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DUALITY IN THE OPTIMAL CONTROL PROBLEMS FOR HYPERBOLIC SYSTEMS

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ABSTRACT. In this paper we deal with the duality theory of optimality for an optimal control problem governed by a class of second order evolution equations. First we establish the dual control systems by using conjugate functions and then associate them to the original optimization problem.

1. Introduction

The purposes of this paper are to establish the dual control systems and discuss relations between their optimal values for the following control problem :

$$\inf \quad J(y,u) = \frac{1}{2}\|y(T) - z_d\|_H^2 + \int_0^T h(u(t))dt \tag{1.1}$$

subject to

$$\begin{cases} \ddot{y}(t) + A(t)y(t) = Bu(t) \quad \text{a.e. } t \in (0,T), \\ y(0) = 0, \dot{y}(0) = 0, \\ u \in \mathcal{U}_{ad}, \end{cases} \tag{1.2}$$

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where $\ddot{y} = \frac{d^2y}{dt^2}$ and z_d is the desired value in H. The precise hypothesis on the system (1.1) and (1.2) will be given in the next section. Recently, many authors have studied optimal control problems of various practical systems[2,4,6,7,8,9,11]. The optimal control problem is to find existence condition of infimum cost functional of primal systems. Central to these developments is to establishing of optimality systems which characterize the optimal control. Invoking the necessary and sufficient conditions for the optimality systems, some authors constructed duality theory for various systems[4,8,9,11]. The purpose of the duality theory is to prove that the infimum cost functional of given systems is equal to supremum cost functional of dual systems. Thus we can solve the optimal control problem of given systems by finding a solution of the optimal control problem of dual systems. Chandra, Craven and Husain[5] and Bector and Husain[3] established duality results for a variational programing and a multiobjective programing, respectively, under differentiable convexity conditions. In a similar fashion, Patel[10] and Xiuhong[12] constructed duality results for the same programing under general convexity, called invex, than Chandra and Craven. On the other hand, Tanimoto[11] and Park and Lee[9] proved a duality theorem by making use of Hamiltonians for the non-well-posed distributed systems with convex and continuously Gâteaux differentiable cost functionals under an infinte-dimensional framework. Our approach is based on the infinite-dimensional version of duality theorem for the optimality systems and the dual control systems presented here take on a similar form as in [7]. However, the dual cost criterion founded in this paper is different from that of Tanimoto[11] and Park and Lee[9]. Motiviated by Barbu and Precupanu[2], we establish the dual cost problems for the systems (1.2) with more general cost criterions- nondifferentiable convex functionals- via conjugate functions and complete the duality theory. To this end, we need to invoke the necessary and sufficient conditions of the optimal control problem (1.1) and (1.2) proved in [7]. The plan of the paper is as follows. The assumptions are formulated in section 2 and the dual control systems of the primal systems (1.1) and (1.2) are contructed and their relationships is proved in section 3.

2. Preliminaries

First we explain the notations used in this paper. Let $||\cdot||_V$ and $||\cdot||_H$ denote the norms on V and H, respectively. V^* denotes the dual space of V, $<\cdot,\cdot>_{V^*,V}$ the dual pairing between V^* and V and $(\cdot,\cdot)$ the scalar product on H identified with its own dual. Assume that $V \hookrightarrow H \equiv H^* \hookrightarrow V^*$ is a Gelfand triple space with continuous, dense and compact imbedding. This implies that $<\phi,\varphi>_{V^*,V}=(\phi,\varphi)$ for $\phi \in H, \varphi \in V$ and there exists $k_1 > 0$ such that $||\phi||_H \leq k_1||\phi||_V$ for $\phi \in V$. $\rightarrow$ denotes strong convergence and $\rightharpoonup$ denotes weak convergence

Let $T > 0$ be fixed.

We will need the following assumptions concerning the data of (1.2).

