ITERATION PROCESSES WITH ERRORS FOR NONLINEAR EQUATIONS INVOLVING α-STRONGLY ACCRETIVE OPERATORS IN BANACH SPACES

Jong Soo Jung

ABSTRACT Let X be a real Banach space and $A : X \to 2^X$ be an α-strongly accretive operator. It is proved that if the duality mapping J of X satisfies Condition (I) with additional conditions, then the Ishikawa and Mann iteration processes with errors converge strongly to the unique solution of operator equation $z \in Ax$ In addition, the convergence of the Ishikawa and Mann iteration processes with errors for α-strongly pseudo-contractive operators is given

1. Introduction

Let X be a real Banach space with norm $\|\cdot\|$ whose dual space is denoted by X^*. The normalized duality mapping J from X into the family of nonempty subset of X^* is defined by

$$J(x) = \{j \in X^* : \langle x, j \rangle = \|x\|^2, \ \|j\| = \|x\|\}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is an immediate consequence of the Hahn-Banach theorem that $J(x)$ is nonempty for each $x \in X$.

We say that J satisfies *Condition* (I) if there exists a function $\Phi :$ $X \to [0, \infty)$ such that for u, $v \in X$,

$$\sup\{\|j - j^*\| : j \in J(u), \; j^* \in J(v)\} \leq \Phi(u - v).$$

This concept was introduced by Calvert and Gupta in [4, Definition 1.1]. They actually showed that that if $X = L^p(\Omega)$ with Ω a bounded subset of $\mathbb{R}^N$, then the duality mapping

$$J : L^p(\Omega) \to L^q(\Omega), \qquad \left(\frac{1}{p} + \frac{1}{q} = 1 \text{ and } 2 \leq p < \infty\right),$$

defined by $J(u) = |u|^{p-1}\text{sgn }u\|u\|_p^{2-p}$, satisfies Condition (I) as a Lipschizian mapping. In this case, it follows that $\lim_{u \to 0} \Phi(u) = 0$. Condition (I) was also used by Morales [16] and Torrejón [21].

An operator $A : D(A) \subset X \to 2^X$ with domain $D(A)$ and range $R(A)$ is said to be *k-accretive* ($k \in \mathbb{R}$) if for each x, $y \in D(A)$ there exists $j \in J(x - y)$ such that

$$(1.1) \qquad\qquad \langle u - v, j \rangle \geq k\|x - y\|^2$$

for all $u \in Ax$ and $v \in Ay$. For $k > 0$ in inequality (1), we say that A is *strongly accretive*, while for $k = 0$, A is simply called *accretive*. In addition, if the range of $I + \lambda A$ is precisely X for all $\lambda > 0$, then A is said to be *m-accretive*. Let $\alpha : [0, \infty) \to [0, \infty)$ be a function which is continuous and strictly increasing with $\alpha(0) = 0$ and $\alpha(r) > 0$ for $r > 0$. An operator $A : D(A) \subset X \to 2^X$ is called α-*strongly accretive* if for each x, $y \in D(A)$ there exists $j \in J(x - y)$ such that

$$\langle u - v, j \rangle \geq \alpha(\|x - y\|)\|x - y\|$$

for all $u \in Ax$ and $v \in Ay$.

Along with the family of k-accretive mappings, we find a family of operators intimately related to it which is known as *k-pseudo-contractive* (see [14]). This latter family is formed by mappings written as $I - A$ where I is the identity and A is k-accretive. In the single-valued case,

an operator T is said to be *k-pseudo-contractive* if for each x, $y \in D(T)$ there exists $j \in J(x - y)$ such that

$$\langle Tx - Ty, j \rangle \leq k \|x - y\|^2.$$

Once again if $k < 1$, T is called *strongly pseudo-contractive*, while if $k = 1$, T is called *pseudo-contractive*. We also say that T is α-*strongly pseudo-contractive* if $I - T$ is α-strongly accretive (see [15], [17]).

Incidentally, these operators were introduced by Browder [2], while the notion of accretive operators was independently introduced by Browder [2] and Kato [10]. In the case $X = H$ is a Hilbert space, one of the earliest problems in the theory of accretive operators was to solve the equation $z = x + Ax$ for a given $z \in H$ and A accretive operator (see for instance [3, 8, 13]). In [2], Browder actually proved that if A is locally Lipschitzian and accretive with $D(A) = X$, then A is m-accretive. In particular, for any $z \in X$, the equation $z = x + Ax$ has a unique solution. This result was later generalized by Martin [12] to continuous accretive operators and extended by Morales [15] to the multi-valued case, respectively.

Recently, the theory of single (multi)-valued accretive and single (multi)-valued strongly accretive operators in connection with the Ishikawa and Mann iteration process have been studied by many authors in the attempt of approximating fixed points of some nonlinear operator equations in Banach spaces (see [5], [6], [20], [21], [22]). Some further extensions of these iterative methods by adding an error term have also been explored (see [11], [22]).

