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SOME GEOMETRIC PROPERTIES AND EXHAUSTION OF JOHN DISK

. KIWON KIM

1. Introduction

Suppose that D is a domain in the complex plane $\mathbb{C}$. Let $D^* = \mathbb{C}\backslash\overline{D}$ be the exterior of D in $\mathbb{C}$ and let $B(z,r) = \{\zeta : |\zeta - z| < r\}$ for $z \in \mathbb{C}$ and $r > 0$.

A simply connected bounded domain $D \subset \mathbb{C}$ is said to be a *c-John disk* if there exist a point $z_0 \in D$ and a constant $c \geq 1$ such that each point $z_1 \in D$ can be joined to z_0 by an arc γ in D satisfying

$$\ell(\gamma(z_1, z)) \leq c \operatorname{dist}(z, \partial D)$$

for each $z \in \gamma$, where $\ell(\gamma(z_1, z))$ is the euclidean length of the subarc of γ with endpoints z_1, z. We call z_0 a *John center*, c a *John constant* and γ a *c-John arc*. We say that D is *John* if it is c-John disk for some c. A bounded domain $D \subset \mathbb{C}$ is John if and only if each pair of points $z_1, z_2 \in D$ can be joined by an arc γ which satisfies

$$\min_{j=1,2} \ell(\gamma(z_j, z)) \leq c \operatorname{dist}(z, \partial D) \tag{1.1}$$

for all $z \in \gamma$. We call γ a *double c-cone arc*. This definition can be used to define the unbounded John disks $D \subset \mathbb{C}$ as well [NV, 2.26].

In section 2, we present two sets of conditions which describe John disks D in terms of their geometry and in terms of the function f which maps the unit disk $\mathbb{B}$ conformally onto D, [GHM], [NV], [P]. From the

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first set we obtain a charaterization for John disks which is a Jordan domain in $\overline{\mathbb{C}}$. From the second set we obtain a fact that a simply connected bounded John disk can be exhausted by Jordan domains

$$D_j = f(\mathbb{B}(0, r_j)), \qquad 0 < r_j < 1$$

which are also c-John disks. Thus for the simplicity we restrict our attention to the case where D is a Jordan domain in $\mathbb{C}$. However the results in this paper also hold for bounded simply connected domains in $\mathbb{C}$.

2. Exterior c-LLC and exhaustion of John disk

The first set of geometric conditions for John disk which is a Jordan domain in $\overline{\mathbb{C}}$ are as follows [GHM] and [NV]:

PROPOSITION 2.1. *If D is a Jordan domain in $\overline{\mathbb{C}}$, then the following conditions are equivalent, where the constants in each condition need not be the same:*

(1) *D is a c-John disk.*
(2) *For every $z \in \mathbb{C}$ and $r > 0$, any two points in $D \setminus \overline{\mathbb{B}}(z, r)$ can be joined by a continuum in $D \setminus \overline{\mathbb{B}}(z, \frac{r}{c})$, i.e., D is $c - LLC_2$.*
(3) *For every $z \in \mathbb{C}$ and $r > 0$, any two points in $\{\overline{\mathbb{C}} \setminus D\} \cap \overline{\mathbb{B}}(z, r)$ can be joined by a continuum in $\{\overline{\mathbb{C}} \setminus D\} \cap \overline{\mathbb{B}}(z, cr)$, i.e., $\overline{\mathbb{C}} \setminus D$ is $c - LLC_1$.*

THEOREM 2.2. *If D is a Jordan domain in $\overline{\mathbb{C}}$, then the condition (3) in proposition 2.1 is equivalent to D^* is $c - LLC_1$, where the constants depend on each other.*

To prove theorem 2.2, we need a lemma.

LEMMA 2.3 *[W, Proposition 4.1] Let E be a set in $\overline{\mathbb{C}}$. Then*

(1) *if ∂E is linearly locally connected, then so is $\overline{E}$;*
(2) *if E is linearly locally connected, then so is $\overline{E}$.*

