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FUZZY D–CONTINUOUS FUNCTIONS

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ABSTRACT In this paper, fuzzy D-continuous function is defined Some basic properties of this continuity are summarized, and sufficient conditions on domain and/or ranges implying fuzzy D-continuity of fuzzy D-continuous functions are given Also fuzzy D-regular space is defined and by using fuzzy D-continuity, the condition which is equivalent to fuzzy D-regular space, is given

1. Introduction

The concept of fuzzy sets was introduced by Zadeh in his classicical paper [12]. Therefore many investigations have been carried out, in the general theorical field and also in different application sides, based on this concept. The idea of fuzzy topological spaces was introduced by Chang [1]. The idea is more or less a generalization of ordinary topological spaces. Different aspect of such spaces have been developed by several investigations.

In this paper, we first generalize the idea of continuity as a local property in fuzzy setting. Then we generalize mainly the concept of D-continuity of a function due to J. K. Kohli [4] in fuzzy setting. It can be seen that fuzzy continuity implies fuzzy D-continuity. Also it can be seen that weaker forms of fuzzy continuity implies weaker forms of fuzzy D-continuty, but not conversely Finally, it can be seen that

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fuzzy continuity and fuzzy D-continuity are equivalent in case when the range space of function is fuzzy D-regular space.

Helderman [3] introduced some new regularity axioms and studied the class of D-regular spaces. Also the class of D-Hausdorff spaces, was introduced by J. K. Kohli [4], was shown to constitute on appropriate class of spaces in which D-continuous functions have strongly closed graphs. Then it turns out that the class of D-regular spaces is precisely the class of spaces in which the concepts of a continuous function and D-continuous function coincide [4, Theorem 4.1.]. In this paper, the class of fuzzy D-regular spaces is introduced and some properties are studied in Section 3. Also it can be seen that the class of fuzzy D-regular spaces is precisely the class spaces in which the concepts of a fuzzy continuous function and fuzzy D-continuous function coincide [see Theorem 25]

Preliminaries

DEFINITION 1 Let X be a nonempty set. Then a *fuzzy set* in X is an element in $[0,1]^X$, i.e. a function from X into $[0,1]$ ([1])

DEFINITION 2 Let α and β be two fuzzy sets in X. Then we have the following properties for fuzzy sets α and β :

$$\alpha \leq \beta \Leftrightarrow \alpha(x) \leq \beta(x) \text{ for all } x \in X,$$

$$\alpha = \beta \Leftrightarrow \alpha(x) = \beta(x) \text{ for all } x \in X,$$

$$\mu = \alpha \vee \beta \Leftrightarrow \mu(x) = \max\{\alpha(x), \beta(x)\} \text{ for all } x \in X,$$

$$\delta = \alpha \wedge \beta \Leftrightarrow \delta(x) = \min\{\alpha(x), \beta(x)\} \text{ for all } x \in X,$$

$$\alpha = \beta' \Leftrightarrow \alpha(x) = 1 - \beta(x) \text{ for all } x \in X.$$

More generally, for a family of fuzzy sets $\mu = \{\mu_i \mid i \in I\}$, the intersection $\beta = \wedge\mu_i$ and the union $\alpha = \vee\mu_i$, are defined as $\alpha(x) = Sup\{\mu_i(x) : x \in X\}$ and $\beta(x) = Inf\{\mu_i(x) : x \in X\}$, for $x \in X$ ([1]).

DEFINITION 3 A fuzzy set in X is called a *fuzzy point* if it takes the value 0 for all $y \in X$ except one, say, $x \in X$ If it's value at x is $\alpha(0 < \alpha \leq 1)$, we denote this fuzzy point by x_α where the point x is called its *support*. We can write the fuzzy point x_α, with

$$x_\alpha(y) = \begin{cases} \alpha & ; \text{ if } y = x \\ 0 & ; \text{ if } y \neq x \end{cases}$$

and we can denote the support of x with *supp* $x_\alpha = x$ ([7]).

DEFINITION 4 A *fuzzy topology* is a family τ of fuzzy sets in X which satifises the following conditions:

(a) $0, 1 \in \tau$;
(b) If $\alpha, \beta \in \tau$, then $\alpha \wedge \beta \in \tau$;
(c) If $\mu_i \in \tau$, for each $i \in I$, then $\vee_{i\in I}, \quad \mu_i \in \tau$,

τ is called a *fuzzy topology* for X, and the pair (X, τ) is a *fuzzy topological space* (shortly f t.s.). Every member of τ is called a *fuzzy open set*. A fuzzy set is called a *fuzzy closed set* iff its complement is open ([7]).

DEFINITION 5. Let (X, τ) be a f.t.s. and $\alpha \in I^X$ The *closure* of α is denoted $\overline{\alpha}$ and given by $\overline{\alpha} = \wedge\{\beta : \beta$ is a fuzzy closed set and $\alpha \leq \beta\}$. The *interior* of α is denoted by *int*α or $\overset{\circ}{\alpha}$ and given by $\overset{\circ}{\alpha} = \vee\{\beta : \beta$ is a fuzzy open set and $\beta \leq \alpha\}$ ([5]).

DEFINITION 6 A fuzzy set α in a f.t.s., (X, τ) is called a *neighborhood* of fuzzy point x_∂ if there exists $\beta \in \tau$ such that $x_\partial \in \beta$ and $\beta \leq \alpha$. A neighborhood α of x_∂ is said to be *open* if α is fuzzy open. The family consisting of all the neighborhoods of x_∂ is called the *system* of neighborhoods of x_∂ ([2]).

