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CERTAIN CLASSES OF SERIES IDENTITIES INVOLVING BINOMIAL COEFFICIENTS

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1. Introduction and Preliminaries

Lots of formulas for series involving binomial coefficients have been developed in many ways.

Vowe and Seiffert [7] showed the following series identity:

$$\begin{aligned}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{1}{2^k(n+k+1)} \\ = \frac{2^n(n-1)!\, n!}{(2n)!}-\frac{1}{n\cdot 2^n} \qquad (n\in \mathbf{N}:=\{1,\ 2,\ 3,\ \cdots\})\end{aligned} \tag{1.1}$$

by evaluating the Eulerian integral

$$\int_0^1 \left(1-\frac{t}{2}\right)^{n-1} t^n\, dt. \tag{1.2}$$

Srivastava [6] evaluated (1.1) with the aid of a summation formula involved in the hypergeometric series ${}_2F_1$ which is due to Kummer [3] (see also Rainville [5, p. 69, Exercise 3]):

$$\begin{aligned}{}_2F_1\big(a,\ 1-a\ ;\ b\ ;\ \frac{1}{2}\big) = \frac{\Gamma(\frac{b}{2})\Gamma(\frac{b+1}{2})}{\Gamma(\frac{b+a}{2})\Gamma(\frac{b-a+1}{2})} \\ (b\neq 0,\ -1,\ -2,\ \cdots)\end{aligned} \tag{1.3}$$

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where the so-called hypergeometric series ${}_2F_1$ (also denoted by F) is defined by

$$
{}_2F_1(a,\ b\ ;\ c\ ;\ z) = {}_2F_1\left[\begin{array}{c} a,\ b\ ; \\ c\ ; \end{array} z\right] := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{1.4}
$$

where a, b and c are arbitrary complex constants and $(\alpha)_n$ denotes the Pochhammer symbol (or the generalized factorial, since $(1)_n = n!$) defined by

$$
(\alpha)_n := \begin{cases} 1 & (n = 0) \\ \alpha(\alpha+1)\cdots(\alpha+n-1) & (n \in \mathbf{N}). \end{cases} \tag{1.5}
$$

Very recently Choi, Zörnig and Rathie [2] obtained following formulas similar to (1.1):

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{k}{2^k (n+k)(n+k+1)} = \frac{2^{n-1}(n!)^2}{(2n+1)!} - \frac{1}{2^{n+1}}; \tag{1.6}
$$

$$
\begin{aligned} &\sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} \frac{k}{2^k (n+k)(n+k+1)} \\ &\qquad = \frac{3 \cdot 2^n \cdot (n!)^2}{(n-1)(2n)!} - \frac{n+2}{(n-1)2^{n-1}}, \end{aligned} \tag{1.7}
$$

by using summation formula contiguous to (1.3) (see Lavoie *et al.* [4]) and contiguous function relations (see Cho *et al.* [1]).

In fact, we are trying to give more general series identities including (1.6) and (1.7) as special cases by making use of known (presumably new) summation formulas for ${}_2F_1$.

From the fundamental functional relation of the Gamma function Γ, $\Gamma(z+1) = z\Gamma(z)$, we have

$$
\begin{aligned} (\alpha)_n &= \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}; \\ \Gamma(n+1) &= n! \qquad (n \in \mathbf{N} \cup \{0\}), \end{aligned} \tag{1.8}
$$

where Γ is the well-known Gamma function whose Weierstrass canonical product form is given by

$$\{\Gamma(z)\}^{-1} = ze^{\gamma z} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}, \tag{1.9}$$

γ being the Euler-Mascheroni's constant defined by

$$\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) \cong 0.577215664\cdots . \tag{1.10}$$

From definition (1.5) and (1.8), we can easily deduce the following formula:

$$(\alpha)_{n-k} = \frac{(-1)^k (\alpha)_n}{(1-\alpha-n)_k}, \tag{1.11}$$

which, for $\alpha = 1$, yields immediately

$$(-n)_k = \begin{cases} \dfrac{(-1)^k n!}{(n-k)!} & \text{if } 0 \le k \le n \\ 0 & \text{if } k > n. \end{cases} \tag{1.12}$$

