Three-Dimensional Performance Analysis of a Thermally Asymmetric Rectangular Fin

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Key words: 3-D analytical method, Fin effectiveness, Fin efficiency, Biot number

Abstract

Fin effectiveness and efficiency of a thermally asymmetric rectangular fin are represented as a function of non-dimensional fin length, width, fip tip surface Biot number and the ratio of fin bottom surface Biot number to top surface Biot number. For this analysis, three dimensional separation of variables method is used. One of the results shows that fin effectiveness can be increased or decreased depending on the fin length as the fin tip surface Biot number increases while fin efficiency decreases without depending on that as the fin tip surface Biot number increases.

Nomenclature

Bi1 : fin top surface Biot number, $h_1 l/k$

Bi2 : fin bottom surface Biot number, $h_2 l/k$

Bi3 : fin left surface Biot number, $h_3 l/k$

Bi4 : fin right surface Biot number, $h_4 l/k$

Bi5 : fin tip surface Biot number, $h_5 l/k$

h_1 : fin top surface heat transfer coefficient [W/m^2℃]

h_2 : fin bottom surface heat transfer coefficient [W/m^2℃]

h_3 : fin left surface heat transfer coefficient [W/m^2℃]

h_4 : fin right surface heat transfer coefficient [W/m^2℃]

h_5 : fin tip surface heat transfer coefficient [W/m^2℃]

k : thermal conductivity [W/m℃]

l : one half fin height at the base [m]

L' : fin length (base to tip) [m]

L : non-dimensional fin length, L'/l

Q : heat loss from a rectangular fin [W]

T : fin temperature [℃]

T_w : fin base temperature [℃]

T_∞ : ambient temperature [℃]

w' : one half fin width [m]

w : non-dimensional a half fin width, w'/l

x' : length directional variable [m]

x : non-dimensional length directional variable, x'/l

y' : height directional variable [m]

y : non-dimensional height directional vari-

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able, y'/l

z' : width directional variable [m]

z : non-dimensional width directional variable, z'/l

Greek characters

θ_0 : adjusted temperature, $(T_w - T_\infty)$

θ : non-dimensional temperature, $(T-T_\infty)/(T_w-T_\infty)$

λ_m : eigenvalues (m=1, 2, 3, ⋯)

μ_n : eigenvalues (n=1, 2, 3, ⋯)

ρ_{nm} : eigenvalues ($\sqrt{\lambda_m^2+\mu_n^2}$)

Subscripts

1 : top surface

2 : bottom surface

3 : left surface

4 : right surface

5 : tip surface

w : wall

∞ : ambient

1. Introduction

Fins are widely used to enhance the rate of heat transfer to a surrounding fluid in many applications such as the cooling of combustion engines, air conditioning systems, many kind of heat exchangers and so on. As a result, a great deal of attention has been directed to fin problems such as performances of fins or heat exchangers with fins. For example, Burmeister[1] analyzes triangular fin performance using heat balance integral method, Look[2] have studied a thermally asymmetric rectangular, Kang[3] analyzed asymmetric various shapes of trapezoidal fins, Rosman et al.,[4] Prasad[5-6] and Maltson et al.[7] have studied performance of plate fin heat exchanger. Also Ullmann and Kalman[8] researched annular fins. Usually most of the studies on the fin or heat exchangers with fins assume that the heat transfer coefficients for all surfaces of the fin or heat exchangers are the same and are analysed by one- or two-dimensional method. But no literature seems to be available which presents performance of a thermally asymmetric rectangular fin by using three-dimensional analysis.

This paper presents the performance of a three dimensional thermally asymmetric rectangular fin. In this study the top surface Biot number, Bi1, is equal to or larger than the bottom surface Biot number, Bi2 and the left surface Biot number, Bi3, is equal to or larger than the right surface Biot number, Bi4, and Bi5, at the fin tip, has various values. Actually side surface Biot number can be larger or smaller than top surface Biot number but left surface Biot number is arbitrarily set to be equal to top surface Biot number in this study. Fin effectiveness and efficiency are investigated as a function of the non-dimensional fin length, width, fin tip surface Biot number and the Bi2/Bi1 ratio using the three-dimensional separation of variables method. For simplicity, the root temperature and the thermal conductivity of the fin's material are assumed constant. Also, the heat transfer process is considered to be steady state.