DEFINITION 7 A fuzzy set α in a f t s., (X, τ) is called Q*-neighborhood* of x_∂ if there exists $\beta \in \tau$ such that $x_\partial \in \beta$ and $\beta \leq \alpha$. The family consisting of all the Q-neighborhds of x_∂ is called the *system* of Q-neighborhds of x_∂. For fuzzy sets here α and β ,$\alpha \in \beta$ mean that $\alpha(y) + \beta(y) > 1$ for at least one point y in X ([2]).

DEFINITION 8. Let X and Y be two f.t.s. and let f be a function from X to Y. Also let β be a fuzzy set in Y. Then the *inverse* of β, written as $f^{-1}(\beta)$, is a fuzzy set in X which is defined by $f^{-1}(\beta)(x) = \beta(f(x))$ for all x in X.

Conversely, let α be a fuzzy set in X. The image of α, written as $f(\alpha)$, is a fuzzy set in Y which is defined by

$$f(\alpha)(y) = \begin{cases} \sup\limits_{z \in f^{-1}(y)} \{\alpha(x)\} : & \text{if } f^{-1}(y) \text{ is nonempty} \\ \quad 0 \quad : & \text{otherwise,} \end{cases}$$

for all y in Y where $f^{-1}(y) = \{x : f(x) = y\}$ ([5]).

DEFINITION 9. Let (X, τ) and (Y, τ') be two f.t.s., f : $X \to Y$ be a function and x_α be a fuzzy point in X. For each Q-neighborhood μ of $f(x_\alpha)$, if there is a Q-neighborhood δ of x_α, such that $f(\delta) \leq \mu$, then it is called that f is *fuzzy continuous* at x_α ([8]).

DEFINITION 10 Let $S = \{S_n : n \in D\}$ be a fuzzy net in X. S is said to be *quasi-coincident* with α if for each $n \in D$, S_n is quasi-coincident with α. Also S is said to be *eventually quasi-coincident* with α if there is an element m of D such that, if $n \in D$ and $n \geq m$ then S_n is quasi-coincident with α ([7]).

DEFINITION 11 A net S in a f.t.s., (X, τ) is said to be *converge* to a fuzzy point x_α in X relavite to τ if S is eventually quasi-coincident with each Q-neighborhood of x_α ([7]).

DEFINITION 12. Let (X, τ) be a f.t.s. and $\mu \in I^X$. If $\mathring{\bar{\mu}} = \mu$, then it is called that μ is a *fuzzy regular open set* (f.r.o.) in X. If $\mu = \bar{\mathring{\mu}}$ then it is called that μ is a *fuzzy regular closed* set (f.r.c.) ([5]).

DEFINITION 13 A f.t.s. X is called a *fuzzy regular space* if for each fuzzy point x_α in X, and if for every fuzzy open set μ properly $x_\alpha \in \mu$, there exists a fuzzy open set η in X such that $x_\alpha \in \eta$ and $\eta \leq \bar{\eta} \leq \mu$ ([6]).

DEFINITION 14 A f.t.s. X is called a *fuzzy almost regular space* if for each f.r.o. set μ in X and for each fuzzy point x_α properly x_α quasi-coincident with μ, there exists a fuzzy regular open set β in X such that $x_\alpha \in \beta$ and $\beta \leq \bar{\beta} \leq \mu$ ([6]).

DEFINITION 15 A f.t.s. X called a *fuzzy semi-regular space* if for each fuzzy open set μ in X and for every fuzzy point $x_\alpha \in \mu$, there exists a fuzzy open set β in X such that $x_\alpha \in \beta$ and $\beta \leq \mathring{\bar{\beta}} \leq \mu$ ([6]).

It can be seen from above definitions that a fuzzy regular space is a fuzzy semi-regular and a fuzzy almost regular space.

DEFINITION 16 A fuzzy almost regular space is a *fuzzy semi-regular space* if and only if it is a fuzzy regular space ([6]).

DEFINITION 17 A function $f : X \to Y$ is said to be *fuzzy continuous* (f.c.) at x_α if for each fuzzy open set β in Y with $f(x_\alpha) \in \beta$, there is a fuzzy open set μ with $x_\alpha \in \mu$ such that $f(\mu) \leq \beta$. The function which is fuzzy continuous at each point is called *fuzzy continuous* ([5]).

DEFINITION 18 A function $f : X \to Y$ is said to be *fuzzy almost continuous*(f.a.c.) at x_α if for each fuzzy open set β in Y with $f(x_\alpha) \in \beta$, there is a fuzzy open set μ with $x_\alpha \in \mu$ such that $f(\mu) \leq \mathring{\bar{\beta}}$. The function which is fuzzy continuous at each point is called *fuzzy continuous* ([5])

2. Fuzzy D-continuous funtions

DEFINITION 19 A fuzzy set in X is a *fuzzy G_δ-set* if it is a countable intersection of fuzzy open sets

DEFINITION 20 A fuzzy set in X is a *fuzzy F_σ-set* if it is a countable union of fuzzy closed sets.

The coplement of a fuzzy G_δ-set is a fuzzy F_σ-set and vice versa.

LEMMA 1 *A fuzzy F_σ-set can be written as the union of an increasing sequence $\sigma_1 \leq \sigma_2 \leq \cdots$ of fuzzy closed sets (Hence, a fuzzy G_δ-set can be written as the intersection of a decreasing sequence of fuzzy open sets.).*

PROOF. It is clear from the definitions 17 and 18.