The binomial coefficient is defined and written in the following form:

$$\begin{aligned} \binom{\alpha}{k} &:= \frac{\alpha(\alpha-1)\ \cdots\ (\alpha-k+1)}{k!} \\ &= \frac{\Gamma(\alpha+1)}{k!\ \Gamma(\alpha-k+1)} = \frac{(-1)^k(-\alpha)_k}{k!}, \end{aligned} \tag{1.13}$$

from which it follows that

$$\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} = \frac{(-1)^n}{(1-\alpha)_n} \quad (\alpha \ne 0,\ \pm 1,\ \pm 2,\ \cdots). \tag{1.14}$$

From (1.5), it is not difficult to show that

$$(\alpha)_{2n} = 2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n \qquad (n \in \mathbf{N} \cup \{0\}), \tag{1.15}$$

which, in view of (1.8), follows also from Legendre's duplication formula for the Gamma function

$$\Gamma\left(\frac{1}{2}\right)\Gamma(2z) = 2^{2z-1}\,\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \tag{1.16}$$

in which $\Gamma\left(\frac{1}{2}\right)$ is evaluated as $\sqrt{\pi}$.

In this paper we are aiming at providing the following formulas similar to (1.1), (1.6) and (1.7) by making use of known summation formulas for ${}_2F_1$ and some of their contiguous relations.

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{2^k} = \frac{1}{2^n}; \tag{1.17}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{n+k-1}{2^k} = -\frac{1}{2^n}; \tag{1.18}$$

(1.19)

$$\sum_{k=0}^{n-4}(-1)^k\binom{n-4}{k}\frac{k}{2^k(n+k)(n+k+1)} = \frac{5n\cdot 2^n\cdot\left((n-1)!\right)^2}{(n-3)(n-2)(2n-2)!} - \frac{(n^2+5n-4)\cdot 2^{3-n}}{(n-3)(n-2)};$$

(1.20)

$$\sum_{k=0}^{n-5}(-1)^k\binom{n-5}{k}\frac{1}{2^k(n+k)} = \frac{1}{2n-5}\left\{\frac{2^{n-1}(n-5)!\,(n-1)!}{(2n-6)!} - \frac{(4n^3-38n^2+110n-100)2^{5-n}}{(n-5)(n-4)(n-3)}\right\};$$

(1.21)
$$\sum_{k=0}^{n-5}(-1)^k\binom{n-5}{k}\frac{1}{2^k(n+k)(n+k+1)}$$
$$=\frac{(n-5)!}{2^{n-3}}\left\{\frac{2(n^2+n-4)!}{(n-2)!}-\frac{(n-1)(2n-3)2^{2n-1}(n-1)!}{(2n-2)!}\right\};$$

(1.22)
$$\sum_{k=0}^{n-3}(-1)^k\binom{n-3}{k}\frac{1}{2^k(n+k)(n+k+1)(n+k+2)}$$
$$=\frac{(n+1)\cdot 2^{n-1}(n-3)!\,(n-1)!}{(2n-1)!}-\frac{2^{2-n}}{(n-1)(n-2)};$$

(1.23)
$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(n+k-3)(n+k-4)}{2^k}=\frac{6-n}{2^{n-1}};$$

(1.24)
$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(n+k-6)(n+k-5)(n+k-4)}{2^k}=\frac{3(n-5)}{2^{n-3}};$$

(1.25)
$$\sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}\frac{n+k+2l-2}{2^k\prod_{j=1}^{2l}(n+k+j-1)}$$
$$=\frac{(n-2)!}{2^{n-2}}\left\{\frac{2^{2n-2}(n+l-1)!}{(k-1)!\,(2n+2l-3)!}-\frac{l!}{(2l-1)!\,(n+l-2)!}\right\}$$
$$(l,\ n\in \mathbf{N}).$$

We also point out relevant connections of the series identities presented here with those given elsewhere.

2. Series Identities and Proofs

For simplicity in printing, we use the notations

$$\begin{aligned} F &= {}_2F_1(a,\ b\ ;\ c\ ;\ z), \\ F(a+) &= F(a+1,\ b\ ;\ c\ ;\ z), \\ F(a-) &= F(a-1,\ b\ ;\ c\ ;\ z), \\ F(a+,b-) &= F(a+1,\ b-1\ ;\ c\ ;\ z) \end{aligned}$$

together with similar notations $F(b+)$, $F(a-,c+)$ and so on.