2. Three-Dimensional Analysis

Geometry of a rectangular fin with all different heat transfer coefficient is shown in Fig. 1.

Assuming no heat source, constant properties and steady state, three-dimensional governing differential equation in terms of dimensionless temperature, θ for this geometry is given as

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}=0 \tag{1}$$

Six boundary conditions are required to solve

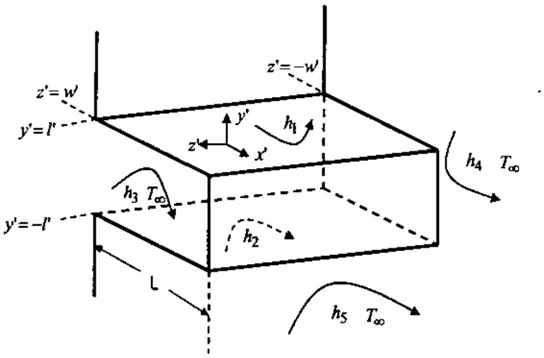

Fig. 1 Geometry of a thermally asymmetric rectangular fin.

equation (1). These conditions are shown as equations (2)~(7).

$$\theta = 1 \quad \text{at} \quad x = 0 \tag{2}$$

$$\frac{\partial \theta}{\partial x} + \mathrm{Bi5} \cdot \theta = 0 \quad \text{at} \quad x = L \tag{3}$$

$$\frac{\partial \theta}{\partial y} + \mathrm{Bi1} \cdot \theta = 0 \quad \text{at} \quad y = 1 \tag{4}$$

$$\frac{\partial \theta}{\partial y} - \mathrm{Bi2} \cdot \theta = 0 \quad \text{at} \quad y = -1 \tag{5}$$

$$\frac{\partial \theta}{\partial z} + \mathrm{Bi3} \cdot \theta = 0 \quad \text{at} \quad z = w \tag{6}$$

$$\frac{\partial \theta}{\partial z} - \mathrm{Bi4} \cdot \theta = 0 \quad \text{at} \quad z = -w \tag{7}$$

The solution for the temperature distribution $\theta(x, y, z)$ within the fin obtained with equations (2)~(5) is

$$\theta(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} N_{nm} \cdot f(x) \cdot f(y) \cdot f(z) \tag{8}$$

where

$$N_{nm} = \frac{4 \sin \lambda_n \cdot \sin(u_m \cdot w)}{f_n \cdot g_m} \tag{9}$$

$$f(x) = \cosh(\rho_{nm} \cdot x) - C_{nm} \cdot \sinh(\rho_{nm} \cdot x) \tag{10}$$

$$C_{nm} = \frac{\rho_{nm} \cdot \tanh(\rho_{nm} \cdot L) + \mathrm{Bi5}}{\rho_{nm} + \mathrm{Bi5} \cdot \tanh(\rho_{nm} \cdot L)} \tag{11}$$

$$\rho_{nm} = \sqrt{(\lambda_n^2 + \mu_m^2)} \tag{12}$$

$$f(y) = \cos(\lambda_n \cdot y) + A_n \cdot \sin(\lambda_n \cdot y) \tag{13}$$

$$A_n = \frac{\lambda_n \cdot \tan \lambda_n - \mathrm{Bi1}}{\lambda_n + \mathrm{Bi1} \cdot \tan \lambda_n} \tag{14}$$

$$f(z) = \cos(\mu_m z) + B_m \cdot \sin(\mu_m z) \tag{15}$$

$$B_m = \frac{\mu_m \cdot \tan(\mu_m \cdot w) - \mathrm{Bi3}}{\mu_m + \mathrm{Bi3} \cdot \sin(\mu_m \cdot w)} \tag{16}$$

$$f_n = \lambda_n + \frac{1}{2}\sin(2\lambda_n) + A_n^2 \cdot \left\{\lambda_n - \frac{1}{2}\sin(2\lambda_n)\right\} \tag{17}$$

$$g_m = \mu_m w + \frac{1}{2}\sin(\mu_m w) + B_m^2 \cdot \left\{\mu_m w - \frac{1}{2}\sin(\mu_m w)\right\} \tag{18}$$