DEFINITION 21 A function $f : X \to Y$ is said to be *fuzzy D-continuous* (f.D.c.) at x_α if for each fuzzy open F_σ-set β in Y with $f(x_\alpha) \in \beta$, there is a fuzzy open set μ with $x_\alpha \in \mu$ such that $f(\mu) \leq \beta$. The function which is fuzzy D-continuous at each point is called *fuzzy D-continuous.*

THEOREM 1. *Let $f : X \to Y$ be a function. If f is fuzzy continuous, then f is f.D.c.*

PROOF. Since each fuzzy open F_σ-set is fuzzy open, the proof is clear.

EXAMPLE 1 *Let X be a nonempty set and $\tau = \{0, 1, \alpha\}$ be a fuzzy topology on X with $\alpha(x) = \frac{11}{30}$. Let $\tau' = \{0, 1, \beta_n : n \in N\}$ where for each $n \in N$ and for all $x \in X$, $\beta_n(x) = \frac{1}{n}$ for all $x \in X$. Then the identity mapping $f : (X, \tau) \to (X, \tau')$ is f.D.c. at $x_{(\frac{21}{30})}$, but not f.c. at $x_{(\frac{21}{30})}$.*

PROOF. For $\beta_3 \in \tau'$ with $f(x_{\frac{21}{30}}) \in \beta_3$, $x_{\frac{21}{30}} \in \alpha$ and $f(\alpha) \nleq \beta_3$ so is not f.c. at $x_{\frac{21}{30}}$. But, if we obtain the family of nonempty fuzzy open F_σ-sets Φ in (X, τ'), then we arrive that $\Phi = \{1, \beta_2\}$. Thus, for $f(x_{\frac{21}{30}}) \in 1$, $x_{\frac{21}{30}} \in 1$ and $f(1) \leq 1$ and for $f(x_{\frac{21}{30}}) \in \beta_2$, $x_{\frac{21}{30}} \in \alpha$ and $f(\alpha) \leq \beta_2$ so f is f.D.c. at $x_{\frac{21}{30}}$.

THEOREM 2 *Let $f : X \to Y$ be a function. Then the following statements are equivalent :*

(a) *f is f.D.c. ;*
(b) *If β is a fuzzy open F_σ-set in Y, then $f^{-1}(\beta)$ is a fuzzy open set in X ;*
(c) *If σ a fuzzy closed G_δ-set in Y, then $f^{-1}(\sigma)$ is a fuzzy closed set in X.*

PROOF $(a) \Rightarrow (b)$: If β is a fuzzy open F_σ-set in Y, then for each fuzzy point x_α in X with $x_\alpha \in f^{-1}(\beta)$, $f(x_\alpha) \in \beta$. From (a), there is a fuzzy open set μ in X with $x_\alpha \in \mu$ such that $f(\mu) \leq \beta$. Thus $x_\alpha \in \mu$

and $\mu \leq f^{-1}(\beta)$ so $f^{-1}(\beta)$ is fuzzy of x_α. Thus $f^{-1}(\beta)$ is a fuzzy open set in X.
$(b) \Rightarrow (a)$: Let β be a fuzzy open F_σ-set in Y with $f(x_\alpha) \in \beta$. From (b), $f^{-1}(\beta)$ is a fuzzy open set in X with $x_\alpha \in f^{-1}(\beta)$. Thus $f(\mu) \leq \beta$ with $f^{-1}(\beta) = \mu$.
$(b) \Rightarrow (c)$: Let σ be a fuzzy closed G_δ-set in Y, then $1 - \sigma$ is a fuzzy open F_σ-set and so from (b), $f^{-1}(1 - \sigma) = 1 - f^{-1}(\sigma)$ is fuzzy open. Thus $f^{-1}(\sigma)$ is fuzzy closed in X.
$(c) \Rightarrow (b)$: Let β be a fuzzy open F_σ-set. Then $1 - \beta$ is a fuzzy closed G_δ-set and so $f^{-1}(1 - \beta) = 1 - f^{-1}(\beta)$ is fuzzy closed. Thus $f^{-1}(\beta)$ is a fuzzy open set in X.

PROPOSITION 1. *Let $f . X \rightarrow Y$ be a function. If f is f.D.c., then for each fuzzy point x_α in X and each fuzzy net $\{S_n : n \in D\}$ which converges to x_α, the fuzzy net $\{f(S_n) : n \in D\}$ is eventually quasi-coincident with each fuzzy open F_σ-set β with $f(x_\alpha) \in \beta$.*

PROOF. By the theorem 2., f is f.D.c. $\Leftrightarrow$ if β is a fuzzy open F_σ-set in Y, then $f^{-1}(\beta)$ is a fuzzy open set in X Now, let $\{S_n : n \in D\}$ be a fuzzy net in X which converges to x_α and let β be a fuzzy open F_σ-set in Y with $f(x_\alpha) \in \beta$. Then $f^{-1}(\beta)$ is a fuzzy open set with $x_\alpha \in f^{-1}(\beta)$. Thus $\{S_n : n \in D\}$ is eventually quasi-coincident with $f^{-1}(\beta)$. Hence $\{f(S_n) : n \in D\}$ is eventually quasi-coincident with β.

DEFINITION 22. Let $f . X \rightarrow Y$ be any function. Then the function $g : X \rightarrow X \times Y$, defined by $g(x) = (x, f(x))$, is called the *graph function* with respect to f ([11]).