The main purpose of this paper is to study the convergences of the so-called Ishikawa and Mann iteration processes with errors to approximate the unique solution of the operator equation of the α-strongly accretive operator A under the condition that the duality mapping J satisfies Condition (I). As a consequence of main result, we obtain the convergence of the Ishikawa and Mann iteration processes with errors for α-strongly pseudo-contractive operators. We should mention that since it is not known whether the duality mapping J actually satisfies a global condition like (I) even in Banach spaces with uniformly convex dual spaces, our results may be, in a sense, independent of the previous related results.

2. Preliminaries and Lemmas

We recall the Ishikawa and Mann iteration processes with errors.

Firstly, Liu [11] introduced the iteration processes which he called Ishikawa and Mann iteration processes "with errors" for nonlinear strongly accretive mappings as follows:

(A) For K a nonempty subset of a real Banach space X and a mapping $T : K \to X$, the sequence $\{x_n\}$ defined by $x_0 \in K$,

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n + u_n \\ \quad y_n = (1 - \beta_n)x_n + \beta_n T x_n + v_n, \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are some real sequences in $[0,1]$ satisfying appropriate conditions $\sum_{n=0}^{\infty} \|u_n\| < \infty$, $\sum_{n=0}^{\infty} \|v_n\| < \infty$, is called the *Ishikawa iteration process with errors.*

(B) With K, X and S as in part (A), the sequence $\{x_n\}$ defined by $x_0 \in K$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n + u_n, \quad n \geq 0,$$

where $\{\alpha_n\}$ and $\{u_n\}$ satisfy conditions as in part (A), is called *Mann iteration process with errors.*

However, the conditions $\sum_{n=0}^{\infty} \|u_n\| < \infty$, $\sum_{n=0}^{\infty} \|v_n\| < \infty$ on error terms introduced in (A) and (B) imply that the errors tend to zero and this is incompatible with the randomness of errors. Recently, Xu [21] improved the Ishikawa and Mann iteration processes with errors of Liu [11] under the randomness of errors as follows:

(C) Let K be a nonempty convex subset of X and $T : K \to K$ a mapping. For any given $x_0 \in K$, the sequence $\{x_n\}$ defined by

$$\begin{cases} x_{n+1} = c_n x_n + d_n T y_n + r_n u_n \\ \quad y_n = c'_n x_n + d'_n T x_n + r'_n v_n, \quad n \geq 0, \end{cases}$$

where $\{c_n\}, \{d_n\}, \{r_n\}, \{c'_n\}, \{d'_n\}, \{r'_n\}$ are some real sequences in $[0,1]$ such that $c_n + d_n + r_n = 1 = c'_n + d'_n + r'_n$ and $\{u_n\}, \{v_n\}$ are

bounded sequences in K for all integers $n \geq 0$, is called the *Ishikawa iteration process with errors.*

(D) In particular, if $c'_n = r'_n = 0$ for all $n \geq 0$, the $\{x_n\}$ defined by

$$x_0 \in K, \quad x_{n+1} = c_n x_n + d_n T x_n + r_n u_n, \quad n \geq 0,$$

is called *Mann iteration process with errors.*

But, if the operator T has bounded range and one imposes the condition that $\sum r_n < \infty$ and $\sum r'_n < \infty$, the iteration processes (C) and (D) with $\alpha_n := d_n + r_n$ and $\beta_n := d'_n + r'_n$ reduce the type of processes (A) and (B). So there is no loss of generality in studying the iteration process (A) and (B) instead of the processes (C) and (D).

In the sequel, we need the following lemmas for the proof of our main results. The first lemma is actually Lemma 1 of Petryshyn [19]. Also Asplund [1] proved a general result for single-valued duality mappings, which can be used to derive this lemma.

LEMMA 1 *Let X be a real Banach space and let J be the normalized duality mapping. Then for any given x, $y \in X$, we have*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j\rangle$$

for all $j \in J(x + y)$.

PROOF. Let x, $y \in X$ and $j \in J(x + y)$. Then

$$
\begin{aligned}
\|x + y\|^2 &= \langle x + y, j\rangle \\
&= \langle x, j\rangle + \langle y, j\rangle \\
&\leq \frac{1}{2}(\|x\|^2 + \|j\|^2) + \langle y, j\rangle.
\end{aligned}
$$

Therefore

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j\rangle.$$

LEMMA 2 ([11]). *Let* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *be three nonnegative real sequences satisfying*

$$a_{n+1} \le (1 - t_n)a_n + b_n + c_n, \quad n \ge n_0,$$

where n_0 *is some positive integer,* $0 \le t_n < 1$, $\sum_{n=0}^{\infty} t_n = \infty$, $b_n = o(t_n)$ *and* $\sum_{n=0}^{\infty} c_n < \infty$. *Then* $\lim_{n \to \infty} a_n = 0$.