(H1) $a(t;\phi,\varphi), t \in [0,T] : V \times V \rightarrow \mathbb{R}$: a bilinear form such that

(1) $a(t;\phi,\varphi) = a(t;\varphi,\phi)$ for all $\phi,\varphi \in V$, there exists $c_1 > 0$ such that

(2) $|a(t;\phi,\varphi)| \leq c_1||\phi||_V||\varphi||_V$ for all $\phi,\varphi \in V$ and there exists $\alpha > 0$ such that

(3) $a(t;\phi,\phi) \geq \alpha||\phi||_V^2$ for all $\phi \in V$.

Then we can define the operator $A(t) \in \mathcal{L}(V,V^*)$ for $t \in [0,T]$ deduced by the relation

(4) $a(t;\phi,\varphi) = \langle A(t)\phi,\varphi\rangle_{V^*,V}$ for all $\phi,\varphi \in V$.

(H2) (1) U is another real Hibert space with the inner product $(\cdot,\cdot)_U$ and the norm $||\cdot||_U$.

(2) $B : U \longrightarrow H$ is a linear continuous operator.

(3) $\mathcal{U}_{ad}$ is a nonempty closed convex subset in $L^2(0,T,U)$.

(H3) $h : U \longrightarrow \bar{\mathbb{R}}$ is convex and lowersemicontinuous. Moreover there exist $c_3 > 0$ and $c_4 \in \mathbb{R}$ such that $h(u) \geq c_3||u||_U^2 + c_4,\ \forall u \in U$.

We define a Hibert space $W(0,T)$ by

$$W(0,T) = \{y|y \in L^2(0,T;V), \dot{y} \in L^2(0,T;H), \ddot{y} \in L^2(0,T;V^*)\}$$

with the norm

$$||y||_{W(0,T)} = (||y||^2_{L^2(0,T,V)} + ||y'||^2_{L^2(0,T,H)} + ||\ddot{y}||^2_{L^2(0,T,V^*)})^{\frac{1}{2}}.$$

We consider the following optimal control problem (P) :

$$\inf \ J(y,u) = \frac{1}{2}||y(T) - z_d||_H^2 + \int_0^T h(u(t))dt$$

subject to

$$\begin{cases} \ddot{y}(t) + A(t)y(t) = Bu(t) \quad \text{a.e. } t \in (0,T), \\ y(0) = 0, \dot{y}(0) = 0, \\ u \in \mathcal{U}_{ad}. \end{cases} \tag{2.1}$$

We associate another optimal control problem to (P), which is called the dual problem of (P). In order to establish it, we introduce the following definitions and the relative results. For details, we refer the readers to [1].

Let X be a real Banach space with dual X^* and φ be a proper convex function from X to $(-\infty, \infty]$.

DEFINITION 2.1 The function φ^* on X^* defined by

$$\varphi^*(x^*) = \sup\{(x, x^*) - \varphi(x) | x \in X\}$$

is called the conjugate of φ.

LEMMA 2.1 *Let φ be a lower-semicontinuous proper convex function on X, then the conjugate φ^* is convex, lower-semicontinuous and proper on the dual space X^*.*

DEFINITION 2.2 For every $x \in X$, we denote by $\partial\varphi(x)$ the set of all $x^* \in X^*$ such that

$$\varphi(x) \le \varphi(y) + (x - y, x^*), \ \forall y \in X.$$

Such elements x^* are called subgradients of φ at x and $\partial\varphi(x)$ is called the subdifferential of φ at x.