THEOREM 3. *Let $f : X \rightarrow Y$ be a function such that the graph function g is f.D.c.. Then f is f.D.c..*

PROOF. Let x_α be a fuzzy point in X and let β be a fuzzy open F_σ-set with $f(x_\alpha) \in \beta$. Since $1 - \beta$ is a fuzzy closed G_δ-set, $1 \times (1 - \beta) = (1 \times 1) - P_2^{-1}(\beta)$ is a fuzzy closed G_δ-set. Thus $P_2^{-1}(\beta)$ is a fuzzy open F_σ-set of $X \times Y$. Since g is f.D.c., there is a fuzzy open set μ with $x_\alpha \in \mu$ such that $g(\mu) \leq P_2^{-1}(\beta)$. It follows that $P_2(g(\mu)) = f(\mu)$ and $f(\mu) \leq \beta$ and $g(x_\alpha) \in P_2^{-1}(\beta)$, and so f is f.D.c..

THEOREM 4. *Let $f : X \to Y$ be any function and A be a subset of X. If f is f.D.c., then the induced $f_{|A} : A \to Y$ is f.D.c..*

PROOF. Let $x_\alpha{}'$ be a fuzzy point in A with

$$x_\alpha{}'(y) = \begin{cases} \alpha & ; \text{ if } \; y = x \\ 0 & ; \text{ if } \; y \neq x \end{cases}$$

and β be any fuzzy open F_σ-set in Y with $f_{|A}(x_\alpha{}') \in \beta$.

If we define fuzzy point x_α in X as

$$x_\alpha(y) = \begin{cases} \alpha & ; \text{ if } \; y = x \\ 0 & ; \text{ if } \; y \neq x, \end{cases}$$

then $f(x_\alpha) \in \beta$. Since f is f.D.c. from X to Y, there exists a fuzzy open set μ with $x_\alpha \in \mu$ such that $f(\mu) \leq \beta$. Then $\mu_{|A} = \mu_A$ is fuzzy open in A and $f_{|A}(\mu_A) \leq \beta$ where $\mu_A : A \to [0, 1]$. Thus $f_{|A}$ is f.D.c..

THEOREM 5. *If $f : X \to Y$ is f.c. and $g : Y \to Z$ is f.D.c., then $g \circ f$ is f.D.c..*

PROOF. Let σ be a fuzzy closed G_δ-set in Z. Then $g^{-1}(\sigma)$ is fuzzy closed in Y and since f is f.c., $(g \circ f)^{-1}(\sigma) = f^{-1}(g^{-1}(\sigma))$ is fuzzy closed in X. Thus $g \circ f$ is f.D.c..

THEOREM 6. *Let $f : X \to Y$ be either a fuzzy open or a fuzzy closed surjection and let $g : Y \to Z$ be any function such that $g \circ f$ is f.D.c.. Then g is f.D.c..*

PROOF. Suppose f is fuzzy open (respectively, fuzzy closed), and let β be a fuzzy open F_σ-set in Z (respectively, β be a fuzzy closed G_σ-set). Since $g \circ f$ is f.D.c., $(g \circ f)^{-1}(\beta) = f^{-1}(g^{-1}(\beta))$ is fuzzy open (respectively, fuzzy closed) and since f is a surjection, $f(f^{-1}(g^{-1}(\beta))) = g^{-1}(\beta)$ is fuzzy open (respectively, fuzzy closed) and consequently g is f.D.c..

DEFINITION 23 Let (X,τ) be a fuzzy topological space. Let R be equivalence relation on X. Let X/R be the quotient set, and let $P : X \to X/R$ be the projection (quotietent map). Let ν be the family of fuzzy sets in X, defined by $\nu = \{\beta \mid P^{-1}(\beta) \in \tau\}$. Then ν is, obviously a fuzzy topology, called the *quotient fuzzy topology* for X/R and $(X/R, \nu)$ is called the *quotient fuzzy space* of (X,τ) (relative to the quotient map). Here P is f.c. ([8]).

THEOREM 7 *Let $f : X \to Y$ be a quotient map. Then a function $g : Y \to Z$ is f.D.c. if and only if $g \circ f$ is f.D.c..*

PROOF $(\Rightarrow)$: It is immediate from the Theorem 6.

$(\Leftarrow)$: Let β be a fuzzy open F_σ-set in Z. Then $(g \circ f)^{-1}(\beta) = f^{-1}(g^{-1}(\beta))$ is fuzzy open in X. Since f is a quotient map, $g^{-1}(\beta)$ is fuzzy open in Y and so g is f.D.c..

THEOREM 8 *For each $\alpha \in I$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a function, and let $f : \prod X_\alpha \to Y\alpha$ be a function defined by $f[(x_\alpha)] = (f_\alpha(x_\alpha))$ for each fuzzy point (x_α) in $\prod X_\alpha$. If f is f.D.c., then each f_α is f.D c..*

PROOF Let $\alpha_0 \in I$, and let σ_{α_0} be a fuzzy closed G_δ-set in Y_{α_0}. Then $\sigma_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} 1_\alpha$ is a fuzzy closed G_δ-set in $\prod Y_\alpha$ where $\alpha_0 \in I$. Since f is f.D c ,by the Theorem 2, $f^{-1}(\sigma_{\alpha_0} \times (\prod 1_\alpha)) = f^{-1}(\sigma_{\alpha_0}) \times (\prod 1_\alpha)$ is fuzzy closed in $\prod X_\alpha$ where $\alpha \neq \alpha_0$. Consequently, $f_{\alpha_0}^{-1}(\sigma_{\alpha_0})$ is fuzzy closed in X_{α_0} and so f_{α_0} is f.D.c..