3. Main results

We now begin with the first main result of this paper.

THEOREM 1 *Let* X *be a Banach space whose duality mapping* J *satisfies Condition (I) with a function* $\Phi : X \to [0, \infty)$. *Let* $A : X \to 2^X$ *be* α-*strongly accretive. Suppose that the equation* $z \in Ax$ *has a solution for each* $z \in X$. *Let* $\{u_n\}$ *and* $\{v_n\}$ *be two sequences in* X *and let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be two real sequences in* $[0, 1]$ *satisfying*
  (i) $\sum_{n=0}^{\infty} \|u_n\| < \infty$, $\lim_{n \to \infty} \|v_n\| = 0$,
  (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n \to \infty} \alpha_n = 0$,
  (iii) $\lim_{n \to \infty} \beta_n = 0$.
For an arbitrary initial value of x_0 *in* X, *let* $\{x_n\}$ *be the Ishikawa type iterative sequence generated by*

$$(3.1) \qquad \begin{cases} x_{n+1} \in (1 - \alpha_n)x_n + \alpha_n(z + (I - A)y_n) + u_n \\ y_n \in (1 - \beta_n)x_n + \beta_n(z + (I - A)x_n) + v_n, \end{cases}$$

in case that there exist bounded selections $\{w_n\}$ *and* $\{z_n\}$ *with* $w_n \in (I - A)y_n$ *and* $z_n \in (I - A)x_n$. *If one of the following conditions hold:*
  (1) $\lim_{n \to \infty} \|w_n - z_{n+1}\| = 0$;
  (2) $\lim_{n \to \infty} \Phi(p_n) = 0$ *for the sequence* $\{p_n\}$ *with* $\lim_{n \to \infty} p_n = 0$;
  (3) $\sum_{n=0}^{\infty} \alpha_n \Phi(p_n) < \infty$ *for the sequence* $\{p_n\}$ *with* $\lim_{n \to \infty} p_n = 0$,
then $\{x_n\}$ *converges strongly to the unique solution of the equation* $z \in Ax$.

PROOF. Let x^* denote the solution of the equation $z \in Ax$. The uniqueness of a solution of the equation follows from the α-strong accretivity condition of A. For all x, $y \in X$, $u \in z + (I - A)x$, and

$v \in z + (I - A)y$, there exist $\overline{u} \in Ax$, $\overline{v} \in Ay$ such that $u = z + x - \overline{u}$, $v = z + y - \overline{v}$, and hence, since A is α-strongly accretive, we have

$$(3.2) \qquad \begin{aligned} \langle u - v, j_{x,y} \rangle &= \langle z + x - \overline{u} - (z + y - \overline{v}), j_{x,y} \rangle \\ &\leq \|x - y\|^2 - \alpha(\|x - y\|)\|x - y\|, \end{aligned}$$

where $j_{x,y} \in J(x - y)$.

Now due to the choice of w_n and z_n, equation (3.2) can be re-written as

$$(3.3) \qquad \begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n w_n' + u_n \\ y_n = (1 - \beta_n)x_n + \beta_n z_n' + v_n \end{cases}$$

for all $n \geq 0$, where $w_n' = z + w_n$ and $z_n' = z + z_n$. Since the sequences $\{w_n\}$ and $\{z_n\}$ are bounded, we may denote by

$$d = \sup_{n \geq 0} \|w_n' - x^*\| + \sup_{n \geq 0} \|z_n' - x^*\| + \|x_0 - x^*\|$$

and

$$M = d + \sum_{n=0}^{\infty} \|u_n\|.$$

This implies that

$$\begin{aligned} \|x_1 - x^*\| &= \|(1 - \alpha_0)(x_0 - x^*) + \alpha_0(w_0' - x^*) + u_0\| \\ &\leq (1 - \alpha_0)\|x_0 - x^*\| + \alpha_0\|w_0' - x^*\| + \|u_0\| \\ &\leq d + \|u_0\| \leq M. \end{aligned}$$

By induction, we obtain

$$\|x_n - x^*\| \leq M$$

and

$$\|y_n - x^*\| = \|(1 - \beta_n)(x_n - x^*) + \beta_n(z_n' - x^*) + v_n\| \leq M + \|v_n\|$$

for all $n \geq 0$.

