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ON SOME PROPERTIES OF PRETOPOLOGICAL CONVERGENCE STRUCTURES

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ABSTRACT In this paper we introduce generalized q-interior operator and n-th pretopological modification of q Furthermore we establish a characterization of $\pi_n(q) = \lambda(q)$

1. Introduction

A convergence structure q defined by Kent ([4]) is a correspondence between the filters on a given set X and the subsets of X which specifies that filters converge to points of X. For given convergence structure q on a set X, Kent introduced convergence structures with q, which are called the pretopological modification and the topological modification. They are denoted by $\pi(q)$ and $\lambda(q)$, respectively.

A q-interior operator I_q introduced by Choquet ([3]) is a set function which has all of the properties of topological interior operator except idempotency. In this paper, we introduce generalized q-interior operator and n-th pretopological modification of q. They are denoted by I_q^n and $\pi_n(q)$, respectively. Also, we study some properties of them and obtain a characterization of $\pi_n(q) = \lambda(q)$.

2. Preliminaries

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Let X be a set. A nonempty collection Φ of nonempty subsets of X is said to be a *filter* on X if it satisfies the following conditions:

(1) $A \in \Phi$ and $B \in \Phi$ implies $A \cap B \in \Phi$,
(2) $A \in \Phi$ and $A \subset B$ implies $B \in \Phi$.

For a nonempty set X, $F(X)$ denotes the set of all filters on X and $P(X)$ the set of all subsets of X.

A *convergence structure* q on a set X is defined to be a function from $F(X)$ into $P(X)$ satisfying the following conditions:
For each Φ and Ψ in $F(X)$,

(1) $x \in q(\dot{x})$ for each $x \in X$,
(2) if $\Phi \subset \Psi$, then $q(\Phi) \subset q(\Psi)$,
(3) if $x \in q(\Phi)$, then $x \in q(\Phi \cap \dot{x})$,

where $\dot{x}$ denotes the ultrafilter containing $\{x\}$. In this case the pair (X, q) is said to be a *convergence space.* If $x \in q(\Phi)$, we say that Φ *q-converges* to x. The filter $V_q(x)$ obtained by intersecting all filters which *q-converge* to x is said to be a *q-neighborhood filter* at x. If $V_q(x)$ *q-converges* to x for each $x \in X$, then q is said to be a *pretopological convergence structure* on X, and (X, q) a *pretopological convergence space.* The *pretopological convergence structure* q is said to be a *topological convergence structure* if for each $x \in X$, the filter $V_q(x)$ has a filter base $B_q(x)$ with the following property:

$$y \in G \in B_q(x) \text{ implies } G \in B_q(y).$$

Let $C(X)$ be the set of all convergence structures on X, partially ordered as follows:

$$q_1 \leq q_2 \text{ iff } q_2(\Phi) \subset q_1(\Phi) \text{ for all } \Phi \in F(X).$$

If $q_1 \leq q_2$, then we say that q_1 is *coarser* than q_2 and q_2 is *finer* than q_1.

For any $q \in C(X)$, we define the following related convergence structures $\pi(q)$ and $\lambda(q)$:

(1) $x \in \pi(q)(\Phi)$ iff $V_q(x) \subset \Phi$,
(2) $x \in \lambda(q)(\Phi)$ iff $U_q(x) \subset \Phi$,

where $U_q(x)$ is the filter generated by the sets $U \in V_q(x)$ which have the property: $y \in U$ implies $U \in V_q(y)$.

In this case $\pi(q)$ and $\lambda(q)$ are called the the *pretopological modification* and the *topological modification* of q. Also, the pairs $(X, \pi(q))$ and $(X, \lambda(q))$ are called the *pretopological modification* and the *topological modification* of (X, q), respectively.