THEOREM 9. *Let $f : X \to \prod X_\alpha$ be a function into a fuzzy product space. If f is f.D.c., then for each $P_\alpha : \prod X_\alpha \to X_\alpha$, $P_\alpha \circ f$ is f.D.c..*

PROOF Let σ_{α_0} be a fuzzy closed G_δ-set in X_{α_0}.
Then $(P_{\alpha_0} \circ f)^{-1}(\sigma_{\alpha_0}) = f^{-1}(P_{\alpha_0}^{-1}(\sigma_{\alpha_0})) = f^{-1}(\sigma_{\alpha_0} x \prod 1_\alpha)$. Since f is f.D.c. and since $\sigma_{\alpha_0} \times \prod 1_\alpha$ is a fuzzy closed G_δ-set, then $f^{-1}(\sigma_{\alpha_0} \times \prod 1_\alpha)$ is fuzzy closed in X. By the Theorem 2, $P_{\alpha_0} \circ f$ is f.D.c..

DEFINITION 24 A function $f : X \to Y$ is said to be *fuzzy almost D-continuous (f.a.D.c.)* at x_α if for each fuzzy open F_σ-set β with $f(x_\alpha) \in \beta$, there exists a fuzzy open set μ, with $x_\alpha \in \mu$ such that $f(\mu) \leq \overset{\circ}{\overline{\beta}}$. A function which is a f.a D.c. at each point is called *f.a.D.c.*

THEOREM 10. *Let $f : X \to Y$ be a function. If f is f.D.c., then f is f.a.D.c..*

PROOF. Since for each fuzzy open set β, $\beta \leq \overset{\circ}{\overline{\beta}}$, the proof is clear.

THEOREM 11. *Let $f : X \to Y$ be a function. Then the following statements are equavelent :*

(a) *f is f.a.D.c. ;*
(b) *For each fuzzy regular open F_σ-set β in Y, $f^{-1}(\beta)$ is fuzzy open in X ;*
(c) *For each fuzzy regular closed G_σ-set β in Y, $f^{-1}(\beta)$ is fuzzy closed in X ;*
(d) *For each fuzzy regular open F_σ-set β in Y and for each x_α with $f(x_\alpha) \in \beta$, there is a fuzzy open set μ with $x_\alpha \in \mu$ such that $f(\mu) \leq \beta$;*
(e) *For every fuzzy open F_σ-set β in Y, $f^{-1}(\beta) \leq [f^{-1}(\overset{\circ}{\overline{\beta}})]$;*
(f) *For every fuzzy closed G_δ-set β in Y, $\overline{[f^{-1}(\overset{\circ}{\beta})]} \leq f^{-1}(\beta)$.*

PROOF. $(a) \Rightarrow (b)$: Let be β a fuzzy regular open F_σ-set in Y, then for each fuzzy point x_α in X with $x_\alpha \,\mathrm{inf}^{-1}(\beta)$, we have $f(x_\alpha) \in \beta$. From (a), there is a fuzzy open set μ with $x_\alpha \in \mu$ such that $f(\mu) \leq \overset{\circ}{\overline{\beta}}$. Since $\beta = \overset{\circ}{\overline{\beta}}$, $f(\mu) \leq \beta$. Thus $x_\alpha \in \mu$ and $\mu \leq f^{-1}(\beta)$. So $f^{-1}(\beta)$ is Q-neighborhood of x_α and $f^{-1}(\beta)$ is fuzzy open in X.
$(b) \Rightarrow (c)$: Let β be a fuzzy regular closed G_δ-set in Y, then $1 - \beta$ is a fuzzy regular open F_σ-set in Y. So $f^{-1}(1 - \beta) = 1 - f^{-1}(\beta)$ is fuzzy open in X. Thus, from (b), $f^{-1}(\beta)$ is fuzzy closed in X.
$(c) \Rightarrow (d)$: Let β be a fuzzy regular open F_σ-set in Y with $f(x_\alpha) \in \beta$. From (c), $1 - \beta$ is a fuzzy regular closed G_δ-set in Y and $f^{-1}(1 - \beta) = 1 - f^{-1}(\beta)$ is fuzzy closed in X. Thus $f^{-1}(\beta)$ is fuzzy open in X and $x_\alpha \,\mathrm{inf}^{-1}(\beta)$. Let $\mu = f^{-1}(\beta)$, then $f(\mu) \leq \beta$.
$(d) \Rightarrow (b)$: Let β be a fuzzy regular open F_σ-set in Y and $x_\alpha \in f^{-1}(\beta)$. Then, $f(x_\alpha) \in \beta$ and from (d), there is a fuzzy open set μ such that $x_\alpha \in \mu$ and $f(\mu) \leq \beta$. Thus $x_\alpha \in \mu$ and $\mu \leq f^{-1}(\beta)$. So $f^{-1}(\beta)$ is a fuzzy open set in X.

