Eccentric Crack in a Piezoelectric Strip Under Electro-Mechanical Loading

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We consider the problem of determining the singular stresses and electric fields in a piezoelectric ceramic strip containing a Griffith eccentric crack off the center line under anti-plane shear loading with the theory of linear piezoelectricity. Fourier transforms are used to reduce the problem to the solution of two pairs of dual integral equations, which are then expressed to a Fredholm integral equation of the second kind. Numerical values on the stress intensity factor and the energy release rate are obtained, and the influences of the electric fields for piezoelectric ceramics are discussed.

Key Words : Eccentric Crack, Piezoelectric Strip, Field Intensity Factors, Energy Release Rate

Nomenclature

- c_{44} : Elastic modulus measured in a constant electric field
- d_{11} : Dielectric permittivity measured at a constant strain
- D_{ji} : Electric displacement vector in ith region
- e : Eccentricity
- e_{15} : Piezoelectric constant
- E_{ki} : Electric field vector in ith region
- h_1, h_2 : Thickness of the upper and lower region, respectively
- ϕ_i : Electric potential in ith region

1. Introduction

With an increasingly wide application media in engineering, the study on the crack problem in piezoelectric media has received much interest.

Pak (1990) suggested the closed form solutions for an infinite piezoelectric body under anti-plane loading by employing a complex variable approach. Shindo *et al.*(1997) obtained the solution of the infinite strip parallel to the crack under anti-plane loading using integral transform method. Chen and Yu (1997) obtained the solution for an infinite piezoelectric medium containing an interfacial center crack between two dissimilar piezoelectric materials under anti-plane loading. Recently, Kwon and Lee (2000) obtained the solution of piezoelectric rectangular body with a center crack under anti-plane shear loading using integral transform method.

In this paper, we apply the theory of linear piezoelectricity to the electroelastic problem of a finite eccentric crack off the center line in a piezoelectric ceramic strip under anti-plane shear loading. The continuous crack boundary condition is adopted. Fourier transforms are used to reduce the problem to the solution of two pairs of dual integral equations, which are expressed to a Fredholm integral equation of the second kind.

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Numerical results for the stress intensity factor and the energy release rate are shown graphically.

2. Problem Statement and Method of Solution

Consider a piezoelectric medium in the form of an infinitely long strip containing a finite eccentric crack off the center line subjected to the combined mechanical and electric loads as shown in Fig. 1. We will consider four possible cases of boundary conditions at the edges of the strip. A Cartesian coordinates (x, y, z) is attached to the center of the crack. The piezoelectric ceramic strip poled with z-axis occupies the region $(-\infty<x<\infty,\ -h_2\leq y\leq h_1,\ 2h=h_1+h_2)$, and is thick enough in the z-direction to allow a state of anti-plane shear. The crack is situated along the virtual interface $(-a\leq x\leq a,\ y=0)$. Because of the assumed symmetry in geometry and loading, it is sufficient to consider the problem for $0\leq x<\infty$ only.

We consider only the out-of-plane displacement and the in-plane electric fields such that

$$u_x^{(i)}=u_y^{(i)}=0,\ u_z^{(i)}=w^{(i)}(x,\ y) \tag{1}$$

$$E_x^{(i)}=E_x^{(i)}(x,\ y),\ E_y^{(i)}=E_y^{(i)}(x,\ y),\ E_z^{(i)}=0 \tag{2}$$

where $u_k^{(i)}$, $w^{(i)}$ and $E_k^{(i)}$ $(k=x,y,z)$ are displacements, out-of-plane displacement and electric fields, respectively. Superscript $i(i=1,2)$ stands for upper and lower regions, respectively.

Anti-plane governing equations are simplified to

$$c_{44}\nabla^2 w^{(i)}+e_{15}\nabla^2\phi^{(i)}=0 \tag{3}$$

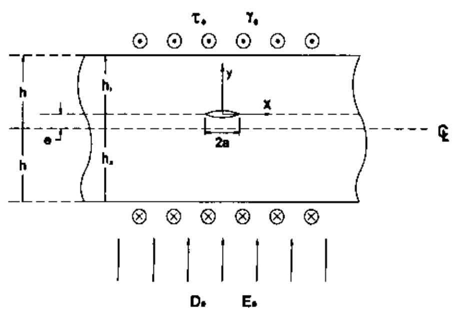

Fig. 1 piezoelectric ceramic strip with an eccentric crack : definition of geometry and loading

$$e_{15}\nabla^2 w^{(i)}-d_{11}\nabla^2\phi^{(i)}=0 \tag{4}$$

where $\nabla^2=\partial^2/\partial x^2+\partial^2/\partial y^2$ and $\phi^{(i)}$, c_{44}, d_{11} and e_{15} are the electric potential, the elastic modulus measured in a constant electric field, the dielectric permittivity measured at a constant strain and the piezoelectric constant, respectively.