Proposition 1([4]). *Let (X, q) be a convergence space. If $(X, \pi(q))$ and $(X, \lambda(q))$ are the pretopological modification and the topological modification of (X, q), respectivly. Then the following statements hold:*

(1) *$\pi(q)$ is the finest pretopological convergence structure coarser than q,*
(2) *$\lambda(q)$ is the finest topological convergence structure coarser than q,*
(3) $\lambda(q) \leq \pi(q) \leq q$.

Let f be a map from a convergence space (X, q) to a convergence space (Y, p). Then f is said to be *continuous* at a point $x \in X$, if the filter $f(\Phi)$ on Y p-converges to $f(x)$ for every filter Φ on X q-converging to x. If f is continuous at every point $x \in X$, then f is said to be *continuous.*

We define a set function $I_q^n : P(X) \to P(X)$ for each $n \in N \cup \{\infty\} \cup \{0\}$, where N is the set of all positive integers, as follows:

(1) $I_q^0(A) = A$,
(2) $I_q^1(A) = I_q(A) = \{x \in X \mid A \in V_q(x)\}$,
(3) $I_q^{n+1}(A) = I_q(I_q^n(A))$, if $n \in N$,
(4) $I_q^\infty(A) = \cap\{I_q^n(A) \mid n \in N\}$.

Proposition 2 ([5]). *For each $n \in N \cup \{\infty\} \cup \{0\}$, I_q^n has the following properties:*

(1) $I_q^n(\emptyset) = \emptyset$, $I_q^n(A) \subset A$,
(2) $I_q^n(X) = X$,
(3) $I_q^n(A \cap B) = I_q^n(A) \cap I_q^n(B)$,
(4) $A \subset B$ *implies* $I_q^n(A) \subset I_q^n(B)$

for each $A, B \subset X$.

But, in general $I_q^n(I_q^n(A)) \neq I_q^n(A)$ for all $A \subset X$.

Define $V_q^n(x) = \{A \subset X \mid x \in I_q^n(A)\}$. Then $V_q^n(x)$ is a filter on X for each $n \in N \cup \{\infty\}$.

Also, we know that for each $n \in N \cup \{\infty\}$

$$I_q^n(A) \supset I_q^{n+1}(A) \supset I_q^{\infty}(A) \text{ for each } A \subset X$$

and

$$V_q^n(x) \supset V_q^{n+1}(x) \supset V_q^{\infty}(x) \text{ for each } x \in X.$$

Define a structure $\pi_n(q)$ for each $n \in N \cup \{\infty\}$ as follows:

$$x \in \pi_n(q)(\Phi) \text{ iff } V_q^n(x) \subset \Phi$$

for each $\Phi \in F(X)$

While, since $V_q^n(x) \subset \dot{x}$, $x \in \pi_n(q)(\dot{x})$ for each $x \in X$. Also, $\Phi \subset \Psi \in F(X)$ implies $\pi_n(q)(\Phi) \subset \pi_n(q)(\Psi)$.

Let $x \in \pi_n(q)(\Phi)$. Then $V_q^n(x) \subset \Phi$. Since $V_q^n(x) \subset \dot{x}$, we obtain $V_q^n(x) \subset \Phi \cap \dot{x}$ and so $x \in \pi_n(q)(\Phi \cap \dot{x})$. Also, $x \in \pi_n(q)(V_q^n(x)) = \pi_n(q)(V_{\pi_n(q)}(x))$ for each $x \in X$. Thus $\pi_n(q)$ is a pretopological convergence structure on X.

In this case $\pi_n(q)$ is called the *n-th pretopological modification* of q Also, $(X, \pi_n(q))$ is called the *n-th pretopological modification* of (X, q).

It is not difficult to show that for each $n \in N \cup \{\infty\}$, the following statements hold:

(1) $V_{\pi_n(q)}(x) = V_q^n(x)$ for all $x \in X$.
(2) $I_{\pi_n(q)}(A) = I_q^n(A)$ for all $A \subset X$.
(3) For each $n \in N$, $q \geq \pi_n(q) \geq \pi_{n+1}(q) \geq \pi_\infty(q)$.