LEMMA 2 2 *The subdifferential $\partial\varphi^*$ of the conjugate function φ^* of φ coincides with $(\partial\varphi)^{-1}$.*

We call the following optimal control problem the dual problem (D) :

$$\sup \quad K(p,y,u) = -\frac{1}{2}||y(T)||_H^2 + \frac{1}{2}||z_d||_H^2 - \int_0^T h^*(-B^*p(t))dt$$

subject to

$$\begin{cases} \ddot{p}(t) + A(t)^*p^*(t) = 0 \quad \text{a.e. } t \in (0.T), \\ p(T) = 0, \dot{p}(T) = -y(T) + z_d, \\ -B^*p(t) \in \partial h(u(t)) \ \text{ a.e. } t \in (0,T), \\ u \in \mathcal{U}_{ad}, y \in W(0,T), \end{cases} \tag{2.2}$$

where $A(t)^*, B^*$ are the adjoint operators for $A(t), B$, respectively, ∂h is the subdifferential of a proper convex lower-semicontinuous function h and h^* is the conjugate of h(see Definition 2.1 and 2.2). It should be note that the control variables of the system (2.2) are y and u.

As for the differential equation in (2.2), we restrict to only solutions which belong to $W(0,T)$. Hence if there exists no solution $(p,y,u) \in W(0,T) \times W(0,T) \times \mathcal{U}_{ad}$ in (2.2), we define the supremum of problem (D) to be $-\infty$.

3. Duality

We first prove a weak duality theorem saying that the infimum of (P) is equal to or greater than the supremum of (D).

THEOREM 3 1 *(Weak Duality) Assume that (H1)-(H3) hold and there exist feasible solutions for (P) and (D). Then the infimum of (P) is equal to or greater than the supremum of (D).*

PROOF Let $(\bar{y},\bar{u}) \in W(0,T) \times \mathcal{U}_{ad}$ and $(p^*,y^*,u^*) \in W(0,T) \times W(0,T) \times \mathcal{U}_{ad}$ be feasible solutions for (P) and (D), respectively. Then by (2.2) and the definition of conjugate h^* of h,

$$\int_0^T h^*(-B^*p^*(t))dt \geq \int_0^T (-B^*p^*(t),\bar{u}(t))_U - h(\bar{u}(t))dt. \tag{3.1}$$

Thus,

$$
\begin{aligned}
&J(\bar{y},\bar{u}) - K(p^*,y^*,u^*)\\
=&\frac{1}{2}||\bar{y}(T) - z_d||_H^2 + \int_0^T h(\bar{u}(t))dt + \frac{1}{2}||y^*(T)||_H^2\\
&-\frac{1}{2}||z_d||_H^2 + \int_0^T h^*(-B^*p^*(t))dt\\
\geq& \frac{1}{2}||\bar{y}(T)||_H^2 - (\bar{y}(T), z_d) + \frac{1}{2}||y^*(T)||_H^2 - \int_0^T (B^*p^*(t), \bar{u}(t))_U dt\\
&(\text{ by } (3.1))\\
=&\frac{1}{2}||\bar{y}(T)||_H^2 - (\bar{y}(T), z_d) + \frac{1}{2}||y^*(T)||_H^2\\
&-\int_0^T \langle p^*(t), \ddot{\bar{y}}(t) + A(t)\bar{y}(t)\rangle_{V,V^*} dt\\
&(\text{ by } (2.1))\\
=&\frac{1}{2}||\bar{y}(T)||_H^2 - (\bar{y}(T), z_d) + \frac{1}{2}||y^*(T)||_H^2\\
&-\Big[(p^*(t), \dot{\bar{y}}(t))\Big]_0^T + \Big[(\dot{p}^*(t), \bar{y}(t))\Big]_0^T\\
&-\int_0^T \langle \ddot{p}^*(t) + A(t)^*p^*(t), \bar{y}(t))\rangle_{V^*,V} dt\\
&(\text{ by integrating by parts})\\
=&\frac{1}{2}||\bar{y}(T)||_H^2 - (\bar{y}(T), z_d) + \frac{1}{2}||y^*(T)||_H^2 + (-y^*(T) + z_d, \bar{y}(T))\\
&(\text{ by } (2.1) \text{ and } (2.2))\\
=&\frac{1}{2}||\bar{y}(T) - y^*(T)||_H^2 \geq 0.
\end{aligned}
\tag{3.2}
$$

Since $(\bar{y},\bar{u})$ and (p^*,y^*,u^*) are arbitrary, (3.2) implies that $\inf(P) \geq \sup(D)$.

