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A NOTE ON D.G. NEAR-RING GROUPS

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1. Introduction

In this paper, we will examine some properties of D.G. near-ring groups and faithful representations of D.G. near-rings. A (left) near-ring R is an algebraic system $(R, +, \cdot)$ with two binary operations $+$ and $\cdot$ such that $(R, +)$ is a group (not necessarily abelian) with neutral element 0, $(R, \cdot)$ is a semigroup and $a(b+c) = ab + ac$ for all a, b, c in R. If R has a unity 1, then R is called *unitary*. A near-ring R with the extra axiom $0a = 0$ for all $a \in R$ is said to be *zero symmetric*. An element d in R is called *distributive* if $(a+b)d = ad + bd$ for all a and b in R.

An *ideal* of R is a subset I of R such that (i) $(I, +)$ is a normal subgroup of $(R, +)$, (ii) $a(I + b) - ab \subset I$ for all $a, b \in R$, (iii) $(I + a)b - ab \subset I$ for all $a, b \in R$. If I satisfies (i) and (ii) then it is called a *left ideal* of R. If I satisfies (i) and (iii) then it is called a *right ideal* of R.

On the other hand, a *(two-sided) R-subgroup* of R is a subset H of R such that (i) $(H, +)$ is a subgroup of $(R, +)$, (ii) $RH \subset H$ and (iii) $HR \subset H$. If H satisfies (i) and (ii) then it is called a *left R-subgroup* of R. If H satisfies (i) and (iii) then it is called a *right R-subgroup* of R

Let $(G, +)$ be a group (not necessarily abelian). In the set

$$M(G) := \{f \mid f : G \longrightarrow G\}$$

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of all the self maps of G, if we define the sum $f+g$ of any two mappings f, g in $M(G)$ by the rule $x(f+g) = xf + xg$ for all $x \in G$ and the product $f \cdot g$ by the rule $x(f \cdot g) = (xf)g$ for all $x \in G$, then $(M(G), +, \cdot)$ becomes a near-ring. It is called the *self map near-ring* of the group G. Also, if we define the set

$$M_0(G) := \{f \in M(G) \mid 0f = 0\},$$

then $(M_0(G), +, \cdot)$ is a zero symmetric near-ring.

Let R and S be two near-rings. Then a mapping θ from R to S is called a *near-ring homomorphism* if (i) $(a+b)\theta = a\theta + b\theta$, (ii) $(ab)\theta = a\theta b\theta$. We can replace homomorphism by momomorphism, epimorphism, isomorphism, endomorphism and automorphism, if these terms have their usual meanings as for rings ([1]).

Let R be any near-ring and G an additive group. Then G is called an *R-group* if there exists a near-ring homomorphism

$$\theta : (R, +, \cdot) \longrightarrow (M(G), +, \cdot).$$

Such a homomorphism θ is called a *representation* of R on G, we write that xr (right scalar multiplication in R) for $x(r\theta)$ for all $x \in G$ and $r \in R$. If R is unitary, then R-group G is called *unitary*. Thus an R-group is an additive group G satisfying (i) $x(a+b) = xa + xb$, (ii) $x(ab) = (xa)b$ and (iii) $x1 = x$ (if R has a unity 1), for all $x \in G$ and $a, b \in R$. Evidently, every near-ring R can be given the structure of an R-group (unitary if R is unitary) by right multiplication in R. Moreover, every group G has a natural $M(G)$-group structure, from the representation of $M(G)$ on G given by applying the $f \in M(G)$ to the $x \in G$ as a scalar multiplication xf.

A representation θ of R on G is called $faithful$ if $Ker\theta = \{0\}$. In this case, we say that G is called a *faithful R-group*.

For an R-group G, a subgroup T of G such that $TR \subset T$ is called an *R-subgroup* of G, and an *R-ideal* of G is a normal subgroup N of G such that $(N+x)a - xa \subset N$ for all $x \in G$, $a \in R$.

A near-ring R is called *distributively generated* (briefly, *D.G.*) by S if $(R, +) = gp < S >$ where S is a semigroup of distributive elements in R (this is motivated by the set of all distributive elements of R is multiplicatively closed and contain the unity of R if it exists), and $gp < S >$ is a group generated by S, we denote it by (R, S). On the other hand, the set of all distributive elements of $M(G)$ are obviously the semigroup $End(G)$ of all endomorphisns of the group G under composition. Here we denote that $E(G)$ is the D.G. near-ring generated by $End(G)$, that is, $E(G)$ is D.G. subnear-ring of $(M_0(G), +, \cdot)$ generated by $End(G)$. It is said to be that $E(G)$ is the *endomorphism near-ring* of the group G.