(1) Let $\lim_{n \to \infty} \|w_n - z_{n+1}\| = 0$. Then it follows that

$$(3.4) \qquad\qquad r_n = \|w_n' - z_{n+1}'\| \to 0$$

as $n \to \infty$. From Lemma 1 and (3.3), we have

$$
(3.5)
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|(1 - \alpha_n)(x_n - x^*) + \alpha_n(w_n' - x^*) + u_n\|^2 \\
&\leq \|(1 - \alpha_n)(x_n - x^*)\|^2 + 2\alpha_n\langle w_n' - x^* + u_n, j_{x_{n+1}, x^*}\rangle \\
&\leq (1 - \alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\langle w_n' - x^*, j_{x_{n+1}, x^*}\rangle + 2M\|u_n\| \\
&\leq (1 - \alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\langle w_n' - z_{n+1}', j_{x_{n+1}, x^*} - j_{y_n, x^*}\rangle \\
&\quad + 2\alpha_n\langle w_n' - z_{n+1}', j_{y_n, x^*}\rangle + 2\alpha_n\langle z_{n+1}' - x^*, j_{x_{n+1}, x^*}\rangle \\
&\quad + 2M\|u_n\| \\
&\leq (1 - \alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\|w_n' - z_{n+1}'\|\Phi(x_{n+1} - y_n) \\
&\quad + 2\alpha_n\|w_n' - z_{n+1}'\|\|y_n - x^*\| + 2\alpha_n\|x_{n+1} - x^*\|^2 \\
&\quad - 2\alpha_n\alpha(\|x_{n+1} - x^*\|)\|x_{n+1} - x^*\| + 2M\|u_n\| \\
&\leq (1 - \alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\|x_{n+1} - x^*\|^2 \\
&\quad - 2\alpha_n\alpha(\|x_{n+1} - x^*\|)\|x_{n+1} - x^*\| + \alpha_n r_n L + 2M\|u_n\|
\end{aligned}
$$

for all $n \geq 0$, where $\Phi(x_{n+1} - y_n) + M + \|v_n\| \leq L < \infty$. Since $\alpha_n \to 0$ as $n \to \infty$, there exists n_1 such that $1/2 < 1 - 2\alpha_n < 1$ for all $n \geq n_1$. It follows from (3.5) that

$$
(3.6)
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\leq \frac{(1 - \alpha_n)^2}{1 - 2\alpha_n}\|x_n - x^*\|^2 + \frac{2L\alpha_n r_n}{1 - 2\alpha_n} \\
&\quad - \frac{2\alpha_n}{1 - 2\alpha_n}\alpha(\|x_{n+1} - x^*\|)\|x_{n+1} - x^*\| + \frac{2M}{1 - 2\alpha_n}\|u_n\| \\
&\leq \|x_n - x^*\|^2 + \frac{\alpha_n}{1 - 2\alpha_n}(M^2\alpha_n + 2Lr_n) \\
&\quad - \frac{2\alpha_n}{1 - 2\alpha_n}\alpha(\|x_{n+1} - x^*\|)\|x_{n+1} - x^*\| + \frac{2M}{1 - 2\alpha_n}\|u_n\|
\end{aligned}
$$

for all $n \geq n_1$. Let $\delta = \inf\{\|x_n - x^*\| : n \geq 0\}$. Now we prove that $\delta = 0$. Suppose that $\delta > 0$. Then $\|x_n - x^*\| \geq \delta > 0$ for all $n \geq 0$. By the strictly increasing property of α, we have $\alpha(\|x_{n+1} - x^*\|) \geq \delta > 0$ for all $n \geq 0$. Since $M^2\alpha_n + 2Lr_n \to 0$ as $n \to \infty$, there exists a positive integer $n_2 \geq n_1$ such that

$$(3.7) \qquad M^2\alpha_n + 2Lr_n < \alpha(\delta)\delta$$

for all $n \geq n_2$. It follows from (3.6) and (3.7) that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 + \frac{\alpha_n}{1 - 2\alpha_n}\alpha(\delta)\delta \\
(3.8) \qquad & \qquad - \frac{2\alpha_n}{1 - 2\alpha_n}\alpha(\delta)\delta + 4M\|u_n\| \\
&\leq \|x_n - x^*\|^2 - \frac{\alpha_n}{1 - 2\alpha_n}\alpha(\delta)\delta + 4M\|u_n\|
\end{aligned}
$$

for all $n \geq 0$. This implies

$$(3.9) \qquad \alpha(\delta)\delta \sum_{n=n_2}^{\infty} \alpha_n < \|x_{n_2} - x^*\|^2 + 4M \sum_{n=n_2}^{\infty} \|u_n\| < \infty,$$

which contradicts the assumption that $\sum_{n=0}^{\infty} \alpha_n = \infty$. Thus $\delta = 0$, and hence there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to 0$ as $j \to \infty$. Since $\{w'_n\}$ is bounded in (3.3), $\alpha_n \to 0$ and $\|u_n\| \to 0$ as $n \to \infty$, we have

$$x_{n_j+1} = (1 - \alpha_{n_j})x_{n_j} + \alpha_{n_j}w'_{n_j} + u_{n_j} \to x^*$$

as $j \to \infty$. By induction, we can prove that

$$x_{n_j+k} \to x^*$$

as $j \to \infty$ for all $k = 1, 2, \cdots$. Therefore we have $x_n \to x^*$ as $n \to \infty$.