The eigenvalues λ_n can be obtained from equation (19) which comes from equations (4) and (5).

$$\frac{\lambda_n \cdot \tan(\lambda_n) - \mathrm{Bi1}}{\lambda_n + \mathrm{Bi1} \cdot \tan(\lambda_n)} = \frac{\mathrm{Bi2} - \lambda_n \cdot \tan(\lambda_n)}{\lambda_n + \mathrm{Bi2} \cdot \tan(\lambda_n)} \tag{19}$$

The eigenvalues μ_m can be obtained from equation (20) which comes from equations (6) and (7).

$$\frac{\mu_m \cdot \tan(\mu_m \cdot w) - \mathrm{Bi3}}{\mu_m + \mathrm{Bi3} \cdot \tan(\mu_m \cdot w)} = \frac{\mathrm{Bi4} - \mu_m \cdot \tan(\mu_m \cdot w)}{\mu_m + \mathrm{Bi4} \cdot \tan(\mu_m \cdot w)} \tag{20}$$

By applying equation (8) to Fourier's law, the heat loss rate conducted into the fin through the fin base is given by

$$Q = 4k \cdot l \cdot \theta_0 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} N_{nm} \cdot \rho_{nm} \cdot C_{nm} \cdot \frac{\sin \lambda_n}{\lambda_n} \cdot \frac{\sin(\mu_m \cdot w)}{\mu_m} \tag{21}$$

Equation for fin effectiveness can be expressed by equation (22) or equation (23) by its definition.

$$\varepsilon_f = \left[\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} N_{nm} \cdot \rho_{nm} \cdot C_{nm} \cdot \frac{4 \sin \lambda_n \cdot \sin(\mu_m \cdot w)}{\lambda_n \cdot \mu_m} \right] / (4w \cdot \mathrm{Bi5}) \tag{22}$$

Heat transfer coefficient at the wall is taken as the heat transfer coefficient at the fin tip in equation (22) while it is taken as the average heat transfer coefficient around the fin in equation (23). In this analysis, equation (23) will be used to obtain the fin effectiveness.

$$\varepsilon_f = \left[\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} N_{nm} \cdot \rho_{nm} \cdot C_{nm} \cdot \frac{4 \sin \lambda_n \cdot \sin(\mu_m \cdot w)}{\lambda_n \cdot \mu_m} \right] / (4w \cdot \mathrm{Bi}m) \tag{23}$$

where

$$\mathrm{Bi}m = \frac{\mathrm{Bi1} + \mathrm{Bi2} + \mathrm{Bi3} + \mathrm{Bi4} + \mathrm{Bi5}}{5}$$

Equation for fin efficiency can be written by equation (24) from its definition.

$$\eta_f = \left[\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} N_{nm} \cdot \rho_{nm} \cdot C_{nm} \cdot \frac{4 \sin \lambda_n \cdot \sin(\mu_m \cdot w)}{\lambda_n \cdot \mu_m} \right] / [2L \cdot w \cdot (\mathrm{Bi1} + \mathrm{Bi2}) + 2L \cdot (\mathrm{Bi3} + \mathrm{Bi4}) + 4w \cdot \mathrm{Bi5}] \tag{24}$$

3. Results and Discussions

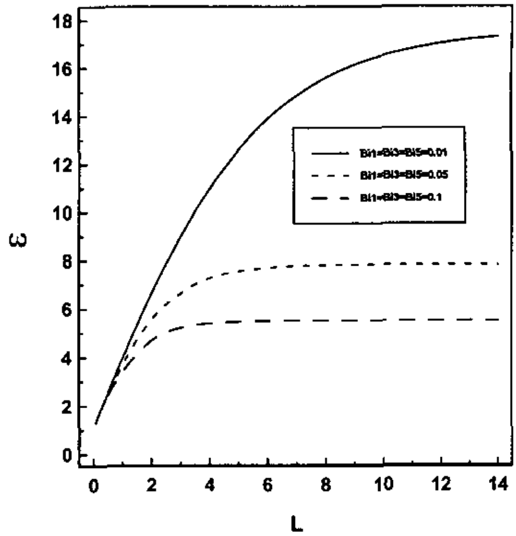

Fig. 2a Fin effectiveness vs. L for w=0.5, Bi2/Bi1=Bi4/Bi3=0.9.