Lavoie *et al.* [4] obtained the following summation formula contiguous to (1.3):

(2.1)

$$\begin{aligned} &{}_2F_1\left(a,\ 2-a\ ;\ b\ ;\ \frac{1}{2}\right) \qquad (b \neq 0,\ -1,\ -2,\ \cdots) \\ &= \frac{\Gamma(\frac{1}{2})\Gamma(b)}{2^{b-2}\cdot(1-a)}\left\{\frac{1}{\Gamma(\frac{b-a}{2})\Gamma(\frac{b+a-1}{2})} - \frac{1}{\Gamma(\frac{b-a+1}{2})\Gamma(\frac{b+a-1}{2})}\right\} \end{aligned}$$

Now, let $S_{\lambda,\mu}$ be the sum in the left side of (2.1).

It is not difficult to express $S_{\lambda,\mu}$ as in the following form:

$$\text{(2.2)} \quad S_{\lambda,\mu} := \sum_{k=0}^{\infty}(-1)^k\binom{\lambda-2}{k}\frac{\binom{\lambda+k-1}{k}}{\binom{\mu+k-1}{k}2^k} \qquad (\mu \neq 0,\ -1,\ -2,\ \cdots).$$

Since

$$\text{(2.3)} \qquad \binom{n-2}{k} = 0 \qquad (k \geq n-1;\ \ n \in \mathbf{N}),$$

the sum in (2.2) will terminate at $k = n-2$ in the special case when $\lambda = n \in \mathbf{N}$.

Some further consequences of the general result (2.2) are worthy of note. Indeed, for every non-negative integer l, we obtain following identity by using $S_{\lambda,\lambda+2l}$

(2.4)

$$\sum_{k=0}^{\infty}(-1)^k\binom{\lambda-2}{k}\frac{1}{2^k\prod_{j=1}^{2l}(\lambda+k+j-1)}$$
$$=\frac{(\lambda-1)!}{2^{\lambda-2}(1-\lambda)}\left\{\frac{2^{2\lambda-2}(\lambda+l-1)!}{(l-1)!\,(2\lambda+2l-2)!}-\frac{l!}{(2l)!\,(\lambda+l-2)!}\right\}$$
$$(\lambda,\neq 0,\ -1,\ -2,\ \cdots).$$

Next, letting $\mu=\lambda+2l+1$ with l replaced by $l-1$, we find that, for $l\in\ \mathbf{N}$,

(2.5)

$$\sum_{k=0}^{\infty}(-1)^k\binom{\lambda-2}{k}\frac{1}{2^k\prod_{j=0}^{2l-2}(\lambda+k+j)}$$
$$=\frac{(\lambda-1)!}{2^{\lambda-1}(1-\lambda)}\left\{\frac{(l-1)!}{(2l-2)!\,(\lambda+l-2)!}-\frac{2^{2\lambda-2}(\lambda+l-2)!}{(l-1)!\,(2\lambda+2l-4)!}\right\}.$$

Upon subtracting (2.4) from (2.5) with the help of (2.3), arrives immediately at our desired identity (1.25).

Setting $l=1$ and $l=2$ in (1.25), we obtain the following special cases:

$$\text{(2.6)}\quad \sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}\frac{1}{2^k(n+k+1)}=\frac{2^{n+1}(n!)^2}{(n-1)(2n)!}-\frac{1}{(n-1)2^{n-2}};$$

(2.7)

$$\sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}\frac{1}{2^k(n+k)(n+k+1)(n+k+3)}$$
$$=\frac{(n-2)!}{2^{n-2}}\left\{\frac{2^{2n-2}(n+1)!}{(2n+1)!}-\frac{1}{3\cdot n!}\right\}$$

and so on.

Recall another summation formula for ${}_2F_1$ contiguous to (1.3) (see Lavoie *et al.* [4]):

(2.8)

$$
\begin{aligned}
&{}_2F_1(a,\ 4-a\ ;\ b\ ;\ \frac{1}{2}) \qquad (b \neq 0,\ -1,\ -2,\ \cdots) \\
&= \frac{\Gamma(\frac{1}{2})\Gamma(b)\Gamma(1-a)}{2^{b-4}\Gamma(4-a)} \left\{ \frac{a-2b+3}{\Gamma\big(\frac{b-a+1}{2}\big)\Gamma\big(\frac{b+a-4}{2}\big)} + \frac{a+2b-7}{\Gamma\big(\frac{b-a}{2}\big)\Gamma\big(\frac{b+a-3}{2}\big)} \right\}.
\end{aligned}
$$

Now let $T_{\lambda,\mu}$ be the sum in the left side of (2.8) and put specific values for $\lambda = n$ and $\mu = n,\ n+1$ and $n+2$.