3. Main Results

By Proposition 1 and the definition of $\pi_n(q)$, we know that

$$q \geq \pi(q) \geq \pi_2(q) \geq \cdots \geq \pi_n(q) \geq \pi_{n+1}(q) \geq \cdots \geq \pi_\infty(q) \geq \lambda(q).$$

THEOREM 3. *Let (X, q) be a pretopological convergence space. Then the following are equivalent:*

(1) *q is a topological convergence structure.*
(2) *I_q is idempotent*

PROOF (1) $\Rightarrow$ (2) : It is clear that $I_q(I_q(A)) \subset I_q(A)$ for all $A \subset X$. We will show that $I_q(A) \subset I_q(I_q(A))$. Let $x \in I_q(A)$. Then $A \in V_q(x)$. Since q is a topological convergence structure, there exists $G \in B_q(x)$ such that $G \subset A$, where $B_q(x)$ is a filter base of $V_q(x)$ which has the following property:

$$y \in H \in B_q(x) \text{ implies } H \in B_q(y).$$

Since $y \in G \Rightarrow G \in B_q(y) \subset V_q(y)$, we obtain $y \in I_q(G)$. Thus $I_q(G) = G$. Since $G = I_q(G) \subset I_q(A)$ and $V_q(x)$ is a filter, $I_q(A) \in V_q(x)$. Thus $x \in I_q(I_q(A))$ and so $I_q(A) = I_q(I_q(A))$. That is I_q is idempotent.

(2) $\Rightarrow$ (1) : Take $B_q(x) = \{B \in V_q(x) \mid I_q(B) = B\}$ for each $x \in X$. Since $I_q(X) = X$, we obtain $B_q(x)$ is not a empty collection. Since $\emptyset \notin V_q(x)$, we obtain $\emptyset \notin B_q(x)$. Let $G_i \in B_q(x)$ for $i \in \{1, 2\}$ Then $G_i \in V_q(x)$ and $I_q(G_i) = G_i$ for $i \in \{1, 2\}$. Since $G_1 \cap G_2 = I_q(G_1) \cap I_q(G_2) = I_q(G_1 \cap G_2)$ and $V_q(x)$ is a filter, we obtain $G_1 \cap G_2 \in B_q(x)$. Also, let $A \in V_q(x)$. Since I_q is idempotent, $I_q(A) = I_q(I_q(A))$ and $I_q(A) \in V_q(x)$. Thus $I_q(A) \in B_q(x)$. Since $I_q(A) \subset A$, $B_q(x)$ is a filter base of $V_q(x)$. Let $y \in H \in B_q(x)$ Since $H = I_q(H)$, we obtain $y \in I_q(H)$. Thus $H \in B_q(y)$. Therefore q is a topological convergence structure.

PROPOSITION 4 *Let (X, q) be a convergence space. Then $\phi(q) = \lambda(q)$ iff I_q is idempotent*

PROOF. Assume that $\pi(q) = \lambda(q)$. Since $\pi(q)$ is a pretopological convergence structure and $\pi(q) = \lambda(q)$, $\pi(q)$ is a topological convergence structure. By Theorem 3, $I_{\pi(q)}$ is idempotent Since $I_{\pi(q)}(A) = I_q(A)$ for all A $\subset$ X, I_q is idempotent. Conversely, let I_q be idempotent. By Theorem 3, q is a topological convergence structure. It is clear that $\lambda(q) = q$ iff q is a topological convergence structure. We know that $q \geq \pi(q) \geq \lambda(q)$. Thus $q = \pi(q) = \lambda(q)$.