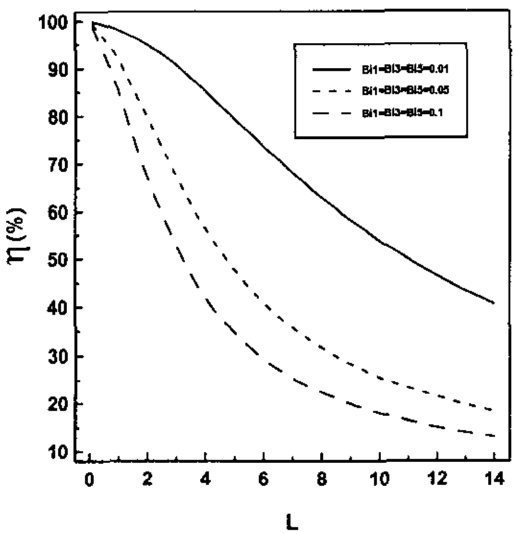

Fig. 2b Fin efficiency vs. L for w=0.5, Bi2/Bi1=Bi4/Bi3=0.9.

Figure 2a presents the fin effectiveness versus the non-dimensional fin length in three cases of the value of Bi1 (=Bi3=Bi5) for w= 0.5, Bi2/Bi1=Bi4/Bi3=0.9. Fin effectiveness increases until about L=4, after then the effect of fin length on ε seems to be independent in case of Bi1=Bi3=Bi5=0.1 while ε increases continuously as L increases from 0.1 to 14 in case of Bi1=Bi3=Bi5=0.01. Figure 2b shows the fin efficiency versus the non-dimensional fin length under the same condition as Fig. 2a. It can be noted that η decreases continuously while ε increases in the range of $L<4$, and then the value looks like constant as the length increases from 0.1 to 14 in case of Bi1 =Bi3=Bi5=0.05 and 0.1. Both the increasing

Table 1 Increasing rate of ε and decreasing rate of η along the fin length for w=0.5, Bi2/Bi1=Bi4/Bi3=0.9

range of L	Bi1=Bi3=Bi5			
	0.01		0.1	
	I. R. of ε (%)	D. R. of η (%)	I. R. of ε (%)	D. R. of η (%)
0.1~ 2.0	395.51	4.97	253.15	32.27
2.0~ 4.0	65.86	10.38	15.60	37.54
4.0~ 6.0	26.29	13.48	1.76	30.28
6.0~ 8.0	12.31	14.59	0.21	23.79
8.0~10.0	6.05	14.44	0.0026	19.30
10.0~12.0	3.03	13.65	0.0003	16.19
12.0~14.0	1.51	12.62	0.00004	13.94

rate of effectiveness and the decreasing rate of efficiency with the variation of L are listed in Table 1.

Table 1 lists the increasing rate of effectiveness and the decreasing rate of efficiency along the fin length in case of Bi1=Bi3=Bi5=0.01 and 0.1 under the same condition of Fig. 2. For Bi1=Bi3=Bi5=0.01, the decreasing rate of efficiency begins to decrease and the increasing rate of effectiveness seems to be small in the range of $8 \leq L \leq 10$. These same phenomena occur in the range of $4 \leq L \leq 6$ in case of Bi1=Bi3=Bi5=0.1. From Figs. 2a, 2b and Table 1, L=8 for Bi1=Bi3=Bi5=0.01 and L=4 for Bi1=Bi3=Bi5=0.1 can be considered as the approximate optimum fin length for given conditions.