PROOF OF THE THEOREM 2 2 If D^* is $c - LLC_1$, then the proof of Lemma 2.3 (2) in [W] gives (3) in proposition 2.1. Suppose next that (3) proposition 2.1 holds, fix $z \in \mathbb{C}$ and $r > 0$, and choose $z_1, z_2 \in D^* \cap \overline{\mathbb{B}}(z, r)$. Since $\overline{\mathbb{C}} \setminus D$ is $c - LLC_1$, there exists an arc γ joining z_1 and z_2 in $\{\overline{\mathbb{C}} \setminus D\} \cap \overline{\mathbb{B}}(z, cr)$. Now since D^* is a Jordan domain, there exists an imbedding $h : \overline{\mathbb{C}} \setminus D \to D^*$ such that $h(z) = z$ for $z \in \overline{\mathbb{C}} \setminus D$ with $d(z, \partial D^*) > \epsilon$, where $0 < \epsilon < \min(d(z_j, \partial D), r)$, $j = 1, 2$. Then for each $z \in \overline{\mathbb{C}} \setminus D$, $|h(z) - z| \leq \epsilon$ and hence $h(\gamma)$ is an arc joining z_1 and z_2 in $D^* \cap \overline{\mathbb{B}}(z, cr + r)$. Then D^* is $c' - LLC_1$ with $c' = c + 1$.

Next let f map the unit disk $\mathbb{B}$ conformally onto the bounded Jordan domain D in $\mathbb{C}$. Then by the Caratheodory extension theorem $f : \mathbb{B} \to D$ admits an extension to a homeomorphism $f : \overline{\mathbb{B}} \to \overline{D}$. The second set of conditions in terms of the conformal mapping $f . \mathbb{B} \to D$ are as follows [P]:

PROPOSITION 2 4 [P] *Suppose that D is a bounded Jordan domain in $\mathbb{C}$, that $w_0 \in D$ and that f is as above with $w_0 = f(0)$. Then the following conditions are equivalent, where the constants c and $\delta > 0$ need not be the same in every condition:*

(1) *D is a c-John disk.*

(2)
$$\frac{|f'(rz)|}{|f'(\rho z)|} \leq c\Big(\frac{1-r}{1-\rho}\Big)^{\delta - 1}$$
for $|z| = 1, \quad 0 \leq \rho \leq r < 1;$

If $\operatorname{dia}(D) \leq q\, d(w_0, \partial D)$, *then the various constants c and δ depend only on q and on each other.*

THEOREM 2 5 *Suppose that D is a bounded simply connected domain in $\mathbb{C}$ and that f maps the unit disk $\mathbb{B}$ conformally onto D with $f(0) = w_0$ for some $w_0 \in D$. If D is a c−John disk, then it can be exhausted by Jordan domains*

$$D_j = f(\mathbb{B}(0, r_j)), \qquad 0 < r_j < 1$$

which are also c-John disks.

PROOF. Let $h : \mathbb{B} \to \mathbb{B}$ be a conformal mapping definded by $h(z) = r_j z$ for $0 < r_j < 1$ and let $g = f \circ h$. Then g maps the unit disk $\mathbb{B}$ conformally onto D with $g(0) = w_0$. Obviously D_j is a Jordan domain. To show that D_j is a c-John disk, it suffices to show by proposition 2.4 (2) that

$$\frac{|g'(rz)|}{|g'(\rho z)|} \le c_1 \Big(\frac{1-r}{1-\rho}\Big)^{\delta-1} \tag{2.6}$$

for some constants c_1 and δ which are the same constants to make D c-John disk, where $|z| = 1, \quad 0 \le \rho \le r < 1$. Since $g' = r_j f'$, $0 \le \rho r_j \le r r_j < r_j < 1$ and since proposition 2.4 (2) also holds for bounded simply connected John disks (see [P, Theorem 1 (4)]), we obtain

$$\frac{|g'(rz)|}{|g'(\rho z)|} = \frac{|f'(rr_j z)|}{|f'(\rho r_j z)|} \le c_1 \Big(\frac{1-rr_j}{1-\rho r_j}\Big)^{\delta-1} \tag{2.7}$$

for some constants c_1 and δ which depend only on c. Since

$$\Big(\frac{1-rt}{1-\rho t}\Big)^{\delta-1}$$

is a nondecreasing function for $0 < t < 1$, we obtain by (2.7)

$$\frac{|g'(rz)|}{|g'(\rho z)|} \le c_1 \Big(\frac{1-rr_j}{1-\rho r_j}\Big)^{\delta-1} \le c_1 \Big(\frac{1-r}{1-\rho}\Big)^{\delta-1}.$$

This yields inequality (2.6).

Thus proposition 2.1 and theorem 2.2 also hold for bounded simply connected domains in $\mathbb{C}$.

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Department of Mathematics
Silla University
Pusan 617-736, Korea
E-mail: kwkim@silla.ac.kr