Cracks in piezoelectric media will be filled with vacuum or air. This requires that both the normal components of electric displacement and the tangential component of the electric field will be continuous across the crack faces.

Based on this concept, the boundary conditions are written as follows,

$$\sigma_{yz}^{(i)}(x,0)=0,\ (0\leq x<a) \tag{5}$$

$$w^{(1)}(x,0^+)=w^{(2)}(x,0^-),\ (a\leq x<\infty) \tag{6}$$

$$D_y^{(1)}(x,0^+)=D_y^{(2)}(x,0^-),\ (0\leq x<a) \tag{7}$$

$$E_x^{(1)}(x,0^+)=E_x^{(2)}(x,0^-),\ (0\leq x<a) \tag{8}$$

$$\phi^{(1)}(x,0^+)=\phi^{(2)}(x,0^-),\ (a\leq x<\infty) \tag{9}$$

$$\begin{aligned}&\sigma_{yz}^{(1)}(x,0^+)=\sigma_{yz}^{(2)}(x,0^-),\ (a\leq x<\infty)\\&D_y^{(1)}(x,0^+)=D_y^{(2)}(x,0^-),\ (a\leq x<\infty)\end{aligned} \tag{10}$$

$$\text{Case 1}:\ \begin{aligned}&\sigma_{yz}^{(1)}(x,h_1)=\sigma_{yz}^{(2)}(x,\ -h_2)=\tau_0\\&D_y^{(1)}(x,h_1)=D_y^{(2)}(x,\ -h_2)=D_0\end{aligned} \tag{11}$$

$$\text{Case 2}:\ \begin{aligned}&\gamma_{yz}^{(1)}(x,h_1)=\gamma_{yz}^{(2)}(x,\ -h_2)=\gamma_0\\&E_y^{(1)}(x,h_1)=E_y^{(2)}(x,\ -h_2)=E_0\end{aligned} \tag{12}$$

$$\text{Case 3}:\ \begin{aligned}&\sigma_{yz}^{(1)}(x,h_1)=\sigma_{yz}^{(2)}(x,\ -h_2)=\tau_0\\&E_y^{(1)}(x,h_1)=E_y^{(2)}(x,\ -h_2)=E_0\end{aligned} \tag{13}$$

$$\text{Case 4}:\ \begin{aligned}&\gamma_{yz}^{(1)}(x,h_1)=\gamma_{yz}^{(2)}(x,\ -h_2)=\gamma_0\\&D_y^{(1)}(x,h_1)=D_y^{(2)}(x,\ -h_2)=D_0\end{aligned} \tag{14}$$

where $\sigma_{yz}^{(i)}$ and $D_y^{(i)}$ are the shear stress and the electric displacement, respectively and τ_0, D_0, γ_0 and E_0 are a uniform shear stress, electric displacement, shear strain and electric field, respectively.

A Fourier transform is applied to Eqs. (3) and (4), and the results are

$$\begin{aligned}w^{(i)}(x,y)=&\frac{2}{\pi}\int_0^\infty\{A_1^{(i)}(s)\cosh(sy)\\&+A_2^{(i)}(s)\sinh(sy)\}\cos(sx)ds+a_0y\end{aligned} \tag{15}$$

$$\begin{aligned}\phi^{(i)}(x,y)=&\frac{2}{\pi}\int_0^\infty\{B_1^{(i)}(s)\cosh(sy)\\&+B_2^{(i)}(s)\sinh(sy)\}\cos(sx)ds-b_0y\end{aligned} \tag{16}$$

where $A_j^{(i)}$, $B_j^{(i)}(j=1,\ 2)$ are the unknowns to be solved. a_0, b_0 are real constants, which will be determined from the edge loading conditions.