Applying the same procedure in the proof of (1.25) to $T_{n,n}$ and $T_{n,n+1} - T_{n,n+2}$, we obtain (1.17) and (1.19).

For the proof of (1.24), recall a contiguous function relation (see Cho *et al.* [1]):

$$
(2.9) \qquad (B-1)F = (B-C)F(B-) + (C-1)F(B-,\ C-).
$$

If we replace $A,\ B,\ C$ and z in (2.9) by $a,\ -a-3,\ b$ and $\frac{1}{2}$ respectively, we obtain

$$
(2.10) \qquad
\begin{aligned}
F\big(a,\ -a-3\ ;\ b\ ;\ \frac{1}{2}\big) &= \frac{a+b+3}{a+4} F\big(a,\ -a-4\ ;\ b\ ;\ \frac{1}{2}\big) \\
&\quad + \frac{1-b}{a+4} F\big(a,\ -a-4\ ;\ b-1\ ;\ \frac{1}{2}\big).
\end{aligned}
$$

Lavoie *et al.* [4] obtained the following summation formula contiguous to (1.3): For $b \neq 0,\ -1,\ -2,\ \cdots$,

(2.11)

$$
\begin{aligned}
&{}_2F_1\big(a,\ -a-4\ ;\ b\ ;\ \frac{1}{2}\big) = \frac{\Gamma(\frac{1}{2})\Gamma(b)}{2^{b+4}} \\
&\times \left\{ \frac{4b^2-2ab-a^2+8b-7a}{\Gamma\big(\frac{b}{2}-\frac{a}{2}+\frac{1}{2}\big)\Gamma\big(\frac{b}{2}+\frac{a}{2}+2\big)} + \frac{4b^2+2ab-a^2+16b-a+12}{\Gamma\big(\frac{b}{2}-\frac{a}{2}\big)\Gamma\big(\frac{b}{2}+\frac{a}{2}+\frac{5}{2}\big)} \right\}.
\end{aligned}
$$

Finally, setting (2.11) in (2.10) and letting $b = a-3$, yields

$$
(2.12) \qquad
\begin{aligned}
&F\big(a,\ -a-3\ ;\ a-3\ ;\ \frac{1}{2}\big) \\
&= \frac{15a^2 \cdot \Gamma(a-2)}{(a+4)\cdot 2^{a+2}\Gamma(a+1)} - \frac{3(a^2-23a+32)\cdot\Gamma(a-3)}{(a+4)\cdot 2^{a+2}\Gamma(a)}
\end{aligned}
$$

which, in view of (1.13), can be written in the equivalent form:
(2.13)

$$\sum_{k=0}^{\infty}(-1)^k\binom{a+3}{k}\frac{(a+k-3)(a+k-2)(a+k-1)}{(a-3)(a-2)(a-1)2^k} = \frac{3}{(a-1)(a-3)2^a}.$$

If we set $a = n \in \mathbf{N}$ in (2.13) and consider (2.3), we immediately reach at the identity (1.24) by letting $n + 3 = n'$ and dropping the prime on n.

Recalling contiguous function relation (see Cho *et al.* [1])

$$(A-1)(1-z)F = (A+B-C-1)F(A-) + (C-B)F(A-,\ B-),$$

and setting $A = a$, $B = 3 - a$, $C = b$ and $z = \frac{1}{2}$, we obtain

$$F\left[a,\ 3-a\ ;\ b\ ;\ \tfrac{1}{2}\right] = \frac{2(2-b)}{a-1}F\left(a-1,\ 3-a\ ;\ b\ ;\ \tfrac{1}{2}\right) + \frac{2(a+b-3)}{a-1}F\left(a-1,\ 2-a\ ;\ b\ ;\ \tfrac{1}{2}\right), \tag{2.14}$$

which, for $b = a + 3$ and applying (1.3) and (2.1), yields our desired identity (1.22) by considering (2.3).

Similarly, other identities can be proved by using ${}_2F_1$ formulas (see [4, p. 297-298]) and contiguous function relations (see [1]).

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