THEOREM 5 *Let (X, q) be a convergence space. Then for each $n \in N \cup \{\infty\}$, the following statements are equivalent:*

(1) $\pi_n(q) = \lambda(q)$.
(2) I_q^n *is idempotent.*

PROOF. (1) $\Rightarrow$ (2) : Assume that $\pi_n(q) = \lambda(q)$. We will show that I_q^n is idempotent. Let $A \subset X$ and $x \in I_q^n(A)$. Then $A \in V_q^n(x)$. Since $\pi_n(q)$ is a topological convergence structure, there exists $G \in B_q^n(x)$ such that $G \subset A$, where $B_q^n(x)$ is a filter base of $V_q^n(x)$ which has the following property:

$$y \in H \in B_q^n(x) \text{ implies } H \in B_q^n(y).$$

Thus $I_q^n(G) = G$. Since $G = I_q^n(G) \subset I_q^n(A)$ and $V_q^n(x)$ is a filter, we obtain $I_q^n(A) \in V_q^n(x)$. Thus $x \in I_q^n(I_q^n(A))$ and so $I_q^n(A) = I_q^n(I_q^n(A))$. That is I_q^n is idempotent.

(2) $\Rightarrow$ (1) : Assume that I_q^n is idempotent. Let $B_q^n(x) = \{G \in V_q^n(x) \mid I_q^n(G) = G\}$ for each $x \in X$. Since $I_q^n(X) = X$, we obtain $X \in B_q^n(x)$. Since $\emptyset \notin V_q^n(x)$, we obtain $\emptyset \notin B_q^n(x)$. Let $G_i \in B_q^n(x)$ for $i \in \{1, 2\}$. Since $G_1 \cap G_2 = I_q^n(G_1) \cap I_q^n(G_2) = I_q^n(G_1 \cap G_2)$ and $V_q^n(x)$ is a filter, we obtain $G_1 \cap G_2 \in B_q^n(x)$. Also, let $A \in V_q^n(x)$. Since I_q^n is idempotent, $I_q^n(A) = I_q^n(I_q^n(A))$ and $I_q^n(A) \in V_q^n(x)$. Thus $I_q^n(A) \in B_q^n(x)$. Since $I_q^n(A) \subset A$, $B_q^n(x)$ is a filter base of $V_q^n(x)$. Let $y \in G \in B_q^n(x)$. Since $H = I_q^n(H)$, we obtain $y \in I_q^n(H)$. Thus $G \in B_q^n(y)$. Therefore $\pi_n(q)$ is a topological convergence structure. Since $\lambda(q)$ is the finest topological convergence structure coarser than q. That is $\pi_n(q) = \lambda(q)$.

In that case $n = \infty$, the proof is similar to in the case $n \in N$.

DEFINITION 6. Let (X, q) be a convergence space. The *length* of q is defined by the smallest positive integer n satisfying $I_q^{n+1}(A) = I_q^n(A)$ for each $A \subset X$. We denote $l(q) = n$.

$$If\ l(q) \neq n \text{ for all } n \in N \text{ and } I_q(I_q^\infty(A)) = I_q^\infty(A) \text{ for all } A \subset X,$$

then we denote $l(q) = \infty$.

THEOREM 7. *Let (X, q) be a convergence space and $n \in N \cup \{\infty\}$. Then the following statements are equivalent:*

(1) I_q^n is idempotent and I_q^m is not idempotent for $m < n$.

(2) $l(q) = n$.

PROOF. At first we will prove in the case $n \in N$.

(1) $\Rightarrow$ (2) : Assume that for each $A \subset X$, $I_q^n(I_q^n(A)) = I_q^n(A)$ and $I_q^m(I_q^m(B)) \neq I_q^m(B)$ for some $B \subset X$ if $m < n$. By the definition of I_q^n,

$$I_q(A) \supset I_q^2(A) \supset \cdots \supset I_q^n(A) \supset I_q^{n+1}(A) \supset \cdots \supset I_q^n(I_q^n(A))$$
$$\supset \cdots \supset I_q^\infty(A) \supset I_q(I_q^\infty(A)) \supset \cdots \supset I_q^\infty(I_q^\infty(A)) \supset \cdots.$$

Since $I_q^n(I_q^n(A)) = I_q^n(A)$, we obtain $I_q^{n+1}(A) = I_q^n(A)$. Suppose that $I_q^{m+1}(A) = I_q^m(A)$ for $m < n$. Then $I_q^m(I_q^m(A)) = I_q^m(A)$ and so I_q^m is idempotent. This is a contradiction. Thus $l(q) = n$.