By applying the edge loading conditions and continuous condition on the crack surface and using the two mixed boundary conditions Eqs. (5)-(9), we obtain the following two simultaneous dual integral equations,

$$\int_0^\infty sF(s)\left[M_A(s)+\frac{e_{15}}{c_{44}}M_B(s)\right]\cos(sx)ds$$
$$=\frac{\pi}{2}\frac{c_0}{c_{44}},\ (0\le x<a)$$
$$\int_0^\infty M_A(s)\cos(sx)ds=0,\ (a\le x<\infty) \qquad (17)$$
$$\int_0^\infty sM_B(s)\sin(sx)ds=0,\ (0\le x<a)$$
$$\int_0^\infty M_B(s)\cos(sx)ds=0,\ (a\le x<\infty) \qquad (18)$$

where

$$A_1^{(1)}(s)-A_1^{(2)}(s)=2M_A(s)$$
$$B_1^{(1)}(s)-B_1^{(2)}(s)=2M_B(s) \qquad (19)$$
$$F(s)=\frac{2\tanh(sh_1)\tanh(sh_2)}{\tanh(sh_1)+\tanh(sh_2)} \qquad (20)$$
$$c_0=c_{44}a_0-e_{15}b_0, \qquad (21)$$
$$=\tau_0,\qquad \text{cases 1 and 3} \qquad (22a)$$
$$=c_{44}\gamma_0-e_{15}E_0,\qquad \text{case 2} \qquad (22b)$$
$$=\frac{(c_{44}d_{11}+e_{15}^2)\gamma_0-e_{15}D_0}{d_{11}},\ \text{case 4} \qquad (22c)$$

To solve the dual integral equations, we define $M_A(s)$ and $M_B(s)$ in the forms,

$$M_A(s)=\frac{\pi}{2}\frac{c_0a^2}{c_{44}}\int_0^1\sqrt{\xi}\Psi(\xi)J_0(sa\xi)d\xi$$
$$M_B(s)=\int_0^1\sqrt{\xi}\,\Phi(\xi)J_0(sa\xi)d\xi \qquad (23)$$

where $J_0(sa\xi)$ is the zero-order Bessel function of the first kind.

Inserting Eqs.(23) into Eqs.(17) and (18), we can find that $\Phi(\xi)=0$ and $\Psi(\xi)$ is given by a Fredholm integral equation of the second kind in the form,

$$\Psi(\xi)+\int_0^1K(\xi,\eta)\Psi(\eta)d\eta=\sqrt{\xi} \qquad (24)$$

where

$$K(\xi,\eta)=\sqrt{\xi\eta}\int_0^\infty s\{F(s/a)-1\}J_0(s\eta)J_0(s\xi)ds \qquad (25)$$

$$F\left(\frac{s}{a}\right)=\tanh\left(\frac{h}{a}s\right)-\frac{2\sinh^2\left(\frac{e}{a}s\right)}{\sinh^2\left(2\frac{h}{a}s\right)} \qquad (26)$$

e denotes the eccentricity.

In case of $e=0$, the obtained results are same as those of Shindo et al.(1997).

Extending the traditional concept of stress intensity factor to other field variables, we have

$$K^T=c_0\sqrt{\pi a}\,\Psi(1),\ K^S=K^T/c_{44}$$
$$K^D=e_{15}K^T/c_{44},\ K^E=0 \qquad (27)$$

where K^T, K^S, K^D and K^E are stress intensity, strain intensity, electric displacement intensity and electric field intensity factor, respectively. From Eq.(27), it is noted that the uniform electric load has no influence on the field singularities, and the electric displacement intensity factor depends on the material constants and the applied mechanical load, τ_0, but not on the applied electric load, D_0 and E_0. These are well agreed with the results of Kwon and Lee (2000) and Gao and Fan (1999).

Evaluating the energy release rate G obtained by Pak (1990), we obtain

$$G=\frac{K^TK^S-K^DK^E}{2}=\frac{\pi a}{2c_{44}}c_0^2\Psi^2(1) \qquad (28)$$

3. Discussions

It is noted from Eq.(28) that the energy release rate is dependent on the electric loading only under constant strain loading and independent of it under constant stress loading, and always has positive values. These are in good agreement with the results of Kwon and Lee (2000).

To examine the effect of electromechanical interactions on the stress intensity factor and the energy release rate, Eq.(24) was computed numerically by Gaussian quadrature formulas. We consider PZT-5H ceramics whose material properties are as follows,

$$c_{44}=3.53\times10^{10}\ (N/m^2),\ e_{15}=17.0(C/m^2).$$
$$d_{11}=151\times10^{-10}\ (C/Vm).$$

Figures 2 and 3 show the variations of the normalized stress intensity factor $K_{III}/c_0\sqrt{\pi a}$ and the energy release rate G/G_∞ against the a/h with various e/h values. $K_{III}/c_0\sqrt{\pi a}$ and G/G_∞ increase with the increase of the a/h and e/h ratios. The normalized energy release rate G/G_{cr} of a PZT-5H ceramic is shown in Fig. 4 as a