We prove next the duality theorem that the infimum of (P) coincides with the supremum of (D).

THEOREM 3.2 *(Strong Duality) Assume that (H1)-(H3) hold and let $(y^*,u^*) \in W(0,T) \times \mathcal{U}_{ad}$ be an optimal pair for (P). Then there exists a $p^* \in W(0,T)$ such that (p^*,y^*,u^*) attains the supremum (D). Furthermore, the infimum of (P) is equal to the supremum of (D).*

PROOF. Let $(y^*,u^*) \in W(0,T) \times \mathcal{U}_{ad}$ be an optimal pair for (P). Then there exists $p^* \in W(0,T)$ satisfying the system (2.2) for $u = u^*, y = y^*$ (see [7]). Then we have,

$$\begin{aligned}
J(y^*,u^*) =& \frac{1}{2}||y^*(T) - z_d||_H^2 + \int_0^T h(u^*(t))dt \\
=& \frac{1}{2}||y^*(T) - z_d||_H^2 + \int_0^T h(u^*(t))dt \\
&+ \int_0^T \langle \ddot{p}^*(t) + A(t)^*p^*(t), y^*(t)\rangle_{V^*,V}dt \\
&(\text{ by (2.2) }) \\
=& \frac{1}{2}||y^*(T)||_H^2 - (y^*(T), z_d) + \frac{1}{2}||z_d||_H^2 + \int_0^T h(u^*(t))dt \\
&+ \Big[(\dot{p}^*(t), y^*(t))\Big]_0^T - \Big[(p^*(t), \dot{y}^*(t))\Big]_0^T \\
&+ \int_0^T \langle p^*(t), \ddot{y}^*(t) + A(t)y^*(t)\rangle_{V,V^*}dt \\
&(\text{ by integrating by parts }) \\
=& -\frac{1}{2}||y^*(T)||_H^2 + \frac{1}{2}||z_d||_H^2 \\
&+ \int_0^T h(u^*(t)) + (B^*p^*(t), u^*(t))_U dt \\
&(\text{ see (2.1) and (2.2) }).
\end{aligned} \tag{3.3}$$

Since $-B^*p^*(t) \in \partial h(u^*(t))$ a.e. $t \in (0,T)$, by definition of ∂h,

$$\begin{aligned} &\int_0^T h(u^*(t)) - h(u(t))dt \\ &\leq \int_0^T (-B^*p^*(t), u^*(t) - u(t))_U dt, \ \ \forall u \in L^2(0,T;U). \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we have

$$\begin{aligned} J(y^*, u^*) :\leq -\frac{1}{2}||y^*(T)||_H^2 + \frac{1}{2}||z_d||_H^2 \\ + \int_0^T h(u(t)) + (B^*p^*(t), u(t))_U dt, \ \ \forall u \in L^2(0,T;U). \end{aligned} \tag{3.5}$$

Since

$$\begin{aligned} h^*(-B^*p^*(t)) &:= \sup\{(-B^*p^*(t), u)_U - h(u) | u \in U\} \\ &= -\inf\{(B^*p^*(t), u)_U + h(u) | u \in U\}, \end{aligned}$$

taking limit-infimum in (3.5) for $u \in L^2(0,T;U)$, we have

$$\begin{aligned} J(y^*, u^*) &:\leq -\frac{1}{2}||y^*(T)||_H^2 + \frac{1}{2}||z_d||_H^2 - \int_0^T h^*(-B^*p^*(t))dt \\ &= K(p^*, y^*, u^*). \end{aligned} \tag{3.6}$$

By (3.6) and Theorem 3.1, we conclude that the infimum of (P) is equal to the supremum of (D) and that (p^*, y^*, u^*) attains the supremum of (D). This completes the proof.

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