$(d) \Rightarrow (e)$: Let β be a fuzzy open F_σ-set in Y. Then $\overset{\circ}{\overline{\beta}}$ is a fuzzy regular open F_σ-set and $f^{-1}(\beta) \leq f^{-1}(\overset{\circ}{\overline{\beta}})$. From $[(d) \Rightarrow (b)]$, $f^{-1}(\overset{\circ}{\overline{\beta}})$ is fuzzy open. Thus $f^{-1}(\beta) \leq \overset{\circ}{[f^{-1}(\overline{\beta})]}$.
$(e) \Rightarrow (f)$: Let β be a fuzzy closed G_δ-set in Y. Then $1-\beta$ is a fuzzy open F_σ-set in Y. From (e), $f^{-1}(1-\beta) = 1 - f^{-1}(\beta) \leq [f^{-1}(\overset{\circ}{\overline{1-\beta}})] = [1 - f^{-1}(\overline{\overset{\circ}{\beta}})] = [1 - f^{-1}(\overline{\overset{\circ}{\beta}})]$. Thus, $f^{-1}(\overline{\overset{\circ}{\beta}}) \leq f^{-1}(\beta)$.
$(f) \Rightarrow (a)$: Let β be a fuzzy open F_σ-set in Y with $f(x_\alpha) \in \beta$. Then $1-\beta$ is fuzzy closed G_δ-set in Y. From (f), $[f^{-1}(\overline{\overset{\circ}{1-\beta}})] \leq 1 - f^{-1}(\beta)$ $\Rightarrow [f^{-1}(1-\overline{\overset{\circ}{\beta}})] \leq 1 - f^{-1}(\beta) \Rightarrow 1 - f^{-1}(\overset{\circ}{\overline{\beta}}) \leq 1 - f^{-1}(\beta) \Rightarrow 1 - f^{-1}(\overset{\circ}{\overline{\beta}}) \leq 1 - f^{-1}(\beta) \Rightarrow f^{-1}(\beta) \leq [f^{-1}(\overset{\circ}{\overline{\beta}})]$ Let $\mu = [f^{-1}(\overset{\circ}{\overline{\beta}})]$, then $x_\alpha \in \mu$ and $f(\mu) \leq \overset{\circ}{\overline{\beta}}$. Thus f is f.a D.c. at x_α.

PROPOSITION 2 *Let $f : X \longrightarrow Y$ be a function. If f is f.a.D.c., then for each fuzzy point x_α in X and for each fuzzy net $S = \{S_n : n \in D\}$ which converges to x_α, the fuzzy net $f(S) = \{f(S_n) : n \in D\}$ is eventually quasi-coincident with each fuzzy regular open F_σ – set, β with $f(x_\alpha) \in \beta$.*

PROOF Let $S = \{S_n : n \in D\}$ be a fuzzy net in X which converges to x_α and let β be a fuzzy regular open F_σ-set in Y with $f(x_\alpha) \in \beta$. Then, from the Theorem 11, $f^{-1}(\beta)$ is a fuzzy open set in X with $x_\alpha \mathrm{inf}^{-1}(\beta)$. Thus, since S is eventually quasi-coincident with $f^{-1}(\beta)$, $f(S) = \{f(S_n) : n \in D\}$ is eventually quasi-coincident with β.

THEOREM 12 *Let $f : X \to Y$ be a f.c mapping. If $g : Y \to Z$ is f.a.D.c., then $g \circ f$ is f.a.D.c..*

PROOF Let β be a fuzzy regular closed G_δ-set in Z. Then, by the Theorem 11, $g^{-1}(\beta)$ is fuzzy closed in Y. Since f is f.c., $f^{-1}(g^{-1}(\beta)) = (g \circ f)^{-1}(\beta)$ is fuzzy closed in X. Thus, $g \circ f$ is f.a.D.c..

COROLLAY 1 *If $f : X \to Y$ is f.c., then f is f.a.D.c..*

PROOF. The proof is clear from the definitions.

THEOREM 13. *If $f : X \to Y$ is f.a.c., then f is f.a.D.c..*

PROOF Since a fuzzy regular open F_σ-set is a fuzzy regular open set, the proof is clear.

DEFINITION 25. A function $f : X \to Y$ is said to be *fuzzy weakly D-continuous* (f.w.D.c.) at x_α if for each fuzzy open set F_σ-set β with $f(x) \in \beta$, there exists a fuzzy open set μ with $x_\alpha \in \mu$ such that $f(\mu) \leq \overline{\beta}$. The function which is a f.w.D.c. in each point is called *f.w.D.c.*.

THEOREM 14 *If $f : X \to Y$ is f.D.c., then f is f.w.D.c..*

PROOF For a fuzzy set β, since $\beta \leq \overline{\beta}$, the proof is clear.

COROLLAY 2 *If $f : X \to Y$ is f.c., then f is f.w.D.c..*

PROOF f is f.c. $\Rightarrow$ f is f.D.c. $\Rightarrow$ f is f.w.D.c..

COROLLAY 3 *If $f : X \to Y$ is f.a.D.c., then f is f.w.D.c..*

PROOF For a fuzzy set β, since $\overset{\circ}{\overline{\beta}} \leq \overline{\beta}$, the proof is clear.

COROLLAY 4 *If $f : X \to Y$ is f.a.c., then f is f.w.D.c..*

PROOF f is f.a.c. $\Rightarrow$ f is f.a.D.c. $\Rightarrow$ f is f.w.D.c..

3. Fuzzy D-regular space and fuzzy D-hausdorff space

DEFINITION 26 Two fuzzy sets β_1 and β_2 in a f.t.s. (X, τ) are said to be *$Q-$seperated* iff there exist fuzzy closed sets μ_i $(i = 1, 2)$ such that $\mu_i \geq \beta_i$ $(i = 1, 2)$ and $\beta_2 \Lambda \mu_1 = \beta_1 \Lambda \mu_2 = 0$. It is obvious that β_1 and β_2 are $Q-$seperated iff $\beta_2 \Lambda \beta_1 = \beta_1 \Lambda \beta_2 = 0$ ([7]).

DEFINITION 27 A fuzzy set β in a f.t.s.(X,τ) is called *fuzzy disconnected* if there exist two nonzero fuzzy sets A and B in supspace $D_o(i.e; supp\beta = D_o)$ such that A and B are $Q-$seperated and $\beta = A \vee B$. A fuzzy set is called *fuzzy connected* if it is not disconnected ([7]).