Suppose that condition (2) or (3) holds. In this case, we follow the approaches of Jung and Morales [9]. For the sake of completeness, we

include its proof under the condition (2) (Similarly, we can also derive the same conclusion under the condition (3)).

Define a sequence $\{r_n\}$ by

$$r_n = |\langle w'_n - x^*, j_{x_{n+1}, x^*} - j_{y_n, x^*} \rangle|$$

for $n \geq 0$. Since $\{x_n\}$, $\{y_n\}$, $\{w'_n\}$ and $\{z'_n\}$ are bounded, by (3.3) and the conditions (i) – (iii), we have

$$x_{n+1} - y_n = (\beta_n - \alpha_n)x_n + \alpha_n w'_n - \beta'_n + u_n - v_n \to 0$$

as $n \to \infty$. By Condition (I) of the duality mapping J with a function Φ satisfying the condition (2), we have

$$
\begin{aligned}
r_n &= |\langle w'_n - x^*, j_{x_{n+1}, x^*} - j_{y_n, x^*} \rangle| \\
&\leq \|w'_n - x^*\| \sup\{\|j_{x_{n+1}, x^*} - j_{y_n, x^*}\| : j_{x_{n+1}, x^*} \in J(x_{n+1} - x^*), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j_{y_n, x^*} \in J(y_n - x^*)\} \\
&\leq M\Phi(x_{n+1} - y_n) \to 0
\end{aligned}
$$

as $n \to \infty$. On the other hand, using Lemma 1 and (3.3), we have

$$
\begin{aligned}
(3.10) \quad &\|y_n - x^*\|^2 \\
&= \|(1 - \beta_n)x_n + \beta_n z'_n + v_n - x^*\|^2 \\
&\leq (1 - \beta_n)^2 \|x_n - x^*\|^2 + 2\beta_n \langle z'_n - x^*, j_{y_n, x^*} \rangle + 2\langle v_n, j_{y_n, x^*} \rangle \\
&\leq (1 - \beta_n)^2 \|x_n - x^*\|^2 + 2(\beta_n \|z'_n - x^*\| + \|v_n\|)\|y_n - x^*\| \\
&\leq \|x_n - x^*\|^2 + 2(\beta_n M + \|v_n\|)(M + \|v_n\|)
\end{aligned}
$$

for all $n \geq 0$. We also have

$$
\begin{aligned}
(3.11) \quad &\|x_{n+1} - x^*\|^2 \\
&= \|(1 - \alpha_n)(x_n - x^*) + \alpha_n(w'_n - x^*) + u_n\|^2 \\
&\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle w'_n - x^*, j_{x_{n+1}, x^*} \rangle \\
&\quad + 2\langle u_n, j_{x_{n+1}, x^*} \rangle \\
&\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle w'_n - x^*, j_{y_n, x^*} \rangle \\
&\quad + 2\alpha_n \langle w'_n - x^*, j_{x_{n+1}, x^*} - j_{y_n, x^*} \rangle + 2\|u_n\|M
\end{aligned}
$$

for all $n \geq 0$, where $j_{x_{n+1},x^*} \in J(x_{n+1} - x^*)$ and $j_{y_n,x^*} \in J(y_n - x^*)$. Thus, using (3.2) , (3.3), (3.10) and (3.11), we obtain

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 \\
&\quad + 2\alpha_n (\|y_n - x^*\|^2 - \alpha(\|y_n - x^*\|)\|y_n - x^*\|) \\
&\quad + 2\alpha_n r_n + 2\|u_n\| M \\[4pt]
&\leq (1 + \alpha_n^2)\|x_n - x^*\|^2 - 2\alpha_n \frac{\alpha(\|y_n - x^*\|)}{\|y_n - x^*\|} \|x_n - x^*\|^2 \\
&\quad + 4\alpha_n(\beta_n M + \|v_n\|)(M + \|u_n\|) + 2\alpha_n r_n + 2\|u_n\| M
\end{aligned}
$$

(3.12)

for all $n \geq 0$.

If $\inf_{n \geq 0} \|y_n - x^*\| > 0$, then there exists $k > 0$ such that

$$
k < \frac{\alpha(\|y_n - x^*\|)}{\|y_n - x^*\|}
$$

for all $n \geq 0$. Hence we have from (3.12)

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\leq (1 + \alpha_n^2 - 2k\alpha_n)\|x_n - x^*\|^2 + 4\alpha_n(\beta_n M + \|v_n\|)(M + \|v_n\|) \\
&\quad + 2\alpha_n r_n + 2\|u_n\| M
\end{aligned}
$$

for all $n \geq 0$. Since $\alpha_n \to 0$ as $n \to \infty$ by (ii), there exists a positive integer n_0 such that $\alpha_n - k \leq 0$ and so $\alpha_n^2 \leq k\alpha_n$ for all $n \geq n_0$. Thus we obtain