Figure 3a presents the fin effectiveness versus the non-dimensional fin width for three different ratios of Bi2/Bi1 and Bi4/Bi3 in case of L=5, Bi1=Bi3=Bi5=0.01 and for all Bi=0.008. These different ratios are arbitrarily chosen. It must be noted that the average values of Biot numbers for three different ratios are equal and the average Biot number is used to calculate the heat loss from the wall. For

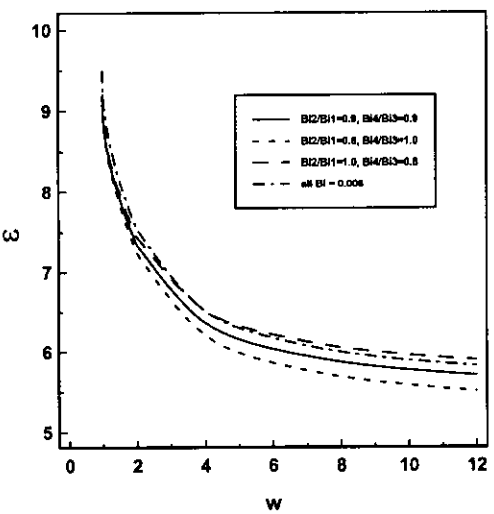

Fig. 3a Fin effectiveness vs. w for L=5, Bi1=Bi3=Bi5=0.01 and all Bi=0.008.

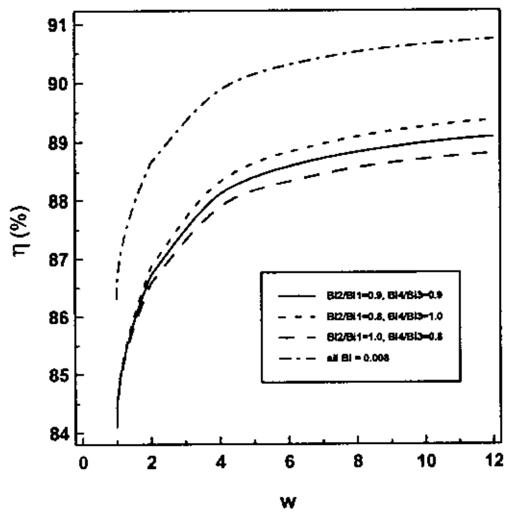

Fig. 3b Fin efficiency vs. w for L=5, Bi1=Bi3=Bi5=0.01 and all Bi=0.008.

Table 2 Relative error between ε_3 (=effectiveness from 3-D analysis when all Biot numbers are equal) and ε_1 (=effectiveness from 1-D analysis)

L	$(\varepsilon_3-\varepsilon_1)/\varepsilon_3\times 100$ (%)			
	Bi=0.01		Bi=0.1	
	w=0.1	w=20	w=0.1	w=20
4	84.07	3.35	72.17	0.93
8	78.19	3.22	69.98	0.83
12	73.99	2.89	69.79	0.82
16	71.79	2.61	69.78	0.82
20	70.73	2.43	69.78	0.82

all three ratios (including the case for all Bi=0.008), ε decreases rapidly and then decreases slowly as width increases. The effectiveness for all Bi=0.008 is larger than that for Bi2/Bi1 =1.0 and Bi4/Bi3=0.8 until w increases to about 4 and these values are reversed after that. Also the effectiveness for Bi2/Bi1=1.0 and Bi4/Bi3=0.8 is the largest among three asymmetric cases and this fact coincides with the physical phenomenon due to high heat loss from large top and bottom surface areas as width increases. Fin efficiency versus the non-dimensional fin width under the same condition as Fig. 3a is represented in Fig. 3b. For all three ratios (including the case for all Bi=0.008), η increases rapidly and then increases slowly as width increases. The efficiency for Bi2/Bi1=1.0 and Bi4/Bi3=0.8 is the smallest and this fact coincides with the physical phenomenon due to temperature decreasing at large top and bottom surface areas as width increases. It can be noted that the efficiency for all Bi=0.008 is remarkably higher than those for other three asymmetric cases.

Table 2 lists the relative error between the effectiveness obtained from 3-D analysis when all Biot numbers are equal and that obtained from 1-D analysis. This table explains that the relative error becomes small as the width and length increase. The relative error for Bi=0.1 is less than that for Bi=0.01 and this fact seems to be due to the effect of heat transfer from the wall.