(2) $\Rightarrow$ (1) : Assume that $l(q) = n$. Then $I_q^n(A) = I_q^{n+1}(A) = I_q(I_q^n(A)) = I_q^2(I_q^n(A)) = \cdots = I_q^n(I_q^n(A))$. Thus I_q^n is idempotent. Also, by the definition of $l(q) = n$, I_q^m is not idempotent for $m < n$. In that case $n = \infty$. By the definition of $l(q) = \infty$, it is clear that (1) $\Leftrightarrow$ (2).

COROLLARY 8. *Let (X, q) be a convergence space and $n \in N \cup \{\infty\}$. Then $\pi_n(q) = \lambda(q)$ and $\pi_m(q) \neq \lambda(q)$ for $m < n$ iff $l(q) = n$.*

PROOF. By Theorem 5 and Theorem 7.

PROPOSITION 9. *Let (X, q) and (Y, p) be convergence spaces and $f : (X, q) \to (Y, p)$ be a map. Then for each $n \in N \cup \{\infty\}$, the following statements are equivalent:*

(1) $f(V_q^n(x)) = V_p^n(f(x))$ *for all* $x \in X$.

(2) $I_q^n(f^{-1}(B)) = f^{-1}(I_p^n(B))$ *for each* $B \subset Y$.

PROOF. (1) $\Rightarrow$ (2) : Assume that $f(V_q^n(x)) = V_p^n(f(x))$ for all $x \in X$. Let $x \in I_q^n(f^{-1}(B))$. Then $f^{-1}(B) \in V_q^n(x)$ and so $B \in f(V_q^n(x))$. Since $f(V_q^n(x)) = V_p^n(f(x))$, $B \in V_p^n(f(x))$. Thus $f(x) \in I_p^n(B)$ and so $x \in f^{-1}(f(x)) \in f^{-1}(I_p^n(B))$. Therefore $I_q^n(f^{-1}(B)) \subset f^{-1}(I_p^n(B))$. The reverse inequality is proved by the counter-order.

(2) $\Rightarrow$ (1) : Assume that $I_q^n(f^{-1}(B)) = f^{-1}(I_p^n(B))$ for each $B \subset Y$. Let $B \in V_p^n(f(x))$. Then $f(x) \in I_p^n(B)$ and so $x \in f^{-1}(I_p^n(B))$. Since $I_q^n(f^{-1}(B)) = f^{-1}(I_p^n(B))$, $x \in I_q^n(f^{-1}(B))$. Thus $f^{-1}(B) \in V_q^n(x)$ and so $B \in f(V_q^n(x))$. Therefore $V_p^n(f(x)) \subset f(V_q^n(x))$. The reverse inequality is proved by the counter-order.

PROPOSITION 10. *Let (X,q) and (Y,p) be convergence spaces. Let $f:(X,q)\to(Y,p)$ be a map. Then the following statements are equivalent:*

(1) $V_p(f(x)) = f(V_q(x))$.
(2) $V_p^n(f(x)) = f(V_q^n(x))$ *for each* $n \in N \cup \{\infty\}$.