(3.13) $\|x_{n+1} - x^*\|^2 \leq (1 - k\alpha_n)\|x_n - x^*\|^2 + b_n + c_n$

for all $n \geq n_0$, where $b_n = \alpha_n[4(\beta_n M + \|v_n\|)(M + \|v_n\|) + 2r_n]$ and $c_n = 2\|u_n\| M$. Let $a_n = \|x_n - x^*\|^2$, $t_n = k\alpha_n$. Then the inequality (3.13) reduces to

$$
a_{n+1} \leq (1 - t_n)a_n + b_n + c_n
$$

for all $n \geq n_0$. By the conditions (i) – (iii), it is easy to see that $\sum_{n=0}^{\infty} t_n = \infty$, $b_n = o(t_n)$, and $\sum_{n=0}^{\infty} c_n < \infty$. It follows from Lemma 2 that $\{x_n\}$ converges strongly to x^* as $n \to \infty$.

Suppose now that $\inf_{n \geq 0} \|y_n - x^*\| = 0$. Then there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that $\lim_{j \to \infty} \|y_{n_j} - x^*\| = 0$. For a given $\varepsilon > 0$ we may choose a positive integer j_0 such that

$$\|x_{j_0} - x^*\| < \frac{\varepsilon}{\sqrt{2}}, \quad \|x_{j_0+1} - y_{j_0}\| < \frac{\varepsilon}{2}, \quad s_j < \varepsilon\alpha(\frac{\varepsilon}{2}) \quad \text{and} \quad \|u_j\| < \frac{\varepsilon^2}{4M}$$

for all $j \geq j_0$, where $s_j = \alpha_j M^2 + 4(\beta_j + \|v_j\|)(M + \|v_j\|) + 2r_j$. Suppose that $\|x_{j_0+1} - x^*\| \geq \varepsilon$. Then

$$\|y_{j_0} - x^*\| \geq \|x_{j_0+1} - x^*\| - \|x_{j_0+1} - y_{j_0}\|$$
$$> \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2},$$

and so

$$\alpha(\|y_{j_0} - x^*\|)\|y_{j_0} - x^*\| > \alpha(\frac{\varepsilon}{2})\frac{\varepsilon}{2}.$$

Since we can also derive

$$\|x_{n+1} - x^*\|^2$$
$$\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \|y_n - x^*\|^2$$
$$\quad - 2\alpha_n\alpha(\|y_n - x^*\|)\|y_n - x^*\| + 2\alpha_n r_n + 2\|u_n\|M$$
$$= (1 + \alpha_n^2)\|x_n - x^*\|^2 - 2\alpha_n\alpha(\|y_n - x^*\|)\|y_n - x^*\|$$
$$\quad + 4\alpha_n(\beta_n M + \|v_n\|)(M + \|v_n\|) + 2\alpha_n r_n + 2\|u_n\|M$$
$$\leq \|x_n - x^*\|^2 - 2\alpha_n\alpha(\|y_n - x^*\|)\|y_n - x^*\|$$
$$\quad + \alpha_n[\alpha_n M^2 + 4(\beta_n M + \|v_n\|)(M + \|v_n\|) + 2r_n] + 2\|u_n\|M$$

for all $n \geq 0$, it follows that

$$\|x_{j_0+1} - x^*\|^2$$
$$\leq \|x_{j_0} - x^*\|^2 - 2\alpha_{j_0}\alpha(\|y_{j_0} - x^*\|)\|y_{j_0} - x^*\| + \alpha_{j_0}s_{j_0} + 2\|u_{j_0}\|M$$
$$< \frac{\varepsilon^2}{2} - 2\alpha_{j_0}\alpha(\frac{\varepsilon}{2})\frac{\varepsilon}{2} + \alpha_{j_0}\varepsilon\alpha(\frac{\varepsilon}{2}) + \frac{\varepsilon^2}{2} = \varepsilon^2,$$

which is a contradiction. Therefore $\|x_{j_0+1} - x^*\| < \varepsilon$ and inductively we have

$$\|x_n - x^*\| < \varepsilon$$

for all $n \geq j_0$. Therefore the sequence $\{x_n\}$ converges strongly to the unique solution of the equation $z \in Ax$.

COROLLARY 1. *Let X, A, J and Φ be as in Theorem 1. Suppose that the equation $z \in Ax$ has a solution for each $z \in X$. Let $\{u_n\}$ be in sequences in X and $\{\alpha_n\}$ be sequences in $[0, 1]$ satisfying the conditions (i) and (ii). For an arbitrary initial value of x_0 in X, let $\{x_n\}$ be the Mann type iterative sequence generated by*

$$x_{n+1} \in (1 - \alpha_n)x_n + \alpha_n(z + (I - A)x_n) + u_n,$$

in case that there exists a bounded selection $\{w_n\}$ with $w_n \in (I - A)x_n$. If one of the following conditions hold:

    (1) $\lim_{n \to \infty} \|w_n - w_{n+1}\| = 0$;

    (2) $\lim_{n \to \infty} \Phi(p_n) = 0$ *for the sequence $\{p_n\}$ with $\lim_{n \to \infty} p_n = 0$;*

    (3) $\sum_{n=0}^{\infty} \alpha_n \Phi(p_n) < \infty$ *for the sequence $\{p_n\}$ with $\lim_{n \to \infty} p_n = 0$,*

then $\{x_n\}$ converges strongly to the unique solution of the equation $z \in Ax$.