Let (R, S) and (T, U) be D.G. near-rings. Then a near-ring homomorphism

$$\theta : (R, S) \longrightarrow (T, U)$$

is called a *D.G. near-ring homomorphism* if $S\theta \subseteq U$. Note that a semigroup homomorphism $\theta : S \longrightarrow U$ is a D.G. near-ring homomorphism if it is a group homomorphism from $(R, +)$ to $(T, +)$ (C. G. Lyons and J. D. P. Meldrum [2], [3]).

Let R be a near-ring and let G be an R-group. If there exists x in G such that $G = xR$, that is, $G = \{xr \mid r \in R\}$, then G is called a *monogenic R-group* and the element x is called a *generator* of G (J. D. P. Meldrum [5], and G. Pilz [6]).

For the remainder concepts and results on near-rings, we refer to J. D. P. Meldrum [5], and G. Pilz [6].

2. Some Properties of D.G. Near-Rings (R, S)-Groups

There is a module like concept as follows: Let (R, S) be a D.G. near-ring. Then an additive group G is called a *D.G. (R, S)-group* if there exists a D.G. near-ring homomorphism

$$\theta : (R, S) \longrightarrow (E(G), End(G))$$

such that $S\theta \subseteq End(G)$. If we write that xr instead of $x(r\theta)$ for all $x \in G$ and $r \in R$, then a D.G. (R, S)-group is an additive group G satisfying the following conditions:

$$x(rs) = (xr)s$$

and

$$x(r+s) = xr + xs,$$

for all $x \in G$ and all $r, s \in R$,

$$(x+y)s = xs + ys,$$

for all $x, y \in G$ and all $s \in S$.

Such a homomorphism θ is called a *D.G. representation* of (R, S). This D.G. representation is said to be *faithful* if $Ker\theta = \{0\}$. In this case, we say that G is called a *faithful D.G. (R, S)-group*.

EXAMPLE 2.1 *If R is a distributive near-ring with unity 1, then R is a ring (See [6, 1.107]). Furthermore, if R is a distributive near-ring with unity 1, then every (R, R)-group is a unitary R-module.*

PROOF Let G be an (R, R)-group. Since G is unitary, $x(2) = x(1+1) = x + x$, for all $x \in G$. Thus we have that

$$x + y + x + y = (x+y)(2) = x(2) + y(2) = x + x + y + y,$$

for all $x, y \in G$. This implies that $(G, +)$ is abelian. Since $R = S$, the set of all distributive elements, $(x+y)r = xr + yr$, for all $x, y \in G$ and all $r \in R$. Hence G becomes a unitary R-module. □

LEMMA 2 2 ([4]) *Let (R, S) be a D.G. near-ring. Then all R-subgroups and all R-homomorphic images of a (R, S)-group are also (R, S)-groups.*

Let G be an R-group and K, K_1 and K_2 be subsets of G. Define

$$(K_1 : K_2) := \{a \in R; K_2 a \subset K_1\}.$$

We abbreviate that for $x \in G$

$$(\{x\} : K_2) =: (x : K_2).$$

Similarly for $(K_1 : x)$. $(0 : K)$ is called the *annihilator* of K, denoted it by $A(K)$. We say that G is a *faithful R-group* or that R *acts faithfully* on G if $A(G) = \{0\}$, that is, $(0 : G) = \{0\}$.

Also, we see that from the previous concepts to elementwise, a subgroup H of G such that $xa \in H$ for all $x \in H, a \in R$, is an *R-subgroup* of G, and an *R-ideal* of G is a normal subgroup N of G such that

$$(x+g)a - ga \in N$$

for all $g \in G, x \in N$ *and* $a \in R$ (J. D. P. Meldrum [5]).