Figure 4a illustrates the fin effectiveness versus fin tip Biot number for three different values of non-dimensional fin length for w=

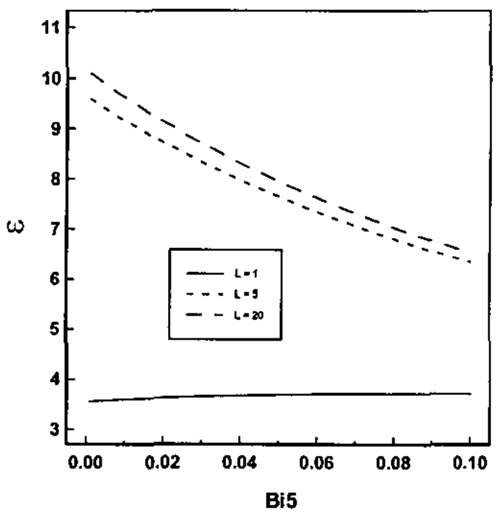

Fig. 4a Fin effectiveness vs. Bi5 for w=0.5, Bi1=Bi3=0.05 and Bi2/Bi1=Bi4/Bi3=0.8.

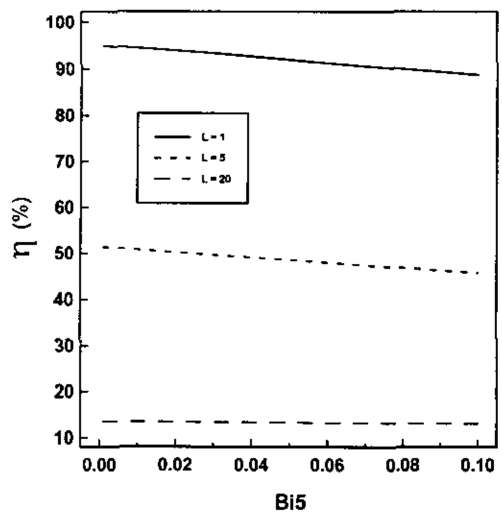

Fig. 4b Fin efficiency vs. Bi5 for w=0.5, Bi1=Bi3=0.05 and Bi2/Bi1=Bi4/Bi3=0.8.

0.5, Bi1 = Bi3 = 0.05 and Bi2/Bi1 = Bi4/Bi3 = 0.8. The fin effectiveness increases for $L=1$ while it decreases for $L=5$, 10 as fin tip Biot number increases. It can be guessed that ε will be independent of the variation of fin tip Biot number for certain value of L which is between 1 and 5 under this asymmetric condition. Figure 4b shows the fin efficiency versus fin tip Biot number under the same asymmetric condition as Fig. 4a. The efficiency decreases linearly as fin tip Biot number increases for $L=1$ and 5. It looks like constant for $L=20$ with the variation of fin tip Biot number and this is explained physically by the fact that the effect of fin tip Biot number on the temperature distribution can be neglected for a long fin.

The fin effectiveness versus the ratio of fin bottom surface Biot number to top surface Biot number for three different values of width in case of $L=5$, Bi4/Bi3 = 0.5 and Bi1 = Bi3 = Bi5 = 0.01 is shown in Fig. 5a. The effectiveness decreases remarkably for $w=0.1$ while it decreases slightly for $w=0.5$ as the ratio of Bi2/Bi1 increases from 0.1 to 1.0. On the contrary it increases as Bi2/Bi1 increases for $w=2$. It can also be guessed that ε will be independent of the variation of Bi2/Bi1 for certain value of w which is between 0.5 and 2 under this asymmetric condition. Figure 5b presents the fin efficiency versus Bi2/Bi1 under the same asymmetric condition as Fig. 5a. It shows that η decreases linearly as Bi2/Bi1 increases for all three values of fin width under given asymmetric condition.

4. Conclusions

The following conclusions can be made from the results.

(1) The fin length can be considered as the approximate optimum length when the decreasing rate of efficiency begins to decrease for given conditions.

(2) Fin effectiveness increases as fin length increases while it decreases as fin width increases and vice versa for fin efficiency.

(3) Fin effectiveness can be increased or decreased depending on the fin length as Bi5 increases while fin efficiency decreases without depending on that as Bi5 increases.