THEOREM 15 *Two fuzzy sets α and β are $Q-$seperated if $A_o \wedge B_o = \emptyset$, $\overline{\alpha}_{A_o \vee B_o} = \overline{\alpha}_{A_o}$, $\overline{\beta}_{A_o \vee B_o} = \overline{\beta}_{B_o}$ ([7]).*

PROPOSITION 3 *Two fuzzy sets α and β are Q-seperated if $\overline{\alpha}_{A_o \vee B_o}$ and $\overline{\beta}_{A_o \vee B_o}$ are Q-seperated ([7]).*

THEOREM 16. *A f.D.c. image of a fuzzy connected space is fuzzy connected.*

PROOF Let $f : X \to Y$ be a f.D.c. surjection from a fuzzy connected space X onto a fuzzy topological space Y. Suppose Y is not fuzzy connected Then, from the Definition 25, there are nonzero fuzzy sets α and β such that α and β are $Q-$seperated and $Y = \alpha \vee \beta$. By the Proposition 3, $\overline{\alpha}$ and $\overline{\beta}$ are $Q-$seperated. Thus $Y = \overline{\alpha} \vee \overline{\beta}$ and $\overline{\alpha} \vee \overline{\beta} = 0$. Hence both $\overline{\alpha}$ and $\overline{\beta}$ are fuzzy clopen sets in Y. This means they are fuzzy closed G_δ-sets in Y. Since f is f.D.c., by the Theorem 2, both $f^{-1}(\overline{\alpha})$ and $f^{-1}(\overline{\beta})$ are fuzzy closed sets in X. Then $1 = f^{-1}(\overline{\alpha}) \vee f^{-1}(\overline{\beta})$ and $f^{-1}(\overline{\alpha}) \wedge f^{-1}(\overline{\beta}) = 0$. Thus $f^{-1}(\overline{\alpha})$ and $f^{-1}(\overline{\beta})$ are $Q-$seperated and (X,τ) is a fuzzy disconnected space. This is a contradiction to the hypothesis.

DEFINITION 28 A function $f : X \to Y$ is said to be *fuzzy connected* if $f(\alpha)$ is fuzzy connected for every fuzzy connected set α in X.

COROLLAY 5 *Every f.D.c. function is a fuzzy connected function.*

PROOF This follows from the Theorem 4 and the Theorem 1.

DEFINITION 29 A f.t.s. (X,τ) is called *fuzzy T_1* if for each $x \in X$ and each $\lambda \in [0,1]$, there exists $\beta \in \tau$ such that $\beta(x) = 1 - \lambda$ and $\beta(y) = 1$ for $y \neq x$ ([7])

PROPOSITION 4. *A f.t.s (X, τ) is fuzzy T_1 if each fuzzy point in X is a fuzzy closed set in X ([9]).*

DEFINITION 30. A f.t.s. (X, τ) is called *fuzzy T_2* (fuzzy Hausdoff) if for any two fuzzy points e and d satisfying $supp e \neq supp d$ there exist $Q-$neighborhoods β and α of e and d, respectively, such that $\beta \wedge \alpha = 0$ ([7]).

THEOREM 17. *Let $f : X \to Y$ be a one-to-one and f.D.c. function such that each singleton in Y is a fuzzy G_δ-set. If Y is fuzzy T_1, then so X.*

PROOF. Since f is f.D.c. and injective for a fuzzy point x_α in X, $f(x_\alpha)$ is a fuzzy point in Y and since Y is fuzzy T_1 and $\{f(x_\alpha)\}$ is a fuzzy G_δ-set in Y, then $\{x_\alpha\}$ is fuzzy closed in X. So X is a fuzzy T_1 space.

THEOREM 18. *Let $f : X \to Y$ be a f.D c. and fuzzy closed function from a fuzzy normal space X onto a fuzzy topological space Y such that each singleton in Y is a fuzzy G_δ-set. If either of the spaces X and Y is fuzzy T_1, then Y is fuzzy Hausdorff.*

PROOF. Case 1. I. The space Y is fuzzy T_1. Let e and d two fuzzy points in Y satisfying $supp e \neq supp d$. Then $\{e\}$ and $\{d\}$ are fuzzy closed G_δ-sets in Y and so, by the Theorem 2, $f^{-1}(e)$ and $f^{-1}(d)$ are fuzzy closed sets in X. By the fuzzy normality of X, there are disjoint fuzzy open sets μ_1 and μ_2 such that $f^{-1}(e) \in \mu_1$ and $f^{-1}(d) \in \mu_2$ and $\mu_1 \wedge \mu_2 = 0$. Since f is fuzzy closed, the sets $\beta_1 = 1 - f(1 - \mu_1)$ and $\beta_2 = 1 - f(1 - \mu_2)$ are fuzzy open in Y. Also $e \in \beta_1$ and $d \in \beta_2$ and $\beta_1 \wedge \beta_2 = 0$, since $f(\mu_1) \leq \beta_1$ and $f(\mu_2) \leq \beta_2$. Thus Y is fuzzy Hausdorff.

Case 2. II. The space X is fuzzy T_1. Let x_α be a fuzzy point in X. Since the singleton $\{x_\alpha\}$ is fuzzy closed, $\{f(x_\alpha)\}$ is a fuzzy closed set in Y. So Y is fuzzy T_1 and the proof is complete in view of case 1.

DEFINITION 31. We call a space *fuzzy D-Hausdorff* if each pair of distinct fuzzy points is quasi-coincident with disjoint fuzzy open F_σ-sets.