PROOF. (2) $\Rightarrow$ (1): It is clear.
(1) $\Rightarrow$ (2) : We will use the mathematical induction to prove above Proposition. Assume that $V_p^k(f(x)) = f(V_q^k(x))$ and let $B \in V_p^{k+1}(f(x))$. Then $f(x) \in I_p^{k+1}(B) = I_p(I_p^k(B))$ and so $I_p^k(B) \in V_p(f(x)) = f(V_q(x))$. By assumption and Proposition 9, $f^{-1}(I_p^k(B)) = I_q^k(f^{-1}(B)) \in V_q(x)$. Thus $x \in I_q(I_q^k(f^{-1}(B)) = I_q^{k+1}(f^{-1}(B))$ and so $f^{-1}(B) \in V_q^{k+1}(x)$. Finally, $B \in f(V_q^{k+1}(x))$. This means $V_p^{k+1}(f(x)) \subset f(V_q^{k+1}(x))$. The reverse inequality is proved by the counter-order.
In that case $n = \infty$, let $B \in V_p^{\infty}(f(x))$. Then $f(x) \in I_p^{\infty}(B)$ and so $f(x) \in I_p^n(B)$ for each $n \in N$. Thus $B \in V_p^n(f(x)) = f(V_q^n(x))$ for each $n \in N$. $B \in \cap\{f(V_q^n(x)) \mid n \in N\} = f(\cap\{V_q^n(x) \mid n \in N\}) = f(V_q^{\infty}(x))$. Finally, $V_p^{\infty}(f(x)) \subset f(V_q^{\infty}(x))$. The reverse inequality is proved by the counter-order.

DEFINITION 11 ([6]). Let (X,q) and (Y,p) be convergence spaces. An onto map $f:(X,q)\to(Y,p)$ is said to be *open* if satisfies the following condition: whenever an ultrafilter Ψ on Y p-converges to y, then for each x in $f^{-1}(y)$ there is a filter Φ which maps on Ψ and q-converges to x.

PROPOSITION 12 *Let (X,q) and (Y,p) be convergence spaces. If a map $f:(X,q)\to(Y,p)$ is onto, continuous and open, then $V_p(f(x)) = f(V_q(x))$ for each $x \in X$.*

PROOF. Since f is continuous, $f(\Phi)$ p-converges to $f(x)$ whenever Φ q-converges to x. Thus $f(V_q(x)) = f(\cap\{\Phi \mid x \in q(\Phi)\}) = \cap\{f(\Phi) \mid x \in q(\Phi)\} \supset \cap\{f(\Phi) \mid f(x) \in p(f(\Phi))\} \supset V_p(f(x))$. Also, we will claim that $f(V_q(x)) \subset V_p(f(x))$. Let $B \in f(V_q(x))$. Then $B = f(A)$ for some $A \in V_q(x)$. Let Ψ be an ultrafilter which p-converges to $f(x)$. Since f is open, there is a filter Φ such that Φ q-converges to x and

$f(\Phi) = \Psi$. Since $A \in \Phi$, we obtain $B = f(A) \in f(\Phi) = \Psi$. Thus B is in each ultrafilter which p-converges to $f(x)$ and so $B \in V_p(f(x))$. Therefore $f(V_q(x)) \subset V_p(f(x))$.

THEOREM 13 *Let (X,q) and (Y,p) be convergence spaces. Let a map $f : (X,q) \to (Y,p)$ be onto, continuous and open. If I_q^n is idempotent, then I_p^n is idempotent.*

PROOF Let $B \subset Y$. Then $f^{-1}(B) \subset X$. Since I_q^n is idempotent, $I_q^n(I_q^n(f^{-1}(B))) = I_q^n(f^{-1}(B))$. By Proposition 9 and Proposition 12, $I_q^n(I_q^n(f^{-1}(B))) = I_q^n(f^{-1}(I_p^n(B))) = f^{-1}(I_p^n(I_p^n(B)))$ and $I_q^n(f^{-1}(B)) = f^{-1}(I_p^n(B))$. Thus $f^{-1}(I_p^n(I_p^n(B))) = f^{-1}(I_p^n(B))$ and so $I_p^n(I_p^n(B)) = I_p^n(B)$. Therefore I_p^n is idempotent.

COROLLARY 14 *Let (X,q) and (Y,p) be convergence spaces. Let a map $f : (X,q) \to (Y,p)$ be onto, continuous and open. Then f preserves the length of convergence structure*

PROOF By Corollary 8 and Theorem 13.

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