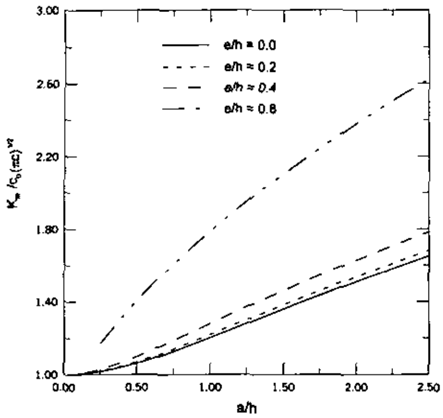

Fig. 2 Stress intensity factor $K_{III}/c_0\sqrt{\pi a}$ versus a/h with various e/h values

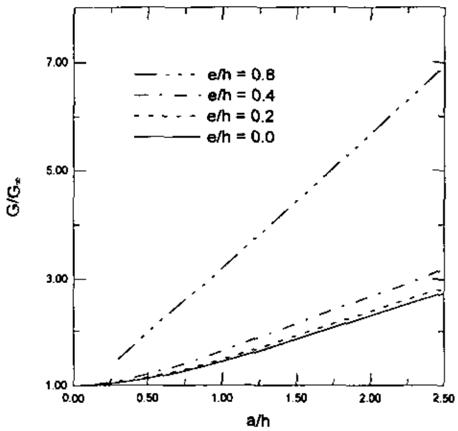

Fig. 3 Energy release rate J/J_∞ versus a/h with various e/h values

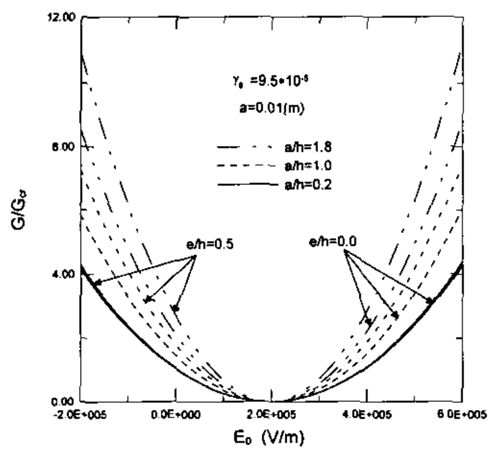

Fig. 4 Energy release rate of PZT-5H (Case 2) at various e/h values

function of the applied electric field E_0 and the a/h ratio for a crack length of $2a=0.02\ m$ and $\gamma_0 = 9.5\times 10^{-5}$ (Case 2) with $e/h=0.0$ and $e/h=0.5$, respectively. As the magnitude of electric field increases from zero, G increases or decreases depending on the directions of the load. But once G is reaches the minimum value, further increase in the electrical load will increase G monotonically. G increases with the increase of the e/h ratio. It can be shown that similar results are obtained in case 4 as a function of the applied electric displacement. D_0 In cases 1 and 3, the electric fields have no effect on G values. The minimum normalized energy release rates can exist with the variation of electrical load but always has positive values.

4. Conclusions

The electroelastic problem of an eccentric crack off the center line in a transversely isotropic piezoelectric ceramic strip under anti-plane shear was analyzed by the continuous crack surface condition and the integral transform approach. The traditional concept of linear elastic fracture mechanics is extended to include the piezoelectric effects and the results are expressed in terms of the stress intensity factor and the energy release rate. The normalized stress intensity factor and the normalized energy release rate increase when the a/h and e/h ratios increase. The energy release rate are dependent on the electric loading only under constant strain loading and independent of it under constant stress loading. In constant strain loadings, the minimum normalized energy release rate can exist with the variation of electrical load but always has positive values.

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References

Chen, Z. T. and Yu, S. W., 1997, "Antiplane Shear Problem for a Crack Between Two Dissimilar Piezoelectric Materials," *Int. J. Fract*, Vol. 86, pp. L9~L12.

Gao C. F. and Fan W. X., 1999, "A General Solution for the Plane Problem in Piezoelectric Media with Collinear Crack," *Int. J. Eng. Sci*., Vol. 37, pp. 347~363.

Kwon, S. M. and Lee, K. Y., 2000, "Analysis of Stress and Electric Fields in a Rectangular Piezoelectric Body with a Center Crack Under Anti-Plane Shear Loading," *Int. J. Solids Struct*., Vol. 37, pp. 4859~4869.

Pak, Y. E., 1990, "Crack Extension Force in a Piezoelectric Material," *ASME, J. Appl. Mech.* Vol. 57, pp. 647~653.

Shindo, Y. et al., 1997, "Singular Stress and Electric Fields of a Piezoelectric Ceramic Strip with a Finite Crack Under Longitudinal Shear," *Acta Mech.* Vol. 120, pp. 31~45.