LEMMA 2.3 . *Let G be an R-group and K_1 and K_2 subsets of G. Then we have the following conditions:*

(1) *If K_1 is a normal subgroup of G, then $(K_1 : K_2)$ is a normal subgroup of a near-ring R.*
(2) *If K_1 is an R-subgroup of G, then $(K_1 : K_2)$ is an R-subgroup of R as an R-group.*
(3) *If K_1 is an ideal of G and K_2 is an R-subgroup of G, then $(K_1 : K_2)$ is a two-sided ideal of R.*

PROOF (1) and (2) are proved by J. D. P. Meldrum [5]. Now, we prove only (3): Using the condition (1), $(K_1 : K_2)$ is a normal subgroup of R. Let $a \in (K_1 : K_2)$ and $r \in R$. Then

$$K_2(ra) = (K_2r)a \subset K_2a \subset K_1,$$

so that $ra \in (K_1 : K_2)$. Whence $(K_1 : K_2)$ is a left ideal of R.

Next, let $r_1, r_2 \in R$ and $a \in (K_1 : K_2)$. Then

$$k\{(a+r_1)r_2 - r_1r_2\} = (ka + kr_1)r_2 - kr_1r_2 \in K_1$$

for all $k \in K_2$, since $K_2a \subset K_1$ and K_1 is an ideal of G. Thus $(K_1 : K_2)$ is a right ideal of R. Therefore $(K_1 : K_2)$ is a two-sided ideal of R. □

COROLLARY 2.4 ([5]). *Let R be a near-ring and G an R-group.*

(1) *For any $x \in G$, $(0 : x)$ is a right ideal of R.*
(2) *For any R-subgroup K of G, $(0 : K)$ is a two-sided ideal of R.*
(3) *For any subset K of G, $(0 : K) = \bigcap_{x \in K} (0 : x)$.*

PROPOSITION 2 5. *Let R be a near-ring and G an R-group. Then we have the following conditions:*

(1) *$A(G)$ is a two-sided ideal of R. Moreover G is a faithful $R/A(G)$-group.*

(2) *For any $x \in G$, we get $xR \cong R/(0:x)$ as R-groups.*

PROOF (1) By Corollary 2.4 and Lemma 2.3, $A(G)$ is a two-sided ideal of R.

We now make G an $R/A(G)$-group by defining, for $x \in R, r+A(G) \in R/A(G)$, the action $x(r+A(G)) = xr$. If $r + A(G) = r' + A(G)$, then $-r' + r \in A(G)$ hence $x(-r'+r) = 0$ for all x in G, that is to say, $xr = xr'$. This tells us that

$$x(r + A(G)) = xr = xr' = x(r' + A(G));$$

thus the action of $R/A(G)$ on G has been shown to be well defined. The verification of the structure of an $R/A(G)$-group is a routine triviality. Finally, to see that G is a faithful $R/A(G)$-group, we note that if $x(r+A(G)) = 0$ for all $x \in G$, then by the definition of $R/A(G)$-group structure, we have $xr = 0$. Hence $r \in A(G)$. This says that only the zero element of $R/A(G)$ annihilates all of G. Thus G is a faithful $R/A(G)$-group.

(2) For any $x \in G$, clearly xR is an R-subgroup of G. The map $\phi : R \longrightarrow xR$ defined by $\phi(r) = xr$ is an R-ephimorphism, so that from the isomorphism theorem, since the kernel of ϕ is $(0:x)$, we deduce that $xR \cong R/(0:x)$ as R-groups. □

PROPOSITION 2.6. *If R is a near-ring and G an R-group, then $R/A(G)$ is isomorphic to a subnear-ring of $M(G)$.*

PROOF. Let $a \in R$. We define $\tau_a : G \longrightarrow G$ by $x\tau_a = xa$ for each $x \in G$. Then τ_a is in $M(G)$. Consider the mapping $\phi : R \longrightarrow M(G)$ defined by $\phi(a) = \tau_a$. Then obviously, we see that

$$\phi(a+b) = \phi(a) + \phi(b) \text{ and } \phi(ab) = \phi(a)\phi(b),$$

that is, ϕ is a near-ring homomorphism from R to $M(G)$.

Next, we must show that $Ker\phi = A(G)$: Indeed, if $a \in Ker\phi$, then $\tau_a = 0$, which implies that $Ga = G\tau_a = 0$, that is, $a \in A(G)$. On the other hand, if $a \in A(G)$, then by the definition of $A(G)$, $Ga = 0$ hence $0 = \tau_a = \phi(a)$, this implies that $a \in Ker\phi$. Therefore from the first isomorphism theorem on R-groups, the image of R is a near-ring isomorphic to $R/A(G)$. Consequently, $R/A(G)$ is isomorphic to a subnear-ring of $M(G)$. □

Thus we can obtain the following important statement as ring theory.