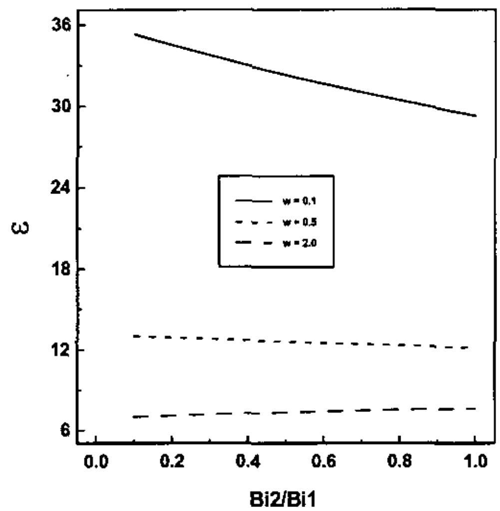

Fig. 5a Fin effectiveness vs. Bi2/Bi1 for $L=5$, Bi1 = Bi3 = Bi5 = 0.01 and Bi4/Bi3 = 0.5.

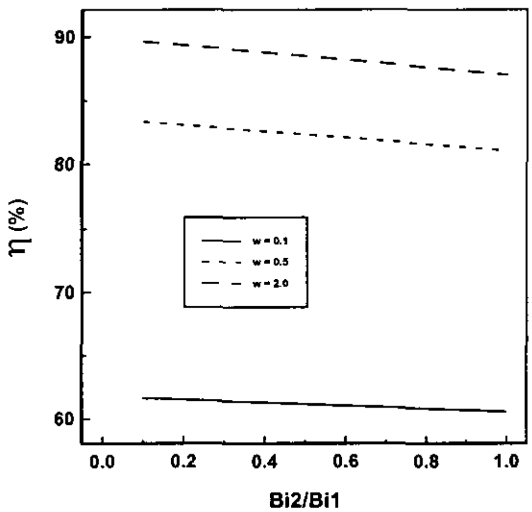

Fig. 5b Fin efficiency vs. Bi2/Bi1 for $L=5$, Bi1 = Bi3 = Bi5 = 0.01 and Bi4/Bi3 = 0.5.

(4) Fin effectiveness can be increased or decreased depending on the fin width as the ratio of Bi2/Bi1 increases while fin efficiency decreases linearly without depending on that as the ratio of Bi2/Bi1 increases.

References

1. Burmeister, L. C., 1979, "Triangular Fin Performance by the Heat Balance Integral Method," ASME J. of Heat Trans., Vol. 101, pp. 562-564.
2. Look, D. C., 1988, "2-D Fin Performance: Bi(top) > Bi(bottom)," ASME J. of Heat Trans., Vol. 111, pp. 780-782.
3. Kang, H. S., 1997, "Comparison of Performance of the Various Shapes of Asymmetric Fins," KSME International Journal, Vol. 11, No. 3, pp. 311-318.
4. Rosman, E. C., Carajilescov, P. and Saboya, F. E. M., 1984, "Performance of One- and Two-Row Tube and Plate Fin Heat Exchangers," ASME J. of Heat Trans., Vol. 106, pp. 627-632.
5. Prasad, B. S. V., 1996, "Fin Efficiency and Mechanisms of Heat Exchange through Fins in Multi-Stream Plate-Fin Heat Exchangers: Formulation," Int. J. Heat Mass Transfer, Vol. 39, pp. 419-428.
6. Prasad, B. S. V., 1997, "Fin Efficiency and Mechanisms of Heat Exchange through Fins in Multi-Stream Plate-Fin Heat Exchangers: Development and Application of a Rating Algorithm," Int. J. Heat Mass Transfer, Vol. 40, pp. 4279-4288.
7. Maltson, J. D., Wilcock, D. and Davenport, C. J., 1989, "Comparative Performance of Rippled Fin Plate Fin and Tube Heat Exchangers," ASME J. of Heat Trans., Vol. 111, pp. 21-28.
8. Ullmann, A. and Kalman, H., 1989, "Efficiency and Optimized Dimensions of Annular Fins of Different Cross-Section Shapes," Int. J. Heat Mass Transfer, Vol. 32, pp. 1105-1110.