REMARK 1 If $A : X \to X$ is a continuous strongly accretive operator, then the existence of a solution of the equation $z = Az$ follows from Martin [11] (see also Morales [14]). Hence we can establish the corresponding results from Theorem 1 and Corollary 1 with $\alpha(t) = kt$ for $k > 0$.

Now we give the convergence of Ishikawa iterative sequence for α-strongly pseudo-contractive operator.

THEOREM 2 *Let X be a Banach space whose duality mapping J satisfies Condition (I) with a function $\Phi : X \to [0, \infty)$. Let $T : X \to 2^X$ be α-strongly pseudo-contractive with a fixed point x^* in X. Let $\{u_n\}, \{v_n\}$ be in sequences in X and $\{\alpha_n\}, \{\beta_n\}$ be sequences in $[0, 1]$ satisfying the conditions (i) – (iii). For an arbitrary initial value of x_0 in X, let $\{x_n\}$ be the Ishikawa type iterative sequence generated by*

$$(3.14) \qquad \begin{cases} x_{n+1} \in (1 - \alpha_n)x_n + \alpha_n T y_n + u_n \\ \quad y_n \in (1 - \beta_n)x_n + \beta_n T x_n + v_n, \end{cases}$$

in case that there exist bounded selections $\{w_n\}$ and $\{z_n\}$ with $w_n \in T y_n$ and $z_n \in T x_n$. If one of the following conditions hold:

(1) $\lim_{n \to \infty} \|w_n - z_{n+1}\| = 0$;

(2) $\lim_{n \to \infty} \Phi(p_n) = 0$ *for the sequence $\{p_n\}$ with $\lim_{n \to \infty} p_n = 0$;*

(3) $\sum_{n=0}^{\infty} \alpha_n \Phi(p_n) < \infty$ *for the sequence $\{p_n\}$ with $\lim_{n \to \infty} p_n = 0$,*

then $\{x_n\}$ converges strongly to the unique fixed point of T.

PROOF   The uniqueness of fixed point of T follows from α-strong pseudo-contractivity condition of T. For any $x, y \in X, u \in Tx, v \in Ty$ there exist $\overline{u} \in (I - T)x, \overline{v} \in (I - T)y$ such that $\overline{u} = x - u, \overline{v} = y - v$. Since $(I - T)$ is α-strongly accretive, we have

$$\begin{aligned} \langle u - v, j_{x,y} \rangle &= \langle x - \overline{u} - (y - \overline{v}), j_{x,y} \rangle \\ &\leq \|x - y\|^2 - \alpha(\|x - y\|)\|x - y\|, \end{aligned}$$

where $j_{x,y} \in J(x - y)$. As in proof of Theorem 1, due to the choice of w_n and z_n, equation (3.14) can be re-written as

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n w_n + u_n \\ \quad y_n = (1 - \beta_n)x_n + \beta_n z_n + v_n \end{cases}$$

for all $n \geq 0$. We can also denote by

$$d = \sup_{n \geq 0} \|w_n - x^*\| + \sup_{n \geq 0} \|z_n - x^*\| + \|x_0 - x^*\|$$

and

$$M = d + \sum_{n=0}^{\infty} \|u_n\|.$$

Now the result follows exactly as in the proof of Theorem 1. This completes the proof.

COROLLARY 2 *Let X, T, J and Φ be as in Theorem 2. Let $\{u_n\}$ be in sequences in X and $\{\alpha_n\}$ be sequences in $[0,1]$ satisfying the conditions (i) and (ii). For an arbitrary initial value x_0 in X, let $\{x_n\}$ be the Mann type iterative sequence generated by*

$$x_{n+1} \in (1 - \alpha_n)x_n + \alpha_n T x_n + u_n,$$

in case that there exist a bounded selection $\{w_n\}$ with $w_n \in Tx_n$. If one of the following conditions hold:
  (1) $\lim_{n \to \infty} \|w_n - w_{n+1}\| = 0$;
  (2) $\lim_{n \to \infty} \Phi(p_n) = 0$ *for the sequence $\{p_n\}$ with $\lim_{n \to \infty} p_n = 0$;*
  (3) $\sum_{n=0}^{\infty} \alpha_n \Phi(p_n) < \infty$ *for the sequence $\{p_n\}$ with $\lim_{n \to \infty} p_n = 0$,*
then $\{x_n\}$ converges strongly to the unique fixed point of T.