COROLLARY 2.7 *If G is a faithful R-group, then R is embedded in $M(G)$.*

PROPOSITION 2.8 *If (R, S) is a D.G. near-ring, then every monogenic R-group is an (R, S)-group.*

PROOF Let G be a monogenic R-group with x as a generator. Then the map $\phi : r \longmapsto xr$ is an R-epimorphism from R to G as R-groups. We see that

$$G \cong R/A(x),$$

where $A(x) = (0 : x) = Ker\phi$. From Lemma 2.2, we obtain that G is an (R, S)-group □

PROPOSITION 2.9 *Let (R, S) be a D.G. near-ring and $(G, +)$ an abelian group. If G is a faithful (R, S)-group, then R is a ring.*

PROOF. Let $x \in G$ and $r, s \in R$. Then, since $(G, +)$ is abelian,

$$x(r + s) = xr + xs = xs + xr = x(s + r).$$

Thus we get that $x\{(r + s) - (s + r)\} = 0$ for all $x \in G$, that is, $(r + s) - (s + r) \in Ker\theta = (0 : G) = A(G)$, where $\theta : R \longrightarrow M(G)$ is a representation of R on G. Since G is faithful (R, S)-group, that is, θ is faithful, $Ker\theta = (0 : G) = \{0\}$. Hence for all $r, s \in R$, $r + s = s + r$. Consequently, $(R, +)$ is an abelian group.

Next we must show that R satisfies the right distributive law. Obviously, we note that for all $r, r' \in R$, all $s \in S$ and $0 \in R$,

$$0s = 0,\ (-r)s = -(rs) = r(-s)\ and\ (r + r')s = rs + r's.$$

On the other hand, for all $x, y \in G$, all $s \in S$ and $0 \in G$,

$$0s = 0,\ (-x)s = -(xs) = x(-s)\ and\ (x + y)s = xs + ys.$$

Let $x \in G$ and $r, s, t \in R$. Then the element t in R is represented by

$$t = \delta_1 s_1 + \delta_2 s_2 + \delta_3 s_3 + \cdots + \delta_n s_n,$$

where $\delta_i = 1$, or -1 and $s_i \in S$ for $1 \leq i \leq n$. Thus, using the above note and $(G, +)$ is abelian, we have the following equalities:

$$\begin{aligned}
x(r + s)t &= (xr + xs)t = (xr + xs)(\delta_1 s_1 + \delta_2 s_2 + \cdots + \delta_n s_n) \\
&= (xr + xs)\delta_1 s_1 + (xr + xs)\delta_2 s_2 + \cdots + (xr + xs)\delta_n s_n \\
&= \delta_1(xr + xs)s_1 + \delta_2(xr + xs)s_2 + \cdots + \delta_n(xr + xs)s_n \\
&= \delta_1(xrs_1 + xss_1) + \delta_2(xrs_2 + xss_2) + \cdots + \delta_n(xrs_n + xss_n) \\
&= \delta_1 xrs_1 + \delta_1 xss_1 + \delta_2 xrs_2 + \delta_2 xss_2 + \cdots + \delta_n xrs_n + \delta_n xss_n \\
&= xr\delta_1 s_1 + xs\delta_1 s_1 + xr\delta_2 s_2 + xs\delta_2 s_2 + \cdots + xr\delta_n s_n + xs\delta_n s_n \\
&= xr(\delta_1 s_1 + \delta_2 s_2 + \cdots + \delta_n s_n) + xs(\delta_1 s_1 + \delta_2 s_2 + \cdots \delta_n s_n) \\
&= xrt + xst = x(rt + st).
\end{aligned}$$

Thus we obtain that $x\{(r + s)t - (rt + st)\} = 0$ for all $x \in G$, namely,

$$(r + s)t - (rt + st) \in (0 : G) = A(G).$$

Since G is faithful, $A(G) = \{0\}$. Applying the first part of this proof, we see that $(r + s)t = rt + st$ for all $r, s, t \in R$, consequently, R satisfies the right distributive law. Hence R becomes a ring. $\square$

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