $\delta > 0$ and bounded positive function h on $\mathbb{N}(n_0)$. This implies from Lemma 2 that

$$|x(n, n_0, x_0)| = \Big|\Big[\int_0^1 \Phi(n, n_0, sx_0)ds\Big]x_0\Big| \leq c|x_0|h(n)h^{-1}(n_0)$$

whenever $|x_0| \leq \delta$ and for all $n \geq n_0$. Hence the zero solution $x = 0$ of (1) is hS. □

For the converse of Theorem 6, we can obtain the following :

THEOREM 7. *Suppose that the zero solution $v = 0$ of (2) is hS and, for $|x| \leq \rho$ with some $\rho > 0$,*

(i) $|f_x(n, x) - f_x(n, 0)| \leq a(n)$
where $a : \mathbb{N}(n_0) \to \mathbb{R}+$,

(ii) $|g_x(n, s, x) - g_x(n, s, 0)| \leq b(n, s)$
where $b : \mathbb{N}(n_0) \times \mathbb{N}(n_0) \to \mathbb{R}^+$,

(iii) $\lambda(n) = h(n)[h^{-1}(n + 1)a(n) + K] \in l_1(n_0)$ *and*

$$\sup_{s \leq \sigma \leq n-1} \sum_{\sigma=s}^{n-1} h^{-1}(\sigma + 1)b(\sigma, s) \leq K$$

for some constant $K > 0$. Then the zero solution $z = 0$ of (3) is also hS.

Proof. Let $z(n) = z(n, n_0, z_0)$ be a solution of (3). We rewrite (3) as

$$\begin{aligned} z(n+1) &= f_x(n, 0)z(n) + \sum_{s=n_0}^{n} g_x(n, s, 0)z(s) \\ &+ [f_x(n, x(n, n_0, x_0)) - f_x(n, 0)]z(n) \\ &+ \sum_{s=n_0}^{n} [g_x(n, s, x(s)) - g_x(n, s, 0)]z(s) \\ &= A(n)z(n) + \sum_{s=n_0}^{n} B(n, s)z(s) + g(n), \end{aligned}$$

where $A(n) = f_x(n, 0)$, $B(n, s) = g_x(n, s, 0)$ and
$g(n) = [f_x(n, x(n)) - f_x(n, 0)]z(n) + \sum_{s=n_0}^{n}[g_x(n, s, x(s)) - g_x(n, s, 0)]z(s)$.
By Lemma 2 and 3, we have

$$\begin{aligned} z(n) &= \Phi(n, n_0, 0)z(n_0) + \sum_{s=n_0}^{n-1} \Phi(n, s+1, 0)[f_x(s, x(s)) - f_x(s, 0)]z(s) \\ &\quad + \sum_{s=n_0}^{n-1} \Phi(n, s+1, 0) \sum_{\sigma=n_0}^{s} [g_x(s, \sigma, x(\sigma)) - g_x(s, \sigma, 0)]z(\sigma) \\ &= \Phi(n, n_0, 0)z(n_0) + \sum_{s=n_0}^{n-1} \Phi(n, s+1, 0)[f_x(s, x(s)) - f_x(s, 0)]z(s) \\ &\quad + \sum_{s=n_0}^{n-1} \sum_{\sigma=s}^{n-1} \Phi(n, \sigma+1, 0)[g_x(\sigma, s, x(s)) - g_x(\sigma, s, 0)]z(s), \end{aligned}$$

for $z_0 \in \mathbb{R}^m$ and $|z_0| \leq \delta$. Thus, by the assumptions and Lemma 4, we have

$$\begin{aligned} |z(n)| &\leq ch(n)h^{-1}(n_0)|z_0| + \sum_{s=n_0}^{n-1} ch(n)h^{-1}(s+1)a(s)|z(s)| \\ &\quad + \sum_{s=n_0}^{n-1} [\sum_{\sigma=s}^{n-1} ch(n)h^{-1}(\sigma+1)b(\sigma, s)]|z(s)| \\ &\leq h(n)[ch^{-1}(n_0)|z(n_0)| + c\sum_{s=n_0}^{n-1} h(s)[h^{-1}(s+1)a(s) + K]\frac{|z(s)|}{h(s)}. \end{aligned}$$

Letting $u(n) = \frac{|z(n)|}{h(n)}$, we obtain

$$u(n) \leq cu(n_0) + c\sum_{s=n_0}^{n-1} h(s)[h^{-1}(s+1)a(s) + K]u(s).$$

Hence, by the discrete Bellman's inequality [1, 6], we obtain

$$\begin{aligned} |z(n)| &\leq ch(n)h^{-1}(n_0)|z_0| \exp\left(c\sum_{s=n_0}^{n-1} \lambda(s)\right) \\ &\leq c_1 h(n)h^{-1}(n_0)|z_0|, \end{aligned}$$

where $\lambda(n) = h(n)(h^{-1}(n+1)a(n) + K)$ and $c_1 = c\exp(c\sum_{s=n_0}^{\infty} \lambda(s))$ is a positive constant. The zero solution $z = 0$ of (3) is hS. This completes the proof. □

Corollary 8. *In addition to the assumptions of* Theorem 7 *assume that*

$$\sup_{n_0 \le s \le n-1} \left\{ h(s)[h^{-1}(s+1)a(s) + \sum_{\sigma=s}^{n-1} h^{-1}(\sigma+1)b(\sigma,s)] \right\} \le K$$

with $cK < 1$, *the zero solution* $z = 0$ *of* (3) *is* hS.

Proof. By the same process in the proof of Theorem 7, we obtain

$$u(n) \le cu(n_0) + cK \sup_{n_0 \le s \le n} u(s).$$

Thus we have

$$\sup_{n_0 \le s \le n} u(s) \le cu(n_0) + cK \sup_{n_0 \le s \le n} u(s).$$

Since $cK < 1$, we have

$$\sup_{n_0 \le s \le n} u(s) \le \frac{cu(n_0)}{1 - cK} = c_1 u(n_0),$$

where $c_1 \ge 1$ and thus we obtain

$$|z(n)| \le c_1 h(n) h^{-1}(n_0) |z_0|,$$

when $|z_0| \le \delta$. Hence $z = 0$ of (3) is hS. □

Corollary 9. *Let the assumptions of* Theorem 7 *hold. Then the zero solution* $x = 0$ *of* (1) *is* hSV.

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