REMARK 2 In case that $T : X \to X$ is a continuous strongly pseudo-contractive operator, the existence of a fixed point of T follows from Deimling [7]. Hence we can also derive the corresponding results from Theorem 2 and Corollary 2 with $\alpha(t) = rt$ for $r \in (0,1)$.

REMARK 3 (i) In contrast to the previous results ([9], [11], [18], [22], [23], [24]), we do not assume that the underlying space X is uniformly smooth. In fact, since it is not known whether the duality mapping J actually satisfies a condition like (I) and the the condition (2) can be replaced by $\Phi(u_n) = \lambda \|u_n\|$ for some $\lambda > 0$ even in uniformly smooth Banach spaces, our results may be independent of the previous related results.

(ii) If $A : X \to CB(X)$ is a uniformly continuous α-strongly accretive operator in Theorem 1, then we can obtain the condition (1), where $CB(X)$ is the family of all bounded closed subsets of X.

(iii) Along with the additional conditions $\sum_{n=0}^{\infty} \alpha_n \beta_n < \infty$ and $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$ in Theorem 1 and 2, using Lemma 1 in [20], we can obtain the same conclusions under only the condition (3).

REFERENCES

[1] E Asplund, *Positivity of duality mappings*, Bull Amer. Math Soc. **73** (1967), 200–203

[2] F.E Browder, *Nonlinear mappings of nonexpansive and accretive type in Banach spaces*, Bull. Amer Math Soc. **73** (1967), 875–882

[3] R E Bruck Jr , *The iterative solution of the equation $y \in x+Tx$ for a monotone operator T in Hilbert space*, Bull Amer Math Soc **79** (1973), 1258–1261

[4] B D Calvert and C.P Gupta, *Nonlinear elliptic boundary value problem in L^p-spaces and sums of ranges of accretive operators*, Nonlinear Anal **2** (1978), 1–26

[5] S S Chang, Y J. Cho, B S Lee, J S Jung and S M. Kang, *Iterative approximations of fixed points and solutions for strongly accretive and strongly pseudo-contractive mappings in Banach spaces*, J Math. Anal Appl **224** (1998), 149–165

[6] C E Chidume, *Global iteration schemes for strongly pseudo-contractive maps*, Proc. Amer Math Soc. **126** (1998), 2641–2649

[7] K Deimling, *Zeros of accretive operators*, Manuscript Math. **13** (1974), 365–374

[8] W G Doston, *An iterative process for nonlinear monotone nonexpansive operators in Hilbert space*, Math Comp **32** (1978), 223–225.

[9] J S. Jung and C H. Morales, *The Mann process for perturbed m-accretive operators in Banach spaces*, Nonlinear Anal **46** (2001), 231–243

[10] T Kato, *Nonlinear semigroups and evolution equations*, J Math. Soc. Japan **19** (1967), 508–520

[11] L S Liu, *Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces*, J Math Anal. Appl **194** (1995), 114–125.

[12] R H Martin, Jr , *A global existence theorem for autonomous differential equations in Banach spaces*, Proc Amer. Math Soc **26** (1970), 307–314

[13] G.J. Minty, *Monotone (nonlinear) operators in Hilbert spaces*, Duke Math J **29** (1962), 341–346

[14] C Morales, *Pseudo-contractive mappings and the Leray-Schauder boundary condition*, Comment. Math Univ Carolin **20** (1979), 745–756

[15] C Morales, *Surjectivity theorems for multi-valued mappings of accretive type*, Comment Math Univ Carolin **26** (1985), 397–413.

[16] C. Morales, *On the range of sums of accretive of continuous operators in Banach spaces*, Nonlinear Anal. **19** (1992), 1–9

[17] C Morales and C Chidume, *Convergence of the steepest descent method for accretive operators*, in press, Proc Amer Math. Soc

[18] M O Osilike, *Iterative solution of nonlinear equations of the Φ-strongly accretive type*, J Math Anal Appl **200** (1996), 259–271

[19] W V Petryshyn, *A characterization of strictly convexity of Banach spaces and other uses of duality mappings*, J Funct. Anal **6** (1970), 282–291

[20] K K Tan and H K Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J Math. Anal Appl **178** (1993), 301–308

[21] R M Torrejón, *Remarks on nonlinear functional equations*, Nonlinear Anal **6** (1982), 197–207

[22] Y Xu, *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations*, J. Math Anal. Appl **224** (1998), 91–101

[23] L C Zeng, *Iterative approximation of solutions to nonlinear equations of strongly accretive operators in Banach spaces*, Nonlinear Anal **31** (1998), 589–598

[24] H Zhou, *Iterative solution of nonlinear equation involving strongly accretive operators without the Lipschitz assumption*, J Math. Anal Appl **213** (1997), 296–307

Department of Mathematics
Dong-A University
Busan 604-714, Korea
E-mail: jungjs@mail.donga.ac.kr