THEOREM 19 *Let $f : X \to Y$ be a f.D c. injection into a fuzzy D-Hausdorff space Y. Then X is fuzzy Hausdorff*

PROOF Let x_α and y_λ be two fuzzy points satisfying $suppx_\alpha \neq suppy_\lambda$. Then $f(x) \neq f(y)$. Since Y is fuzzy D-Hausdorff, there are disjoint fuzzy open F_σ-sets β_1 and β_2 with $\beta_1 \in f(x_\alpha)$ and $\beta_2 \in f(y_\lambda)$, respectively. By the Theorem 2, $f^{-1}(\beta_1)$ and $f^{-1}(\beta_2)$ are disjoint fuzzy open sets with $x_\alpha \in f^{-1}(\beta_1)$ and $y_\lambda \in f^{-1}(\beta_2)$, respectively. Thus X is fuzzy Hausdorff.

Let (X, τ) be a f.t.s. and let Ψ denote the collection of all fuzzy open F_σ-sets in (X, τ). Since the intersection of two fuzzy open F_σ-sets is a fuzzy open F_σ-set, the collection Ψ is a base for a fuzzy topology τ^* on X. Clearly $\tau^* \subset \tau$. Moreover, if each singleton in X is a fuzzy G_δ-set, then (X, τ^*) is fuzzy T_1 whenever (X, τ) is.

DEFINITION 32 A f.t.s (X, τ) is called *fuzzy D-regular*if for each fuzzy point x_α in X and each fuzzy open set μ with $x_\alpha \in \mu$, there is a fuzzy open F_σ-set μ^* such that $x_\alpha \in \mu^*$ and $\mu^* \leq \mu$.

COROLLAY 6 *A f.t.s. (X, τ) is fuzzy D-regular if and only if $\tau = \tau^*$.*

PROOF ($\Rightarrow$): Let (X, τ) be a fuzzy D-regular space and let $\beta \in \tau$. If $\beta \notin \tau^*$, there is a fuzzy point x_α in X such that for every fuzzy open F_σ-set μ^* in X with $x_\alpha \in \mu^*, \mu^* \leq \beta$. But, since X is fuzzy D-regular, this is contradiction. Thus $\beta \in \tau^*$ and $\tau = \tau^*$.
($\Leftarrow$) : Let $\tau = \tau^*$ and let x_α be a fuzzy point in X. Suppose μ be a fuzzy open set in X with $x_\alpha \in \mu$. Since $\tau = \tau^*$, $\mu \in \tau^*$, $x_\alpha \in \mu$ and $\mu \leq \mu$. Thus X is a fuzzy D-regular space.

THEOREM 20 *Let (Y, τ') be a f.t.s.. Then following statements are equivalent :*

(a) *(Y, τ') is fuzzy D-regular ;*
(b) *Every f.D.c. function from a f.t.s. (X, τ) into Y is f.c. ;*
(c) *The identity mapping I from (Y, τ'^*) onto (Y, τ') is f.c..*

PROOF (a)⇒(b) : Let $f(x_\alpha) \in \beta$ and β be a fuzzy open set in Y. Since (Y, τ') is fuzzy D-regular, there is a fuzzy open F_σ-set β^* such that $x_\alpha \in \beta^*$ and $\beta^* \leq \beta$. By the Theorem 2, $f^{-1}(\beta^*)$ is fuzzy open and $x_\alpha \inf^{-1}(\beta^*)$ and $f(f^{-1}(\beta^*)) \leq \beta$. Thus f is f.c..
(b)⇒(c) : Let $f = I, I : (Y, \tau'^*) \to (Y, \tau')$ be the identity mapping. Let $f(x_\alpha) \in \beta$ and β be a fuzzy open F_σ-set in (Y, τ'). Then $x_\alpha \inf^{-1}(\beta)$ and $f^{-1}(\beta) \in \tau'^*$. So $f(f^{-1}(\beta)) \leq \beta$. Thus f is f.D.c.. From (b), f is f.c..
(c)⇒(a) : Let x_α be a fuzzy point and β be a fuzzy open set in (Y, τ') with $x_\alpha \in \beta$. Since $f = I : (Y, \tau'^*) \to (Y, \tau')$ is f.c., there is a fuzzy open F_σ-set μ in (Y, τ'^*) such that $x_\alpha \in \mu$ and $f(\mu) \leq \beta$. Thus $x_\alpha \in \mu \leq \beta$ and (Y, τ') is fuzzy D-regular.

THEOREM 21 *The product of any family $\{X_\alpha : \alpha \in D\}$ of fuzzy D-regular spaces is fuzzy D-regular.*

PROOF To show that $X = \prod X_\alpha$ is f.D-regular, in view of the Theorem 7, it is sufficient to show that every f.D.c. function $f : Y \to X$ is f.c.. Thus it suffices to show that P_α of is f.c. for each α, where P_α denotes the projection onto the α-co-ordinate space. Let σ be a fuzzy closed $G_\sigma-$set in X_i. Then $P_\alpha^{-1}(\sigma x \prod_{i \neq \alpha} 1\alpha)$ is a fuzzy closed G_δ-set in X. Since $(P_\alpha \circ f)^{-1}(\sigma) = f^{-1}(P_\alpha^{-1}(\sigma))$ is fuzzy closed in Y, $P_\alpha \circ f$ is f.D.c.. In view of fuzzy D-regularity of X_α $(for\ each\ \alpha \in D)$, $P_\alpha \circ f$ is f.c. and the proof of the theorem is complete.

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